Commutative Algebra
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Lecture - 15
Properties of Tensor Products

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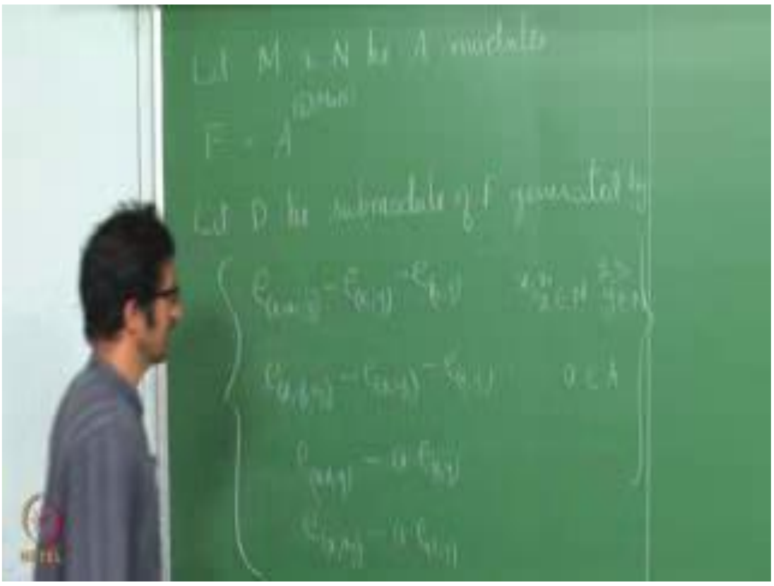

So we were talking about tensor products yesterday. Let us recall the definition. So, let M and N be A modules, F be as many copies of A as number of elements of M cross N. Then let D be the module generated by be the sub module of F generated by e x 1 plus x 2 y, minus x 1 y, minus x 2 y; e x y 1 plus y 2, minus e x y 1, minus e x y 2; e a x y minus a times x y e x a y minus a times e x y; x 1 x 2 x is in M, y 1 y 2 y in N, a in A.

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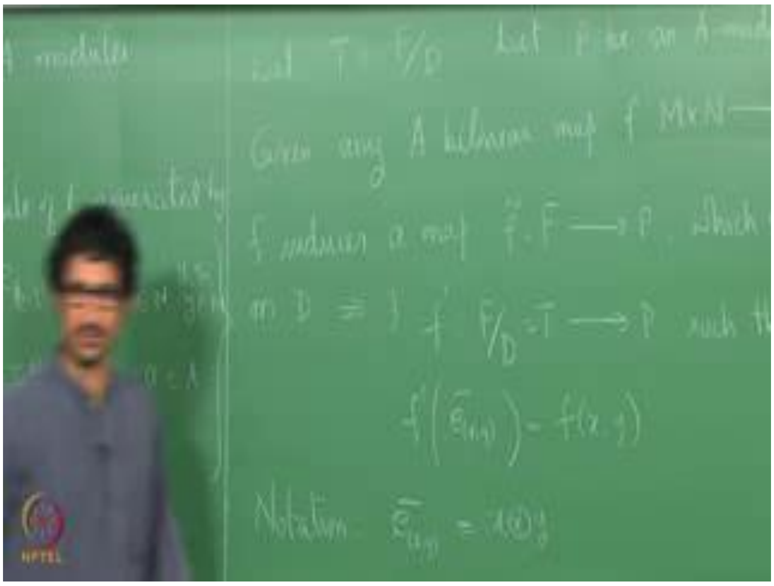

And let D be, let us write T as F mod D. So, given any bilinear map A bilinear map, F from M cross N to P where P is an A module given any M cross N to P, F induces a map F tilde from F to P which vanishes on D, therefore, there exists F prime from F mod D which is T to P or in other words what we have now is that I have given such that F prime of the image of each x y is equal to f of x y, this is in. So, let us set this notation the image of e x y in T this is denoted by x tensor y. These are the generators of T this is called tensor, we call it.

Student: Sir, what is the motivation behind doing all this.

Yeah we will come to that. So, the main motivation is precisely what we said in the beginning that given a bilinear map from M cross N to P there exist a unique module in which it filters through.

Student: What exactly meaning in the filters through?

