Commutative Algebra
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Lecture - 16
Properties of tensor products (Continued)

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We were talking about tensor product. So, last time we were discussing about A tensor M. So, as we saw there is a natural map from A cross M to M, a comma m being mapped to a m, this map is A bilinear. Therefore, this extends to a map from A tensor M to M. So, let us call this map phi, there exists a map phi prime from A tensor M to M phi prime of a tensor m is equal to a m, phi tensor of a, phi prime of a tensor m is equal to phi of a comma m which is a m. Once, this is generated by all elements of the form A tensor M where a is in A and m is in M. So, therefore, once you define it on these elements, it is defined throughout. See now we are trying to say that this is an isomorphism.

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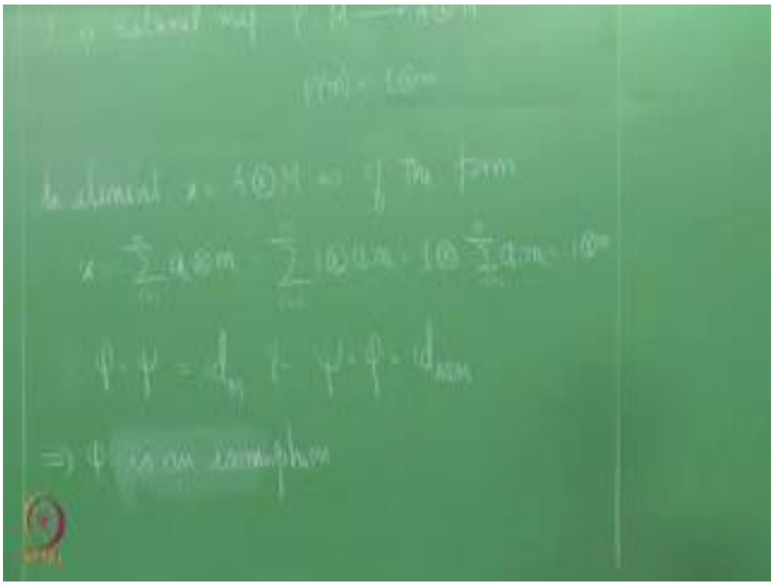

There is a natural map. There exists a natural map. So, let us call that psi, psi from M to A tensor M, what is that natural map, can you think of a natural map.

Student: (Refer Time: 02:28).

Any element here psi of m equal to;

Student: (Refer Time: 02:36).

1 tensor m, this is like embedding a module into a cross product. So, therefore, this is, there is an, I mean this is a map from M to A tensor M, we have mapped from A tensor M to M.

Now, let us look at their composition, what do I get?

Student: sir (Refer Time: 03:10).

So, let me take that up first. See any element here, how does any element here look like? In general if I take 2 A modules M and N, any element of M tensor N looks like summation i tensor n i, where m i belongs to m, and n i belongs to n. Now if I take any element here, an element x in A tensor M is of the form x equal to summation a i tensor m i i from 1 to sum n some finite n, but now look at this elements a i belongs to a, a i tensor m i is equal to, is also equal to 1 tensor, and a i you can take a i outside, a i times 1 tensor m i, but a i times 1 tensor m is same as 1 tensor a i m i. So, this is, I can write it as i from 1 to n 1 tensor a i m i, but now this summation is equal to 1 tensor summation a i m i using the property of tensor product. So, therefore, every element, I can write this in the form 1 tensor m. So, every element of a tensor m is of the form 1 tensor m for some m and n.

Student: (Refer Time: 05:08).

So, therefore, we can, these 2 are inverses of each other. So, therefore, if I take this map, this is surjective and phi tensor; therefore, phi tensors i is equal s identity on m, and psi tensor phi is identity on a tensor m. therefore, phi and psi are inverses of, you know phi is an isomorphic. So, a tensor m is nothing, but simply m. is this clear.

