Commutative Algebra
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Lecture - 14
Homomorphisms and Tensor Products

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Yesterday we proved that if 0 to M prime to M to M double prime is exact, this is exact then for any module N, 0 to Hom a M prime N; M double prime N to Hom a M N to hom a M prime N this is exact. So, just wanted to remark that this is the converse is also true that if this is true that; for any module N this is exact that will imply this is also exact, 1 can you know to say that this is subjective and this is kernel is image 1 can take in appropriately M N prime and so on and prove this.

So, I will simply remark that this is right the modified statement; the sequence is exact if and only if for any module N this is exact. Similarly the sequence 0 to N prime to N to N double prime is exact if and only if Hom for any module M. Hom M N dash to Hom M N to Hom M N double prime is exact. So, this is again usual diagram chasing the exactly the way we did for the case of exact these proves yes last time.

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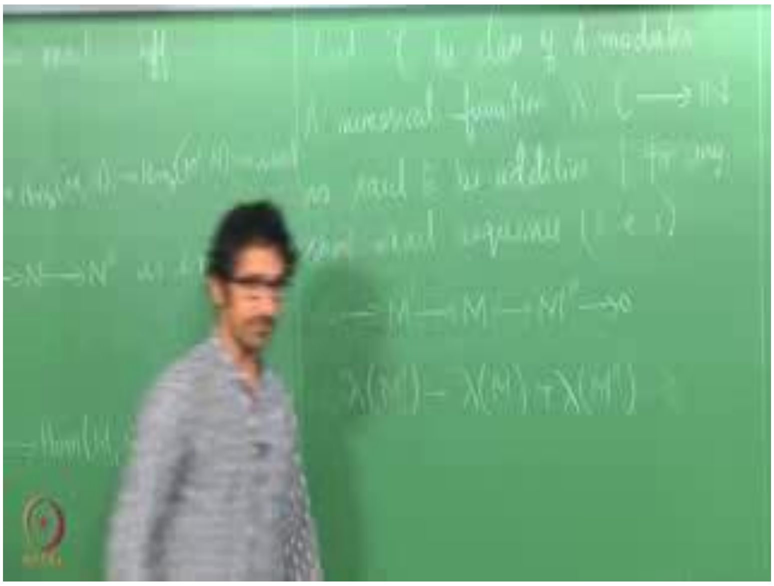

So, suppose you take a let us C be a class of class of A modules; suppose you have you have you know suppose here A is field K you can think of C as finite dimensional vector spaces over K. So, you are basically looking at from the category of A modules we are taking a sub class. A numerical function lambda from C to N is said to be additive if for any exact sequence short exact sequence. So, this I will write like this, 0 to M to M double prime to 0, we have lambda of M prime minus lambda of M plus lambda of M double prime is 0.

So, as I said if you want to have a look at this example, suppose your ring A is a field and C is set of our finite dimensional vector spaces over A; then you can take this lambda as dimension. If you take lambda as dimension then it is a numerical function will it be an additive function.

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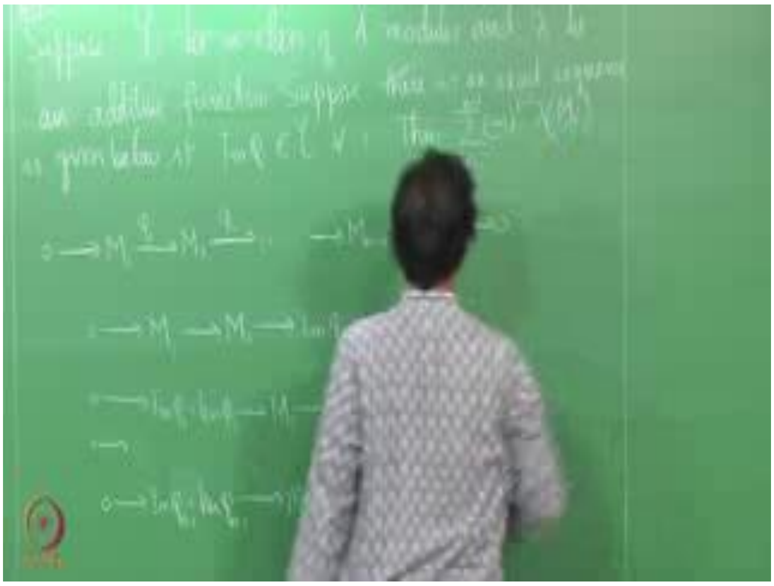

Suppose, you have V 1 to V 2 to V 3 to 0; if you have a vector space if you have 3 vector space is finite dimensional vector spaces. Can we say that dimension of V 1 minus dimension of V 2 plus dimension of V 3 is 0? this is exactly. If this is exact what we know is that V 3. So, this is subjective map therefore, V 3 is isomorphic to V 2 modular kernel, which is the image which is isomorphic to V 1. So, you can write this as isomorphic to V 2 mod V 1. So, therefore, dimension of V 3 is its basically rank nullity theorem.

So rank nullity theorem says is this is if we take C to be; if you are ring a is a field and C is set of all finite dimensional vector spaces over A then lambda equal to dimension of V is an additive function. Suppose I have suppose C be a class of I mean now I am talking about general some class over a general ring be a class of modules of A modules, and lambda be an additive function. Suppose I have a exact sequence, so, I have an exact sequence like this M 1 to M 2, M N minus 1 to M N to 0. So, call this phi 1 phi 2, phi N minus 1. Lambda is given to be an additive function and this is an exact sequence here N could be any finite number; can be or would you like to say something about lambda?

Student: (Refer Time: 08:30).

