Commutative Algebra
Prof. A. V. Jayanthan
Department of Mathematics
Indian Institute of Technology, Madras

Lecture – 17
Tensor product of Algebras

So, we are talking about tensor products, we showed that tensor product is left exact right exact and it is tensor product is not left exact it is a not very difficult to see.

(Refer Slide Time: 00:45)

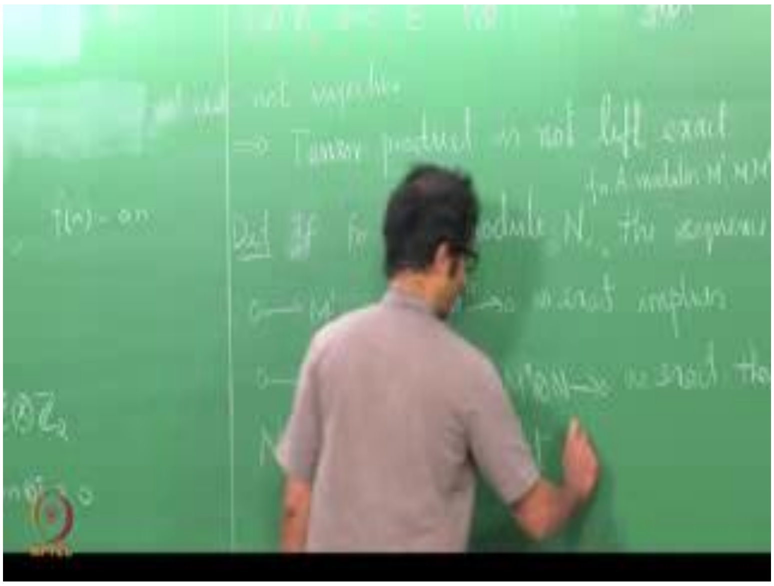
So, what is mean by it is not right exact sorry left exact, there exists I mean if I can find some module I mean some exact sequence such that this is the exact, but M tensor N to M to sorry M tensor N to M double prime tensor N to 0 is not exact. If I can find something like this that will say that it is not so; again when we are dealing with exactness we know that it is left exact. So, if you just hide this much this exactness will imply this exactness that is what we proved last time.

So, if we have to say that it is not left exact, we only have to say that this map there exists an exact sequence where this is injective, but this is not injecting. So, this portion does not really play a role. So, we only have to say that I can find something like this, but that is very easy if you look at Z to Z this is multiplication f of n equal to 0 n, take the multiplication by 2 and take N to be equal to. So, you are A and M, M prime all are Z; N is Z 2, Z naught to Z then what do we get here we have Z tensor Z 2 to Z tensor Z

2; this is map is f tensor one bar; what is the image of f tensor one bar of any N tensor one bar? This would be 2 N tensor one bar, but what is this you can write it as n times n tensor to bar which is 0.

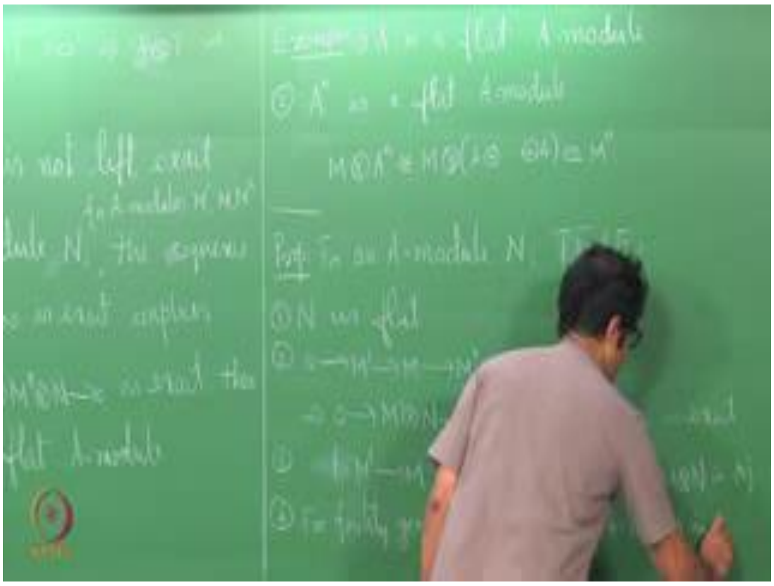
So therefore, this is 0 this implies that f tensor 1 bar is a 0 map.