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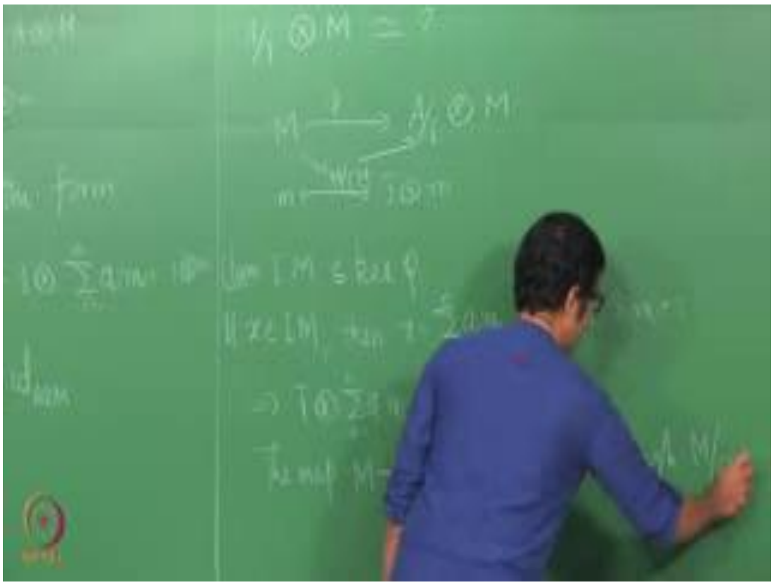

So, therefore, what is A mod I tensor M. see there is a natural map, see you have a natural isomorphism M to A tensor M. there is a natural map from you know M to A mod I tensor M. What is the natural map x going to or m going to 1 bar tensor m, and; obviously, there is a natural map from k. So, now can you give me some elements which vanishes or you know some sub module of kernel of this map. Suppose I call this phi, and you tell me some sub module of kernel of phi. I M is a sub module of kernel of phi right, because if I take any element suppose x belongs to I M then x is of the form summation a i m i i from 1 to n with a i n i and m i in m, but then 1; 1 bar tensor summation a i m i is same as summation i from 1 to n a i bar tensor m i, but a i is in i therefore, a i bar is 0; therefore, this is 0. So, phi takes every element of i m 2 0;

therefore, this extends to a map from m mod i m to this, there exists. Therefore, i m is contained in kernel; therefore, the map M to i m m to a mod i order tensor m extends to; you know filters through M mod I M.

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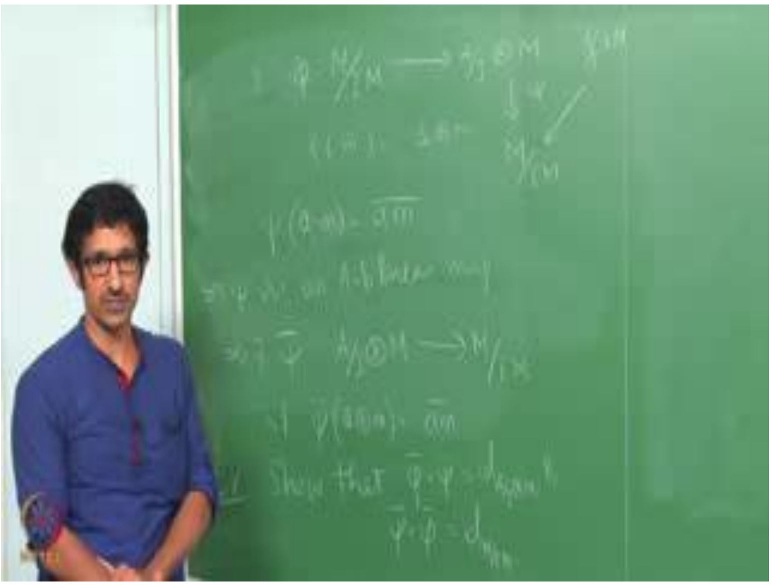

So, there exists, there exists is map M mod I M. So, let me call this phi bar from M mod I M to M mod I tensor M phi of m bar is equal to 1 bar tensor m. we have already seen that this is well defined, because of that. Now, can you think of see we are trying to say. Can you think of a map from here to M mod I M, in there is a natural map.

Student: (Refer Time: 10:37).