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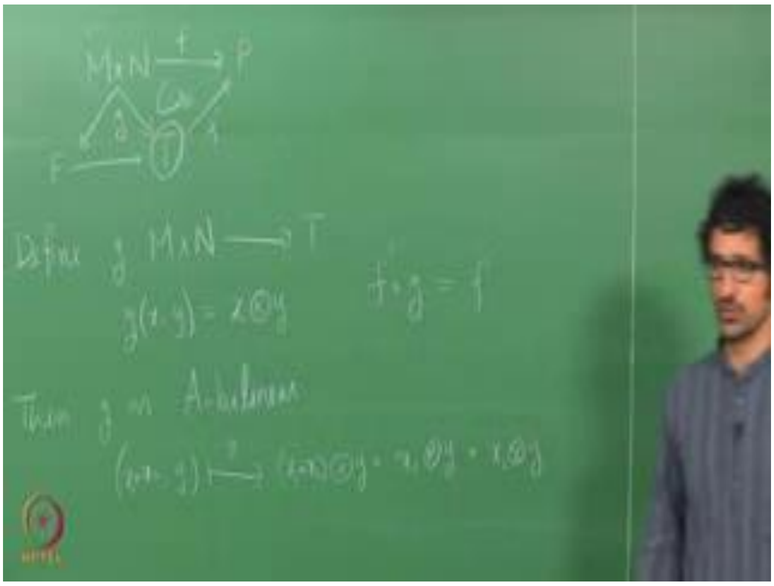

Filters through is, so if I have a map, this is I have M cross N to P. So, we are looking for in some sense universal module which depends on M and N such that given this I have bilinear map which is, this should be a natural map and this is a map that induced by F such that this diagram is commutative. So, this such existence of such a module that gives you know studying the structure of T one can say about M and N, similarly given M and N one can study the structure of T and you know if we go little forward we will see that this construction looks so unnatural, but the modules that one obtained they are very very natural So, the construction was probably and if you look back the history the construction was done after realizing what the module is.

Student: (Refer Time: 07:20).

Yeah, we are going to talk about what g is. So, we have now, this is what we are looking for. So, define there is a natural you know definition g of, g from M cross N to T, g of M N g of x comma y equal to. There is a natural way right g this induces a map into F x comma y going to e x comma y and from F there is a natural quotient map x comma e x comma y mapped to.

Student: e bar.

E bar x comma y, e bar x comma y I am denoting it by x tensor y. So, there is a natural map from M cross N to T which I will denote this by x tensor y. Then this is a map from

M cross into T then g is A-bilinear right I look at where would x plus sorry x 1 plus x 2 comma y be mapped to, this will be mapped to x 1 plus x 2 tensor y I mean this is e bar of x 1 plus x 2 comma y.

Student: (Refer Time: 09:06).

But we are modulo D; modulo D this is e of x 1 comma y.

Student: (Refer Time: 09:12).

Plus I mean. So, this minus that is 0. So, therefore, this is e of x 1 comma y plus e of x 2 comma y bar which is same as x 1 tensor y plus x 2 tensor y. And similarly on the other variable and we have, and a also and scalar multiplication also is taken care of similarly; g of a x comma y is a x tensor y, but modulo D it is same as a times x tensor y. So, therefore, this g is an A-bilinear map from M cross N to T. And by definition F prime composite g is F by definition that is how we have constructed F prime; that is how we have constructed g. So, by definition we have this map F filters through T.

So, what we have done is we have produced an A module called T and we have produced a bilinear map g from M cross N to T such that given any bilinear map F from M cross N to P I have a map from T to P such that F prime composite g is same as F. So, this is this property is called a universal mapping property, given any F bilinear map from here to here there exists a map from this to this, such that, this is already there such that this is commutative this diagram is commutative you start from here go via this or this is the same.

Now, what we are interested in whether this you know can we have several, we have in this case we have constructed a T. Does there exist any other module with the same property that given any bilinear map from here to here do we have a module here through which it filters. So, let me, let us see what happens in that case. Sorry?

Student: F prime?

F prime is well; F prime is not bilinear, bilinear in this sense; in the sense that so if I think of this as 2 variables then it is bilinear. Now I mean usually bilinear is talked about when you have 2 coordinates, but well this is also said bilinear.

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So, suppose I have I mean what we are trying to see is the uniqueness of T. So, suppose there exists a T prime there exists an A module T prime and a map g prime, g prime from M cross N to T prime bilinear map A-bilinear, such that given any A-bilinear map F from M cross N to P. So, I have this, this is T prime g prime, so this is F prime. There exists F prime from T prime to P such that F prime g prime is equal to F. Suppose you have another module T prime which satisfies the universal property, universal mapping property.