(Refer Slide Time: 03:42)



But is this this is not 0 right Z tensor Z 2 is not 0 map, 0 module. 1 Z tensor Z 2 this is non zero and f tensor one bar is 0 this implies this f tensor one bar is not injective. So, this says that the tensor product is not left exact; if a module satisfies the property that for a very exact sequence this is exact, then that module N is called a flat a mod. So, let me if for an A module N, the sequence 0 to M prime to M to M double prime to 0 is exact implies 0 to M prime to N to M tensor N to M prime tensor N to 0 is exact, then N is said to be for A modules M prime M, M double prime; given any A module with this exact sequence the tensor product is exact, then N is said to be a flat a module it is said to be a flat A mod this is.

(Refer Slide Time: 06:28)



Some immediate examples of flat A modules, can you think of a flat a module this is the exact implies this is exact; tensor ring by. So, if I in particular if you start with suppose I take A, N equal to A, M prime tensor N M prime tensor A is M prime itself that is what we proved last time. So, this exact sequence will be same as this exact sequence. So, this exact implies this exact.

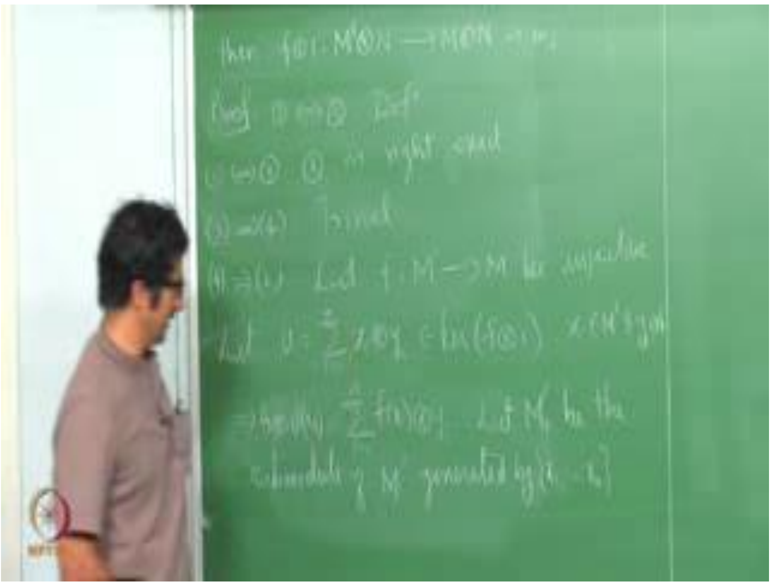
So, A is a flat A module, more generally A power N is a flat a module right. Because M tensor A power N what is this isomorphic to? We have M tensor A direction N times, we have seen that M distributes tensor product distributes over the summation therefore; this is M tensor A direct sum M tensor A direct sum and so on. So, this is isomorphic to M N, N copies of M. So, here what would be the map if I have f?

Student: (Refer Time: 08:15).

It will be if this is f and this is g then this if N is a power N this would be simply N copies of f, and here simply N copies of g therefore, if an element is 0 here f of it will be a particular f of that particular element will be 0, and we will have corresponding elements being 0 because f being injective. So, this is a flat A module; we will see more examples of flat modules little later.

So, to start with we will just look at one characterization of flat modules for an A module and the following are equivalent; N is flat 0 to M prime to M to M double prime to 0 is exact implies M prime tensor N to M tensor N to M double prime tensor N to 0 is exact, 0 to M prime to M is injective implies f tensor one from M prime tensor N to M tensor N is injective.

(Refer Slide Time: 10:56)



For finitely generated modules M prime and M, f from M prime to M is injective. For finitely generated modules M prime and M if this is injective, then f tensor one from M prime tensor N to M tensor N is injective. So, the main thing here is that one if and only 0 is the definition; one if and only if 0 this definition that is how flatness is defined. How do you say to if and only if 3?

We are using that tensor product is right exact. So, given this if I have a sequence the right exactness is already given to you. So, you only have to worry about the left exactness that is what the statement 3 is. So, this is tensor product is right exact; 3 implies for trivial; if it is if that statement holds for any module, it should particularly hold for finitely generated modules. So, only statement that we need to prove in this is 4 implies 3.