A bar, so this is let me call this psi; psi of a m, sorry a bar tensor m up to n bar. This again you know. So, this is can we see whether these 2 are you know, as in the case of isomorphic in between m and a tensor m. what can you say about psi tensor phi bar. See here also you have to make sure that the map is well defined. First of all there is a quotient M mod I, then this tensor products itself is a quotient right. So, this is if I take a map from here to there it is well defined, but this ism you do not really have to do that, because this comes as a natural map from a mod I cross m to M mode I M, you have this natural map a bar comma m going to a m bar. We do not really have to worry about this. So, first one can define a bar m equal to m bar. This is a well-defined natural bilinear a bilinear map.

Then psi is innate by linear map. So, therefore, there exists psi bar from A mod I tensor M to M mod I M; such that psi bar tensor psi bar of a bar tensor m is am bar. This is same as psi of a bar comma m which is m bar. Now show that phi bar tensor phi bar phi bar composite phi bar is identity on a mode i tensor m and psi bar composite phi bar is identity on m mod i m. So, therefore, this is an isomorphism.

So, here you know most often we are, when we have to produce an isomorphism, what we need to prove is. There exists some natural map between them, then there exists a natural inverse. These are the 2 things that we need to prove; that is what exactly we prove. There exists a natural map and there exists another natural map, and these 2 are inverses of each other. In most of the situation that is exactly what wants job to prove the isomorphism.

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So, let me write down some more properties M N P are a modules, then there exists natural isomorphism M tensor N to N tensor M. Before doing this let me mention one more thing. We have defined tensor product for 2 modules; one can be define it for finitely many modules, by extending the definition to instead of bilinear, one talks about multilinear. Linear on each variable, if I have a map from M 1 cross M 2 cross multilinear map from M 1 cross M 2 cross M n to a module phi. Then the unique module through which it filters, is called the, you know tensor product of them.

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So, I can denote if I have m 1 cross I have modules M 1 cross M n etcetera M n, then there exists this unique module M 1 tensor M 2 tensor M N. Once we do this, you know there is a natural question that arises. to start with let us say suppose I have three modules M 1, M 2, M 3, then I have I can form these 2 and take these to form the tensor product, and then take the tensor product with the third one. Take this second and third take the tensor product, and do tensor product with this one and use the pure definition to get M 1 tensor M 2 tensor M 3, how are they related naturally one expects that these are all isomorphic and they are indeed.

So, first I mean I will simply state that, proofs are all exactly like this we have to just find the natural map between these modules and then prove that you know they are there exists inverse; the natural maps which are inverses. So, this is n tensor m map to m tensor m tensor m. Second one m tensor n tensor p this isomorphic to m tensor n tensor p which is isomorphic to m tensor n tensor p the isomorphism given by natural isomorphism m tensor n this will be element here p this is mapped to m tensor m tensor n tensor p and this is, now there is second map is this is being mapped to, these are all natural isomorphism.

Another important property is the distributive of tensor product over the direct sum; that is M direct sum N tensor P to N tensor P direct sum N tensor P given by m comma n tensor p map to m tensor of p comma intensity. I will not go into the details of the proofs, but these are all, I mean all these natural maps given here. They are all the isomorphisms. You only have to get a map from here to here, what is the natural map that you can think of from here to here N tensor M map to m tensor n. So, to say that, see you have to come through the; you know natural I mean to say; see every time this is a quotient. So, every time you have to prove the well definedness, you have to discuss the well definedness, but then if you just define this map and try to prove the well definedness, it is difficult. So, the easiest way is to define from n cross m, and then say that it, you know it is a by a bilinear map; therefore, it filters through this one, use the universal mapping property. So, therefore, always defined from there and then get this natural map from here to here. Similarly in this case and in this case.