We want to say that this is T prime is isomorphic to T. So, how do I say that? See these 2 both of them are defined using a universal mapping property. So, we have T prime is defined using this property, there exists a module T prime with g prime a map from here to here A-bilinear such that any such map filters through T prime. So, we can replace this by our T. So, consider the map, we have 2 pairs M cross N to T to T prime. So, this is our g prime, this is our g.

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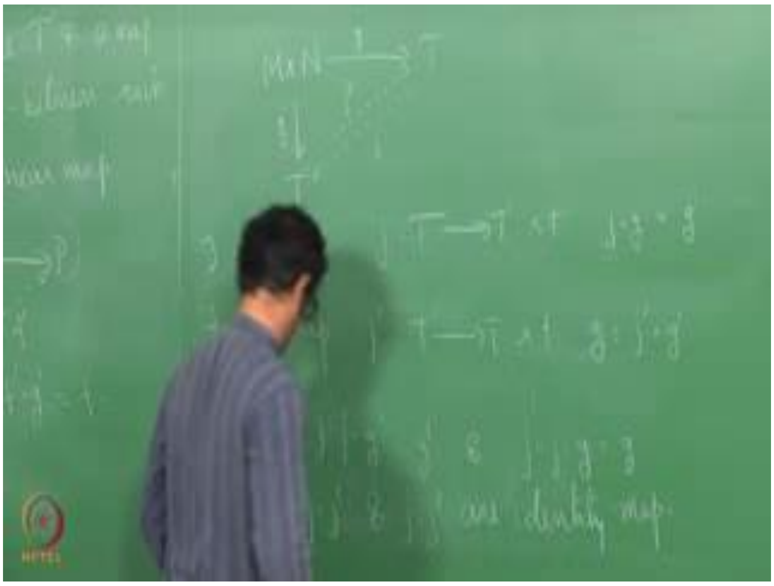

So, first we have proved that this is our T which is F modulo D and this is T prime is the module that we have assumed which exists with respect to this property there exists this module such that given any a bilinear map from M cross N to N a module P there exists f prime such that the composition is f. So, therefore, our aim is to see if I have a map here and a map here. So, let us take the first whatever we have proved here, take this to be P given this g prime I have a map from T to T prime take P equal to T, T prime first; then there exists a map let me call this j from T to T prime such that, this is j such that j composite g is same as g prime. By taking this is we obtained this map j by taking T and P equal to T prime in this diagram.

Now, take P equal to T in this diagram, what do I get? And take this to be this map F to be equal to g. So, we get a map j prime from T prime to T. So, this is j prime, from T prime to T such that g is equal to j prime composite g prime, is that clear. So, therefore, now look at these two, putting j g equal to this one here what do we get? We get j composite j prime composite g prime is g prime and j prime composite j composite g is same as g.

See now look at this g and g prime g and g prime take any x comma y to x tensor y; x tensor y. So, what this says is that j composite j prime acting on x tensor y is x tensor y for every x comma y in.

Student: M and N.

M and N, but they are the generators of x, the module T is generated by.

Student: (Refer Time: 19:41).

These elements, so therefore, this says that j prime composite j and j composite j prime are identity maps which means j and j prime are isomorphisms, j is isomorphisms, j and j prime is the inverse of that which means T and T prime are isomorphic.

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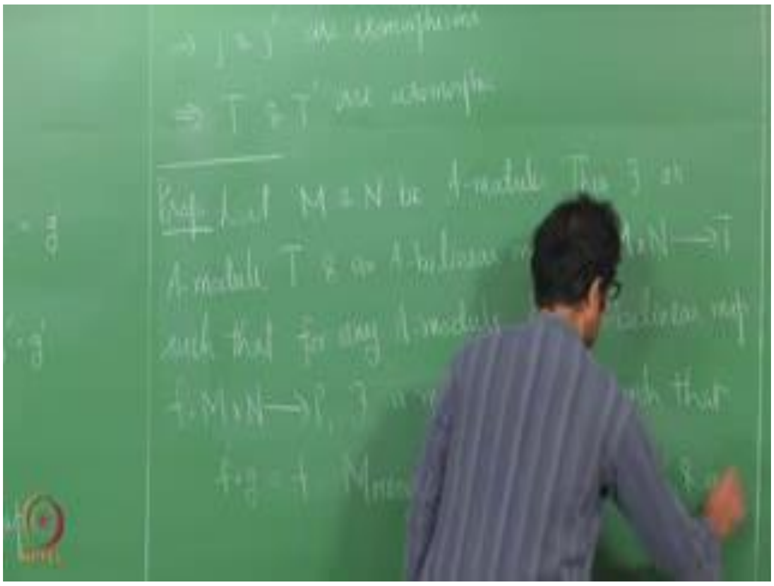