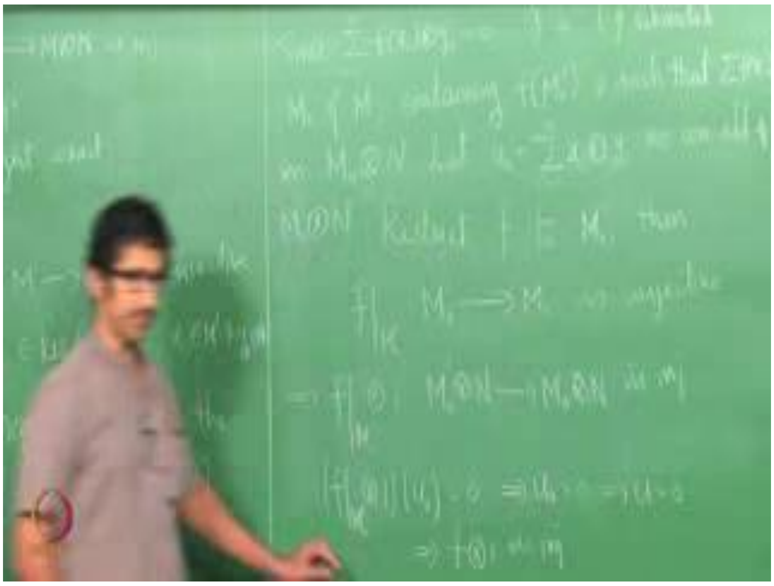
Student: Right exactly means that (Refer Time: 12:41).

This one if this is injective then this is injective is what we know; now to prove to suppose this is given, now we look at this corresponding exact sequence. The right exactness of the tensor products say is that it is exacting this is surjective and kernel of this map is equal to image of this map. So, only remaining thing is whether this is injective or that is given by this one third statement.

Now, 4 to say 4 implies 3 we need to say that see here the fourth statement says that you only need to check the left exactness for finitely generated modules; within the category of finitely generated modules if the tensor product is left exact then the module is flat. By definition you need to check in the category of A modules, but the fourth statement says that you only need to check within the category of finitely generated A modules. So, suppose the fourth statement is true and I have let f from M prime to M be injective, then I have and let us take an element u in the kernel of f tensor 1. I want to say that f tensor one is also injective suppose I start with an element in the kernel of f tensor 1 so; that means, that f of u which is equal sorry f tensor one of u which is equal to I from 1 to N f x I tensor y i. So, here I am taking x i in M prime and y i in N. see I need to bring in finitely generated module.

How do I bring in finitely generated module? Here I want to think of f as a map from a if I can think of f as a map from a finitely generated module to a finitely generated module, then I can say that it is corresponding tensor product will also be injective. So, what is the obvious possibility here? See I am see right now my aim is only to show that u is 0. So, let me take M naught be the sub module of M prime. So, let me write this M naught prime be the sub module of M prime generated by x 1 up to x N these elements, and this element f x i tensor y I this is 0.

(Refer Slide Time: 16:13)



See we have already proved that if I have a linear combination of tensor product equal to 0, then there exists a finitely generated module, finitely generated sub module of you know the corresponding modules, such that the tensor product is 0 in that module. See we have what we have seen earlier was that a tensor product is 0 does not mean that if I take an element.

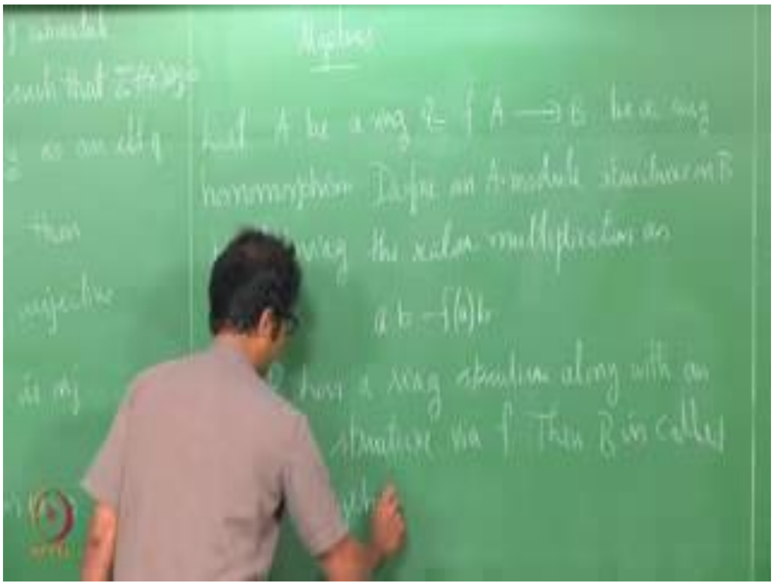
It is representation is 0 in the module M tensor N does not mean that it is representation is 0 in every sub module; we had that example of 0 tensor 1 in 0 tensor of 1 bar in Z tensor Z 2 and 2 Z tensor Z 2; in 2 Z tensor Z 2 it is non-zero. So, therefore, this, but we showed that if such a linear combination is 0 then there exists a finitely generated sub module such that there is this representation is also 0 in that. So, there exists a finitely generated sub module M naught of M, will I put one more condition containing f of M naught prime.