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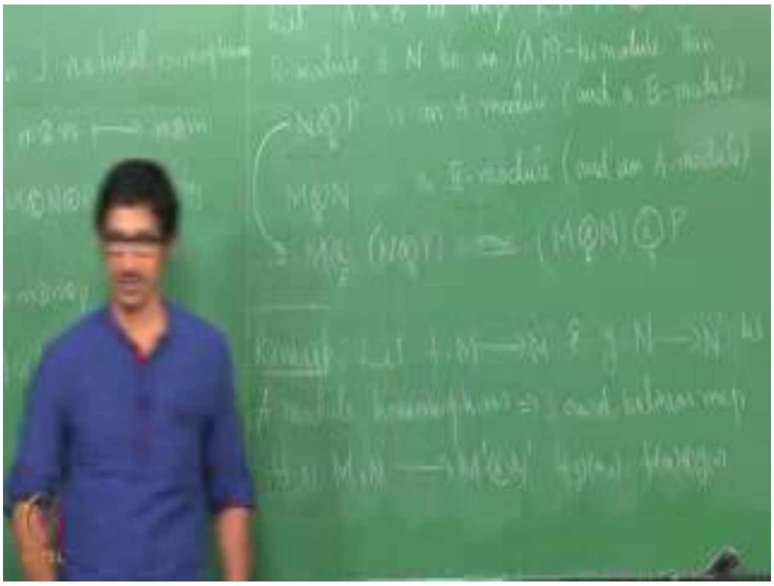

Now, suppose we have, let A and B be rings, M be an A module P be a B module and N be an A B bi module. What do I mean by A B bi module? it is a module over a as well as a module over B. for example, if you take any q vector space, it is a module over q as well as a module over z, or any sub ring of q. So, in particular if you know, if is A sub ring of B then any module over B is also an module over A, but some modules over A need not necessarily be modules over B.

So, when I say A comma B be a bi module, the multiplication is also compatible. Then if I look at M is an a module, then then N tensor P, this is an A module, N is an A module. So, look at this how do you define the module structure here. This is in fact, a bi module right, this is an A module as well as A B module. Moreover; so, I have this, I have M tensor N, this is. See this is naturally a A module, but this is a B module aspects. You can define the scalar multiplication to be scalar multiplication on the right component. So, therefore, this is a B module aspect.

So, using this I can define this gives M tensor N tensor P. So, when I write this under a tensor product, it simply means I am taking the tensor product as this module. And then I have M tensor N, and then tensor P as B module. Then one can say that this. So, I will simply, again this is exercise, these 2 are isomorphic. The natural isomorphism that one can think of, is an isomorphism. So, therefore, this is simply a isomorphism, this is an exercise. Now there is another observation that one makes.

Suppose M N and P be A modules. Well before that, let it be there as remark. Suppose, I have f from M prime and g from N to N prime be A module homeomorphisms. Then this naturally you know gives a homomorphism from M cross N to M prime cross N prime since. So, therefore, there exists a map an a bilinear map f comma g from M cross N to, I will define to m prime tensor n prime a comma b mapping to f a tensor g of b, or f comma g being of M comma N this is equal to f of M tensor g of N. this is a an A bilinear map therefore, I have a map from M tensor N to M prime tensor N prime.

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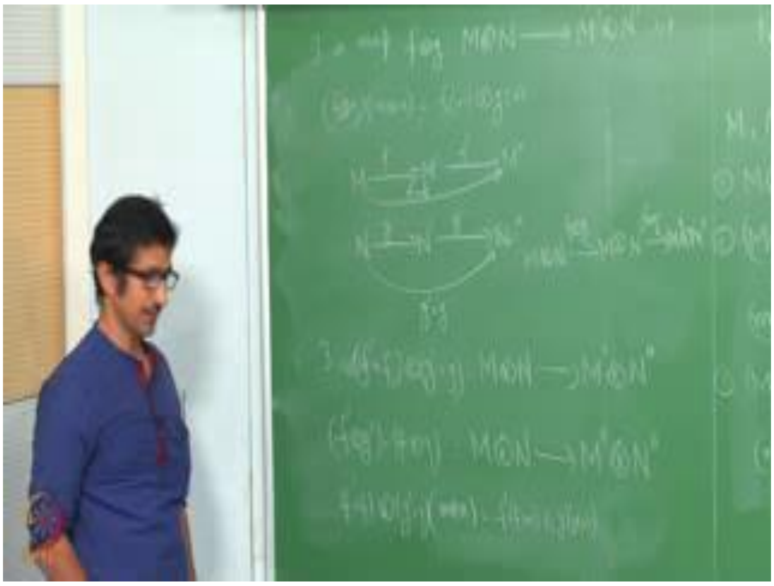

There exists a map I will call this f tensor g the map from M tensor N to M prime tensor n; such that f tensor g acting on m tensor n is equal to f m tensor g m.