So, therefore, let us write down what we have proved so far. Let M and N be A modules then there exists an A module T and an A-bilinear map g from M cross N to T such that for any A module P and A-bilinear and an A-bilinear map F from M cross N to P there exists a map F prime from T to P such that F prime composite g is same as F. Moreover such a module, if there exists another pair another module T prime an A module T prime and A-bilinear map, A-bilinear map g prime from M cross into N to P.

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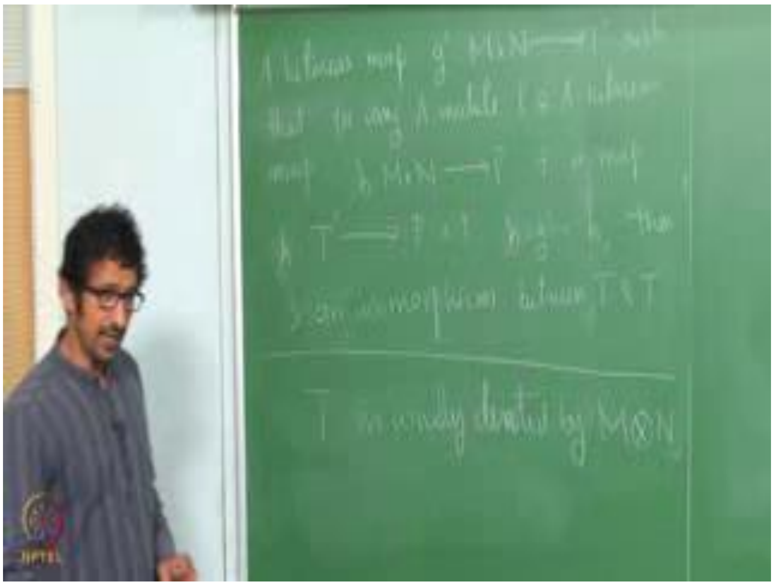

Student: T prime.

Sorry T prime, such that for any A module P and A-bilinear map F from M cross N to P there exists a map f, so maybe I will use this h there exists h prime from T prime to P such that g prime composite other way around, h prime composite g prime plus h. Then if there exist such a map with this property then there exists an isomorphism between T and T prime. So, this element, this module with this property is uniquely determined up to isomorphism, this module is, so usually denoted by this T is usually denoted by M tensor N. There are nice properties of this tensor product to start with see note that this M tensor N, in M cross N every element is of the form x comma y where x belongs to M and y belongs to N, but here not all elements are of that form right. We are see, how did we construct this we looked at f for each x comma y I had a basis element e x y every element f will be a linear combination of elements of e x ys.

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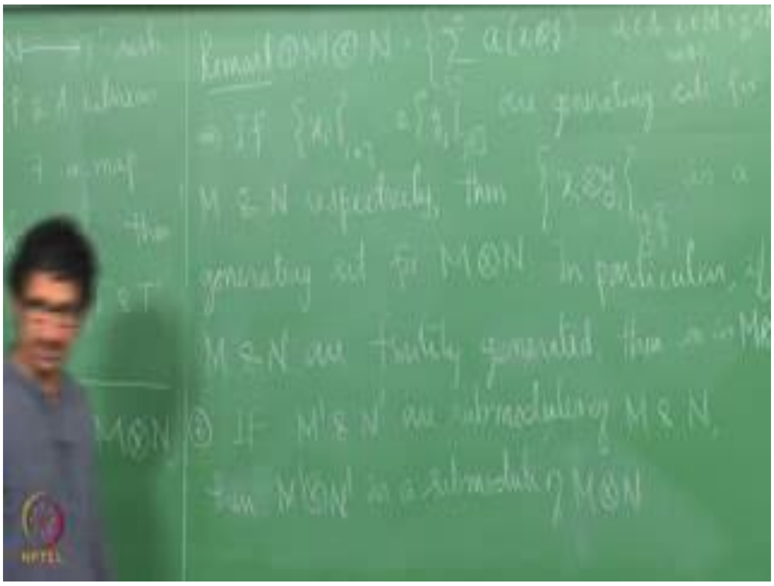