So, I will put 2 conditions here, 2 finite generation conditions; one is the finite generation condition incoming from the proposition that we proved earlier, and this is M naught is M naught prime is finitely generated. So, f of M naught prime is finitely generated therefore, I can take in fact, if I just take f of M prime f of M naught prime and the finitely generated in sub module existing finitely generated sub module here, taken there they can take we can take the sum and take M naught to be that one. So, let just take the finitely generated sub module M naught of M contain f of M naught, and such that summation f x i tensor y i is 0 in M naught tensor N; in this module it is 0 and this sub module it is 0.

So, let u naught be equal to summation x i tensor y I, i from 1 to N, as an element of M naught prime tensor N. I mean you can just see they have as we said they have rebra they could have different representations in different sub modules. So, I just take it as an element of M not tensor in this is. Now restrict f to M naught prime then f restricted to M naught prime there is a map from M naught prime to M naught; because M naught is the module sub finitely generated module containing f of M naught prime, then this is injective therefore, f restricted to M naught prime tensor 1 from M naught prime tensor N to M naught tensor N is injective by hypothesis, the fourth hypothesis says that for finitely generated modules if this is injective then this tensor is an injective.

But now f restricted to M naught prime tensor 1 acting on u naught, this is this is 0 right. That implies that u naught is 0 because it is this is injective; this implies u naught is zero, but now this is 0 in the sub module, if it is 0 in the sub model it has to be 0 in the module because it is you know some element is 0 implies you can you know this is in the quotient module. So, therefore, that as I mean does not matter whether you are considering it here or here this 0 will imply the map the element u is also 0. So, therefore, this says that f tensor one is injective.

(Refer Slide Time: 22:37)



So, this as we go along we will see more examples of a flat modules; right now we what we know is the free modules they are all flat that solving, we will see more examples probably even today. Another concept of a important concept in commutative algebra is of algebras this is pretty similar to rings, but it will have one more structure that let A be a ring and be f from A to B a be ring homomorphism, then this B can be thought of as an A module how do you do that? Define an A module structure on B by defining the scalar multiplication as a times b equal to through the map f, f a times p. So, therefore, b has a ring structure as well as an A module structure. So, any ring b with a homomorphism like this is called an a algebra.

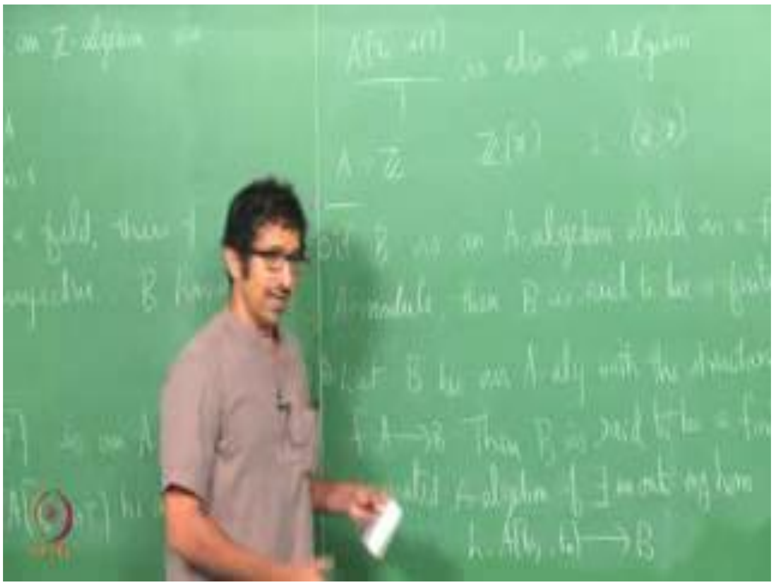
Along with a with an module structure via f then B is called an A algebra. So, this is a trivial example is any ring is in Z algebra.