Student: (Refer Time: 28:10).

Sorry N prime, now, further. So, I have now mapped from M to M prime. Suppose I have this map M double prime I have N to N prime to N double prime. So, this gives me a map from f let me call this f prime this and let me call this g prime. So, therefore, I have a map from M to M double prime, which is f prime composite g. here I have map from N to N double prime which is g prime composite g. So, this map f prime composite g and, f prime composite f and g prime composite g. these 2 maps give me and using this construction, using this observation I get a map from, there exists a map. In fact, there exists 2 maps; there exists f prime composite f tensor g prime composite g from M tensor N to M double prime tensor N double prime. This composition comes from considering this as a homomorphism from M to M double prime, and this as a homophone from N to N double prime.

Now, we can look at from these 2 maps I have mapped from M tensor N to M prime tensor N prime, and from these 2 maps I have mapped from M prime tensor N prime to M double prime N tensor N double prime. So, these 2 maps give me f prime tensor g prime composite f tensor g. this is again map from M tensor N to M double prime tensor N double prime. This as a map from M tensor N to M prime tensor N prime and then to M double prime tensor M triple prime, this is f tensor g this is f prime tensor g prime looking at individual maps. This, the first one I get it looking at that composition, directly looking at the composition. Now, what happens if I took I mean, if I apply this on an elementary tensor; that is f prime composite f tensor g prime composite g acting on an elementary tensor. This is called an elementary tensor element, because this generates the whole thing M tensor N. what will be this. This by definition this will be same as f prime composite f or f prime of f of m tensor g prime of g of n.

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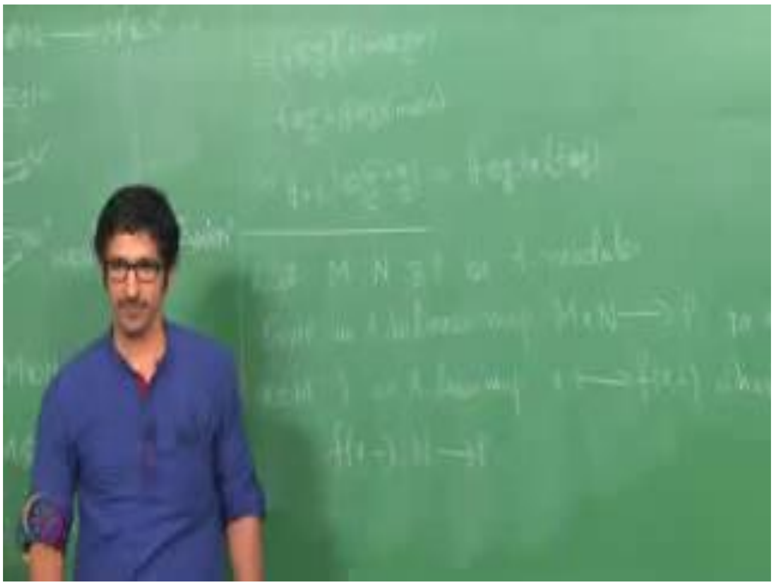

But now f prime of f of m tensor g prime of g of n. This is same as which is equal to f prime tensor g prime of f m tensor g n, but this is same as f prime tensor g prime composite f tensor g acting on the element m tensor n. So, what we have seen now is that.

Student: (Refer Time: 32:57).

Yes, both these maps agree on the generating set, each element of the generating set; therefore, f composite f, f prime composite f these 2 maps, this is same as f composite g. Is this clear?