So, therefore, every element here will be of the form some linear combination of x tensor y. So, M tensor N this is summation a i finite linear combinations like this where a i is in A, x i is in M and y i is in N, and n in N, finite linear combinations of this form. What this says is that, if this is an I have 2 generating sets are generating sets for M and N respectively then this is a generating set for M tensor N.

This x tensor y we will, let me just take some note some properties then I will take 1 or 2 examples, we will see that this M tensor N is you know it has slightly different meaning then what you know not just f, see in the f we are just taking a free product of all I mean as many copies of M cross N are, but here when we are taking modular relations it becomes you know it becomes very different. We will see, we will come back to that.

So, if this is, if these 2 are generating sets for M and N then this is a generating set for M tensor N. So, this says in particular if M and N are finitely generated then so is M tensor N. See even if M and N are finitely generated the module f that we were looking at is far from being finitely generated right. In general see if you look at this F, F was direct sum of as many copies of you know M cross N like this, even if your module M and N are finitely generated this is far from being finitely generated because there are you know each element is a generator right each e x y for x comma y in M cross N is a generator.

Student: (Refer Time: 29:19) finitely generator.

Finitely, so for example, you take Z modulo Z, I am sorry Z over Z this is generated by 1 element, but if you look at in the, if your M is this also equal to N then the free module F will have as many generators as.

Student: Elements in (Refer Time: 29:44).

Elements in Z which is also same as Z cross Z. So, this is far from being finitely generated this is a huge module, but when we take modulus in nice situations the module also becomes the tensor product also becomes nice. So, this, in particular if M and N are finitely generated then, so is M tensor N.

Now another remark this is I would say slightly unexpected when you see this for the first time, in the case of vector spaces if you take a module, if you take a subspace and take any vector there that vector has a very precise meaning or unique meaning whether you consider it as an element of this ring and this subspace or the original space right, but that is not the case for modules. For example, if I take say in the case of this is an element x tensor y in M tensor N. So, this may have, so let me just write like this if M prime and N prime are sub modules of M N then M prime tensor N prime is a sub module of M tensor N.

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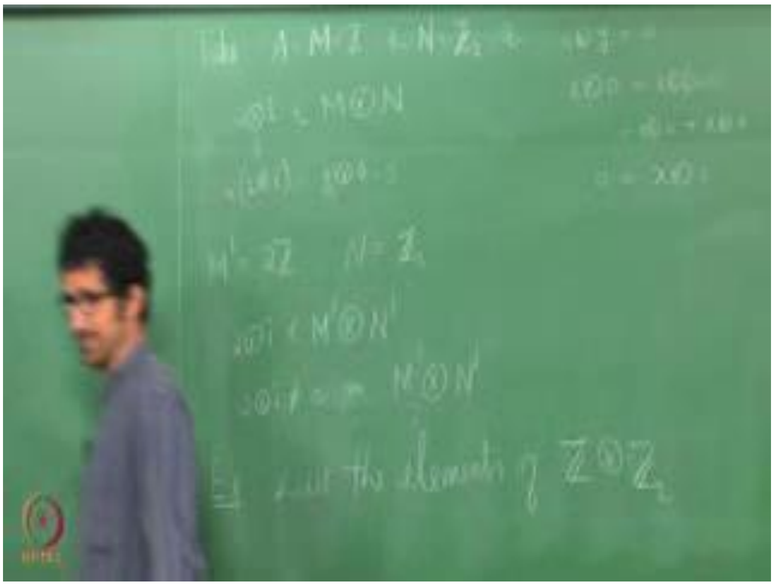

But now, let us look at one example. Take A equal to M equal to Z, and N equal to Z 2 and look at the element 2 tensor 1 bar in M tensor N, because of the property in tensor products this is equal to I can write it as 2 times 1 tensor 1 bar which is same as 2 time sorry 1 tensor 2 times 1 bar which is 2 bar which is 0, but this is, so this is one thing. Suppose you take you know in M tensor N for any x this is 0. I mean, so I have this representation this is 0 as the additive identity of the module M tensor N, how do you see that? This is equal to x tensor 0 plus 0 by properties of tensor product I have this is x tensor 0 plus x tensor 0, this is same as this these are all elements of the module. So, you know add additive inverse of this on both sides use associativity to say that 0 is equal to x tensor 0. So, if you look at this element x tensor 0 that is 0.