(Refer Slide Time: 25:22)

It has we know that it has a ring structure and any competitive ring is a Z module. We have N times. So, I have a map via Z going to any ring A is an Z algebra. This is N going to N times 1; we have this map is called structure map. In fact, when we say B is an A algebra why are the structure map f. So, with this structure map N going to N times 1 this is an a or any ring is N algebra. If k is if A is a field then your ring map will be a ring homomorphism from A to B if it is non 0 it is always injective.

So, you will always have if A is a field, a algebra will always contain a copy of A, then f from A to B assuming it is non 0 it is always injective. Therefore, B has a copy of A; any k algebra will contain a copy of k itself. Some of the familiar k algebras or A in fact, of any ring A, A algebra, x alpha polynomial ring over polynomial ring in what some variables over a is an A algebra.

(Refer Slide Time: 28:39)

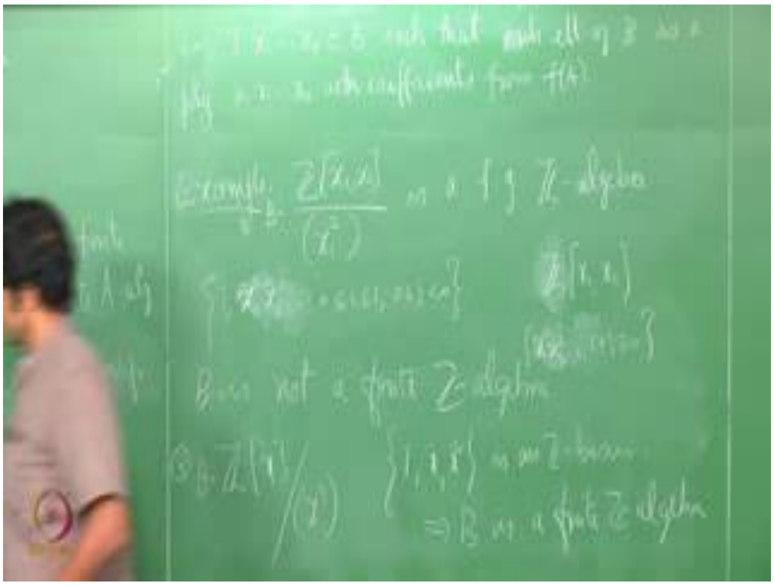


So, in this case also we have a copy of A inside this, A going to this the inclusion map is injective. So, this is an A algebra. In fact, more generally if I have an ideal let I be an ideal, then this is also it is also an A algebra. In this case this A to this one a to A x alpha mod I need not necessarily be injective for example, if you have if your ring A is Z and this is A algebra is Z x and you take ideal I to be 0 comma x, then this map is not injective; A to a Z x mod I is not injective right. So, this is these are some examples of algebras if B is an A algebra, which is a finite A module, then B is said to be a finite A algebra.

There are 2 terminologies one should be careful and if B is; if every element of B if every. So, let B be an a algebra with the structure map f from A to B then B is said to be finitely generated A algebra, if there exists I will simply write like this there exists a on to ring homomorphism h from A polynomial ring t naught of t n to B.

That means I can write B as a; see I mean I can find finitely many elements here such that every element of B is a polynomial on this with coefficients coming from f of a.

(Refer Slide Time: 32:35)



Let me write down that is there exists x 1 up to x n in B such that each element of B is a polynomial in x 1 up to x n with coefficients from f A. We can write A or f a because when you write A by definition it will be f of A. t 1 or in other words B will be some finitely generated polynomial ring modulo an ideal, by first isomorphism theorem if I have in ring on to ring homomorphism like this, B is isomorphic to this modulo. So, if I have this is.

So, for example, Z x 1 x 2 modulo x 1 square, this is a finitely generated Z algebra is this a finite Z algebra is this is the finite Z algebra as a Z module can you tell me a generating set for this, B is this, can you tell me a generating set for this as a Z algebra or Z module I will have one of course, one bar I will denote by bar because you know this is a question ring modulo of. I will have x 1 bar see if you take this Z x 1 x 2 can you tell me a generating set for this as a module over Z; as a module over Z. Can you tell me as a vector space over Q, what is its generating set? As a vector space over Q is this finite dimensional, so what does it is a can you give me a basis.