Now, we make another observation regarding the tensor product. So, let M N and P be A modules. Look at this. So, given an A bilinear map M cross N to P. for each x in M there exists a; an A linear map; x going to f of x comma y. So, f of x, where this is f of x comma, this is a map from N to P. any a bilinear map I have this x from. So, what are we getting, suppose I am given an A bilinear map, I am getting a map from M to. From M I am getting a homomorphism from M to P. for any element in m there exists a homomorphism from N to P.

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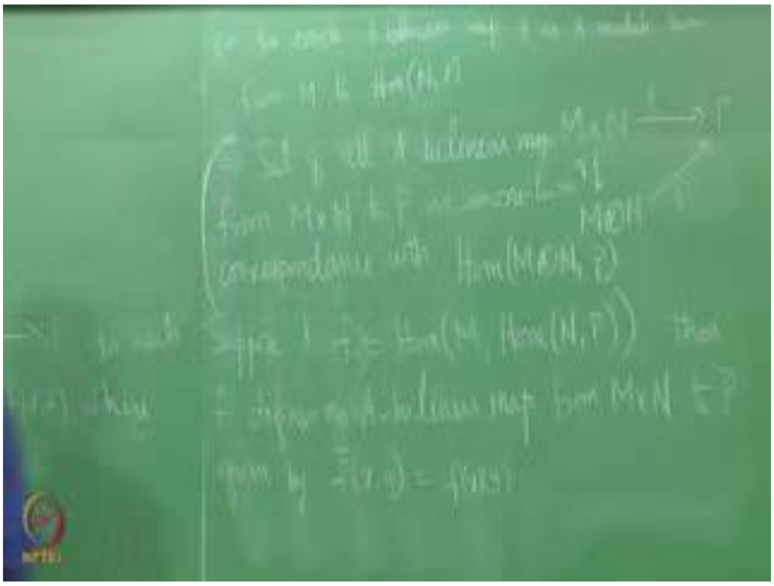

Or in other words we are getting a map from, for each that is; for each A bilinear map there exists an A module homomorphism from M to hom N P. Now, while defining the tensor product, what we have shown is that. Given any A bilinear map from M cross N to P, there exists a unique A module homomorphism from M tensor N to P; such that the unique homomorphism acting on the basic elementary M tensor N is same as f of A tensor N A comma B or whatever. So, this says that. See given any a bilinear map there exists a unique a module homomorphism from M tensor N to P given any bilinear map from here to here, there exists a unique map. So, there exists a unique map from here to here such that, given this given f there exists a unique f prime such that g composite, f prime composite g is same as f.

That means the set of all bilinear maps from M cross N to P, is in one to one correspondence with set of all homomorphism from m tensor into p. this says that set of all A bilinear maps from M cross N to P is in one to one correspondence with hom M tensor N P. we have to observe one more thing here, converse of this. See what we have shown here, is that given any A bilinear map from M cross N to P there exists a homomorphism from M to hom N P.

Now, suppose there exists, this should be coming here actually. Suppose there exists a map let us say f from f n hom M to hom N P. Suppose I have a map, for every element I have a homomorphism from N to P. this will define a bilinear map from M cross N to P then f defines a an A bilinear map from M cross N to P given by. So, f bar of x comma y this should be. For each x f x is a homomorphism from N to P. we want an element in P. So, f of x is a homomorphism from N to P. So, therefore, this acting on y. we know that each of them is linear therefore, this is bilinear.

So, here what we have seen is that the set of all bilinear maps is in one to one correspondence with hom M hom N P. And here we are seeing that set of all A bilinear maps from M cross N to P is in one to one correspondence with hom M tensor N to P. And it is an easy exercise to verify that the natural maps from queen hom M hom N comma P to hom of M tensor N to P, this is an isomorphism.

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So, what are the natural maps, let us let me just write down the natural maps; hom from hom N P to hom of M tensor N to P. What is the natural map? Suppose I have a map f. So, let us call this phi; phi of f. This should be a map, this should be an element in this one or in other words it is a homomorphism from M tensor N to P. So, therefore, we need to say what happens if it acts on every element of. Once you define it on the elementary tensor products, it is defined on the entire M tensor N. So, phi of f acting on any M tensor N should look like, what should be this be f is a map it is.