Student: (Refer Time: 34:54) prove 0 tensor.

Yes 0 tensor y is also 0 for every.

Student: For every unique [FL].

Sorry?

Student: It should be unique.

Yes, see if you take for example, you take Z mod 5 Z, what is 5 bar?

Student: 0.

What is 10 bar?

Student: 0.

Yeah, they are all different representation that is all. These are all different representations of 0. So, this is 2 tensor 1 bar is 0, now I take the sub module M prime equal to 2 Z.

Student: (Refer Time: 35:48).

Sorry?

Student: (Refer Time: 35:51).

No I am taking M prime sub module of M in 2 Z. Now look at this and N prime is same as Z 2 itself I look at then this is there in M prime tensor N prime. Can we do the same thing in M prime? See 2 tensor 1 bar this I cannot write like this in M prime tensor N prime because this is not there this element is not here. So, this equality does not hold in this module this equality holds only here not here. So, therefore, this is not equal to 0 in M prime tensor N prime.

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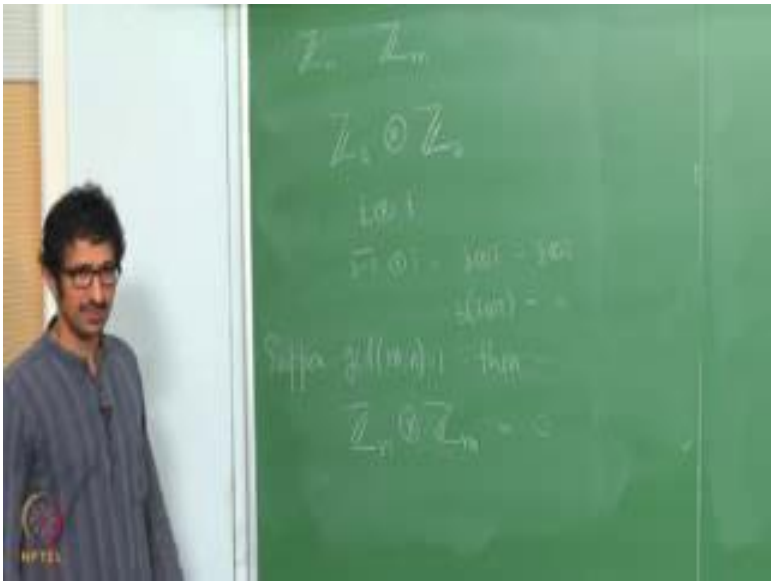

So, in this case exercise lists the elements of M tensor N. You will get a hint on this when we do the next exercise I have Z n I have Z m. So, let us look at Z n tensor Z m, what would this module be? So, let us let us start with Z 2 tensor Z 3. So, let us look at the element 1 tensor 1, but here this 1 I can write it as 3 minus 2, this is 3 bar tensor 1 bar minus 2 bar tensor 1 bar, what can you say about this element, this element, what is 2 bar?

Student: 0.

0. This is?

Student: 0.

And this is?

Student: (Refer Time: 39:09).

This is 0, this I can write as 3 times 1 bar tensor 1 bar, but this I can send it to the other side and say that is also 0.

Student: It could have directly written 1 by plus 1 minus j by (Refer Time: 39:34).

Yes. In this case yes, yeah I wrote this because of now Z n tensor Z m, do you have some idea about this? M minus N is in one look if you look little more carefully.

Student: Sir M minus Z or N minus Z depending on it.

Well, even more general, what is that we are using here? See if I can write 1 as a linear combination of M and N then I can keep N in 1 side M to the other side and say that both of them are 0, and what is that property?

Student: Gcd of M and N.