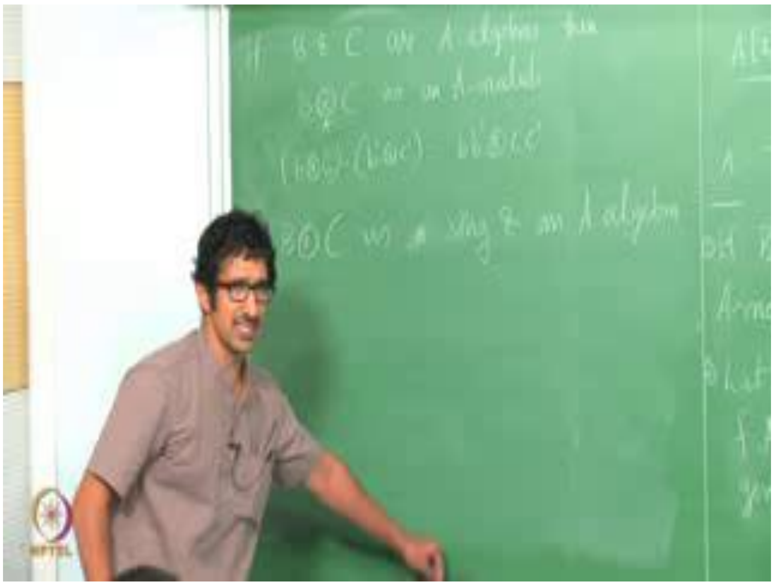
Student: (Refer Time: 35:50).

X 1, x 1 square, can you be can we be little more. So, I can simply write x 1 power i x 2 power j, i plus j bigger than to 0, this is a basis. Now can you give me a Z module as a Z module what is its generating set a minimal generating set? This will work right this will again be a generate Z module generating set right. So, when you take a quotient see you are basically nullifying some of them right, so which all will remain in a minimal generating set.

Student: (Refer Time: 36:56).

X 1, x 1, x 2, x 1 or x 2 j. So, x 1 power i, x 2 power j what is the condition on i and j? 0 less than to I less than to 1, 0 less than to j; so this is also this is as an Z module this is not finite, but this is a finitely generated Z algebra. So, this is not a B is not a finite Z algebra, it is a finitely generated Z algebra. B is not a finite Z algebra and if I take Z x modulo x cube, as a Z module what is it is can you give me a generating set for this one; right this is an Z basis in fact. So, therefore, Z I call this B, B is a finite Z algebra, just wanted to give an example of a finite Z algebra and the finitely generated 0 algebra. The first one is a finitely generated 0 algebra which is not a finite algebra, the second one is a finite algebra.

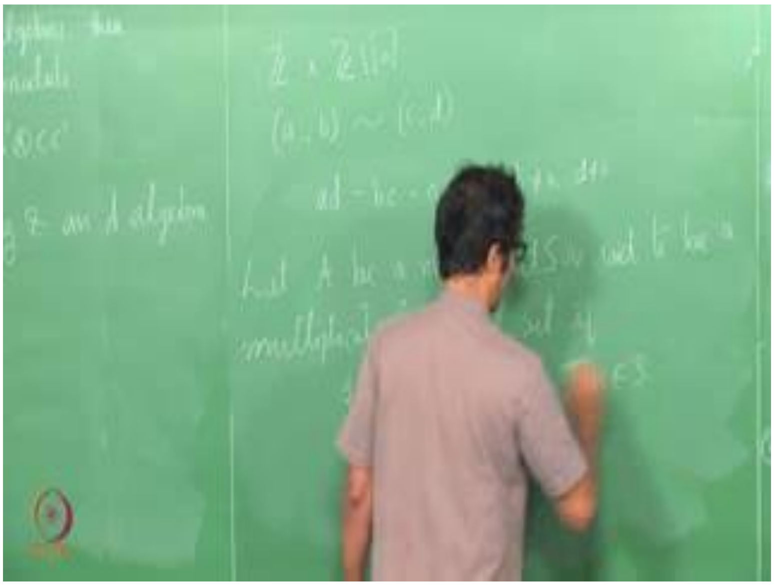
(Refer Slide Time: 39:44)



If it is a finite A algebra it is automatically a finitely generated A algebra, because here we have finitely many elements whose linear combination is A linear combination gives the whole ring, that automatically gives all polynomial combination. So, if B and C are A algebras, then B tensor C this is an A module, as A modules I can take tensor products right? This is an A module and this this has a ring structure by taking the multiplication. So, let me write down this definition of B tensor, see this has a this is an A module though. So, this has a addition in place; now if I take B tensor C, B prime tensor C prime I define it to be B B prime tensor C C.