Student: (Refer Time: 42:56).

That is exactly what we defined there: it should be f m acting on m; that is precisely the map that we defined over there. See we are defining this is. So, defined this is a define the map from here to here. So, for each f here, I am defining what is phi of f phi of f is the homomorphism which acting on M tensor N gives me this, yes. See here. See, so let me phi of f, where should this belong to. This is in hom; that means, let me denote this by M this in hom M tensor N to P, where g is the homomorphism, which takes every element this to. Is that clear now? And I just described the homomorphism directly. So, g is the homomorphism that takes this one. Now I will simply leave it to say that this phi is an isomorphism. this is exactly the map that you know come from here to M cross N to, hom M comma N, hom of N comma P to set of all bilinear maps from.

Student: (Refer Time: 45:09).

Yes, M cross N to P and this is basically the composition of those 2 maps. So, phi is an isomorphism.

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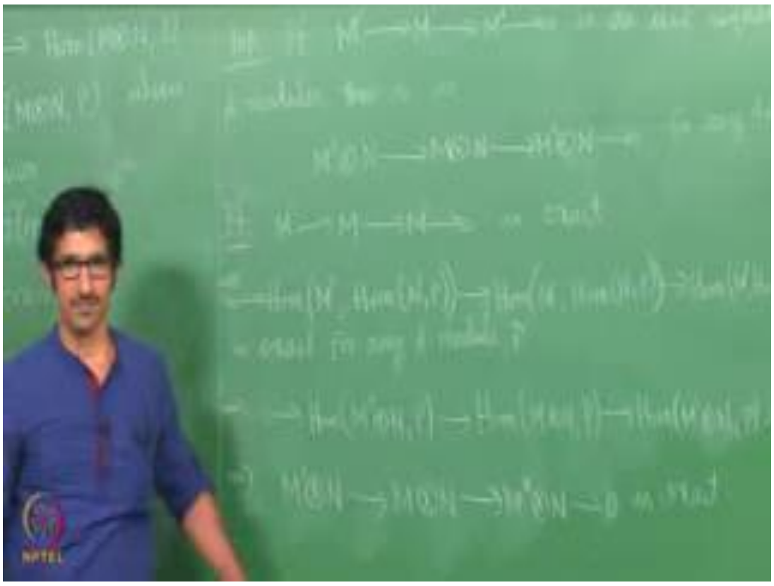

Now there is a very interesting corollary of this observation that I will write this proposition, if M prime to M to M double prime to 0 is an exact sequence of A modules then, so this M prime tensor N to M tensor N to M double prime tensor [noise]; 0, for any A module N. So, this is, in the hom properties we showed that this is exact if and only if;

Student: 0 to hom.

0 to hom.

Student: (Refer Time: 46:47).

M double prime to some module you know that is exact, it is if and only if. So, now, we use that property. this is exact M prime to M to M double prime to 0 is exact, implies that I use it with the hom M double prime to hom N P to hom. So, my module now here is, hom N P, I just take some module P and use this for any module. Let P be an A module, then this is exact. Hom M to hom N P to hom M prime to, sorry there is a 0 here 0 to this; hom M prime to hom N P, this is exact for any A module P, and this implies that 0 to. Now we use this isomorphism hom M double prime tensor N P to hom M prime tensor M, sorry, M tensor N P to hom M prime tensor N to P is exact, but again using the property of hom, the if and only property of hom to say that, this implies M prime tensor N to M M tensor N to M double prime tensor N to P, sorry to 0 is exact.

Student: (Refer Time: 49:16).

Yes this from here to here is it fine. Now this what did we prove here, this is isomorphic to this so;

Student: (Refer Time: 49:28).

Now, this is again you have a map, we proved and we use this. This is exact if and only if the corresponding hom factures hom thing is exact. We proved only one direction, but I added that it is if and only.

Student: (Refer Time: 49:48).

So, therefore, from here to here we use the reverse direction of that result, and say that this is exact. So, this result is in short, said as the tensor products right exact.

We will continue in the next class.