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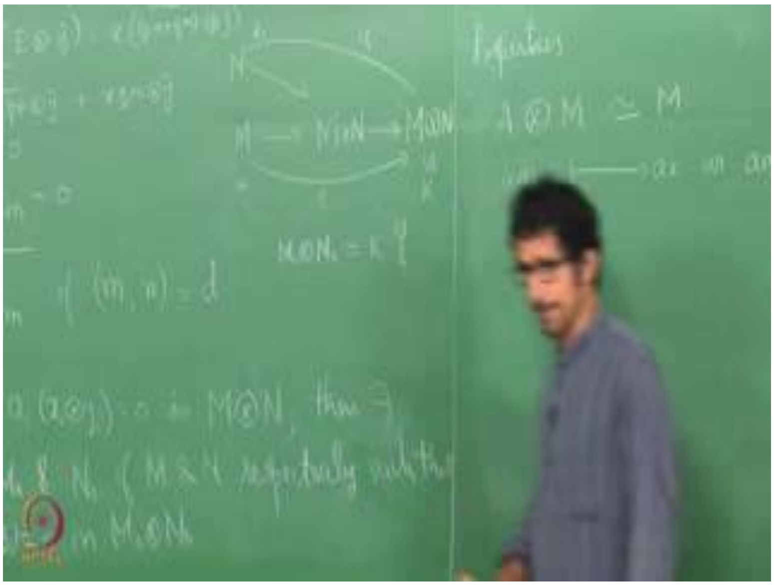

Gcd of M and N are 1. So, if gcd then I want to say that this is 0 right, how do you say that? You take any x bar tensor y bar this I can write it as x times 1 bar tensor y bar, but 1 bar is some x times 1 bar is some pn plus qm tensor I mean whole bar tensor y bar, this is same as xpn tensor y bar plus xqm tensor y bar this is there already this we take it to the other side and say that both of them are 0.

So, in Z m cross Z n, Z m tensor Z n you can make all the elements 0. So, now, think about this exercise or list the elements of in this case. So, let me just write it as Z tensor Z 2, M tensor N means in this example that should be pretty easy now.

Student: (Refer Time: 42:25).

Think about it we will; so, what happens if Z n tensor Z m if gcd of m and n is d; think about this. This is left as an exercise. Think about this what module is this. See this is again an Z module, this is a finitely generated Z module. So, it has to be something that you can easily express. So, think about it what should this, we will come back to that later.

Student: (Refer Time: 43:10).

Will it be d, I can write it as linear combination of this. So, what if there is no d I mean. So, think about it. There is, it is not difficult at all, it is its easy think about it. You had something similar in the case of indicators of ideals. So, now, let us look at 1 or 2 properties of tensor product. So, what we have seen here is that an element is 0 does not mean that it is 0 as element of every sub module, but there is corollary of the construction that - if summation a i x i tensor y i, i from 1 to n this is 0 in M tensor N then there exists sub modules M naught and N naught of M and N respectively such that summation a i x i tensor y i, i from 1 to n is 0 in M naught tensor N naught. So, there exists some sub module in which this is 0.

That is easy to see because if you take this summation, if you take this summation this is 0 means it is in d if you look at. So, you can take it is a finite linear combination of elements of s, so take all the first coordinates and define M naught to be the module generated by those first coordinates and N naught to be the module generated by the second coordinates.

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So, let me just write one of the basic properties of tensor product that if I take, so properties, A tensor M is isomorphic to M that is a tensor x mapping to a x this is an isomorphism.

Student: Sir.

Yeah.

Student: (Refer Time: 46:27).

Sorry. No, this is 0. There exists a module where this is 0 see earlier what we saw is that if this is 0 then that does not imply that this is 0 in every sub module, but here we are saying that there exists a module in which this is 0. There could be modules where this is nonzero sub modules with this being nonzero, but there exists at least 1 module in which this is 0 yeah. So, this is you have again all these you have a map from M to M cross N to M tensor N.

Now, if I take a sub module k and look at, so I have this map and I look at phi inverse of k it is a sub module here, similarly I have N to M tensor N M M cross N M to M tensor N, I have this is phi this is call it psi. So, psi inverse of k will again be a sub module of N and then now think about whether, if I call this to be M naught and this N naught check whether M naught tensor N naught is same as k, whether this is indeed the case and see how we constructed the module M tensor. Well. So, I will think about this proof see whenever we have to prove something like this, how do we prove that? We have to use the universal mapping property from here to, so A cross M I have a map to M this filters through.

Student: A tensor M.

A tensor M and we are saying that this the map A tensor M going to A M is an isomorphism, how do I say that this is indeed the isomorphism? Think about it, I will do that next time.