So, with this multiplication B tensor C is an a, as is a ring and hence an A algebra. I will leave the easy verification to you to complete this. We will see more of A algebras later, but now let us a move on to study what is called localizations. So, localization is a process that we you know generalizing the construction of q from z. So, how did we construct q from z?

(Refer Slide Time: 41:59)



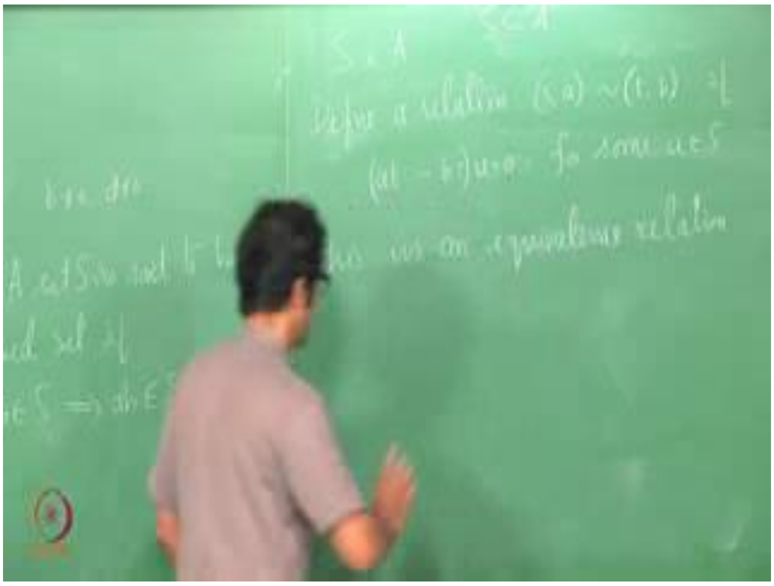
We look at this set Z cross Z, I take you know some a comma b, I say that this is equivalent to c comma d if. So, I mean see with this I have my I have in mind a by b and this is c by d, when do I say these 0 are equal?

Student: A d minus (Refer Time: 42:37).

A d minus b c is 0 right, but here the we have one more condition that neither b nor d is 0 right; b is non 0 d is non 0 or in other words the second component is in fact, Z without 0 right? We are taking this cross product and putting this equivalence relation on this set. Now we imitate this and do it in the case of general rings; see in the case of general rings we want to you know move ahead from a and get to a place there we have more units, see in this by this construction then we prove that this is an equivalence relation and say that the equivalence class is basically q. So, in general the idea is we get a ring containing a which has more units.

So, the idea is this, let A be a ring a set is said to be multiplicatively closed, said to be a multiplicatively closed set if one belongs to. So, a set S, one belongs to S, and a comma b belongs to S implies the product is in S. So, what we will do is taking a multi multiplicative a closed set.

(Refer Slide Time: 44:54)



So, I take look at this S cross A define a relation on S cross A, S comma A is same as t comma b, if a t minus b s is 0 this is what we want. Ultimately what is that we want we want to say that a by s is same as b by t, but again there is one more you know small or in other words we want to say this is 0 and our aim is like this, but this is t a or a t times b s divided by t s.

But at the same time see I want a by s equal to b by t, see I should also get this is same as some a t prime s t prime; this should be equal to this as well right. On other words a s t prime minus; so here. So, let us now look at this condition, this says that b s t prime minus a t t prime this is 0, or in other words this is 0 in the case of Z this 0 will automatically imply that this is 0 but.

Student: (Refer Time: 46:59).

Yes, sorry s is a subset of a. We are inverting elements of s some you know some elements of a. So, in the case of Z once you reach here this will automatically imply this is 0, but in the case of general rings this does not imply that this is 0; if it is not an integral domain right. So, when we are doing this process, when we are making this relation this is probably not enough if you want to say a by s is same as b by t same as a t prime by s t prime. What we want is if I can find some u such that this time is for some u in s ok.

Student: (Refer Time: 48:08).

We are not saying that right now that will be a corollary. If I have 0 in that we will see what happens to this process. So, this one let me stop here by saying that this is an equivalence relation. We will continue from here next class.