Modules in C for example, I am talking about modules these are all M 1, M 2 etcetera are all modules in C. So, first of all this is something that I probably mentioned it in sometime back. See if I have an exact sequence like this I can split this into a lot of short exact sequences right. So, this is. So, 0 to M 1 to M 2 to image of phi 2 to 0 right; this is an exact sequence. Now I have image of phi 2 which is also same as kernel of phi 3, to M 3, to image of phi 4 to 0 right and so on, 0 to. So, I have image of phi N minus 1 is M N itself. So, we can talk about the preview image phi N here; image phi N is sorry phi N minus 1 this is M N to 0, to M N minus 1 to image of phi N minus 2 which is same as kernel phi N minus 1. I have these are all short exact sequences arising from here, this this sequence is exact then I have several short exact sequences.

Suppose your function lambda is additive on C, what can we say here looking at this one? Lambda of M 1 minus lambda of M 2 plus lambda of; so for that we need 1 condition that this is also here; we have started by started by taking M 1 M 2 etcetera. We are just taking a class we are just taking an abstract class. So, we do not know if I take M 1 M 2 and so on and look at an exact sequence like this, that need not necessarily imply that all these are in that class.

So therefore, if I assume that you know all these are in this class, then I can apply lambda the property of lambda on each of them lambda M 1 lambda of M 1 minus lambda of M 2 plus lambda of N 3 is 0 sorry lambda of this is 0, but now from here see lambda of this plus lambda of M 3 plus lambda minus lambda of this minus lambda of M 3, plus lambda of image phi 4 is 0. Or in other words lambda of image phi 2 is same as lambda of M 3 minus lambda of image phi 4.

But now image phi 4; I can get from the next exact, I can keep going down what do I get. So, so let us now complete this proposition, and then this is suppose there is an exact sequence as given below such that image phi I is also in C for all I then I will reserve that conclusion. Let us see what it is and then we will come back and write that.

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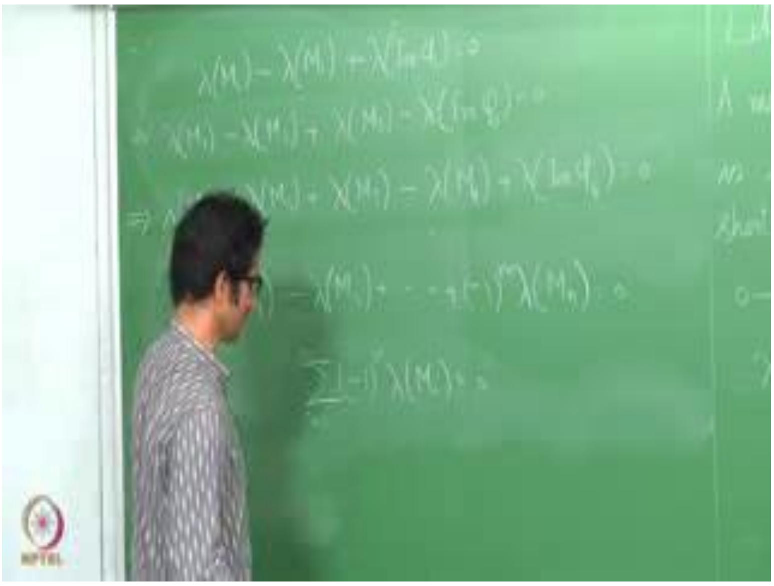

What we have from the first exact sequence we have lambda of M 1 minus lambda of M 2, plus lambda of image phi 2 is 0. Now from the second exact sequence image phi 2 is lambda of M 1 minus lambda of M 2, plus image phi 2 is lambda of image phi 2 is lambda of M 3 minus lambda of image phi 4, this is equal to lambda of M 3 minus lambda of image phi 3 sorry phi 4. But what would be image phi 4 from the next one it will be lambda of M 4 minus lambda of image.

Student: (Refer Time: 14:40).

Phi 5 lambda; so is this is 0 that implies lambda of M 1 minus lambda of M 2 plus lambda M 3, minus this is lambda of M 4, minus of minus plus lambda of image phi 5 equal to 0.

Student: (Refer Time: 15:06).

Sorry it will be.

Student: (Refer Time: 15:12).

It will be 6.

Student: (Refer Time: 15:19).

This will be next will be image 0 to image phi 4 equal to kernel phi 5 to M 5 to image phi 6 it will be 6 (Refer Time: 15:38) you are.

Student: (Refer Time: 15:44).

Sorry no we do not need see at every time now image phi 2 is kernel phi 3.

Student: (Refer Time: 15:56).

No this is this should be 3 image phi 3 itself. This should be 3 and 4 see from M 3 the I mean the map starts from M 3 is phi 3. So, the M 2 image phi 3. So, ultimately what are we going to get? Lambda of M 1 minus lambda of M 2 plus minus 1 power N lambda of M N or in other words summation minus 1 power i lambda M i is 0. i from 1 i minus 1. So, this is N minus 1 to N minus sorry 1 2 N. So, let us complete this this is i from 1 to N minus 1, minus 1 power i minus 1 lambda m.

Student: (Refer Time: 17:10).

Equal to 0: if I have an additive function that the additive function means, if you take a short exact sequence, the alternate sum of the corresponding numerical values is 0. If I have an additive function and if I have a exact sequence coming from the class of modules where lambda is energy function, then what this says is that you have alternate sum can be extended to long exact sequence. I mean not long exact sequence this exact sequence, which is not necessarily the short exact sequence. One typical example is if I take a vector space A sorry if I take field A and finite dimensional vector spaces over A.

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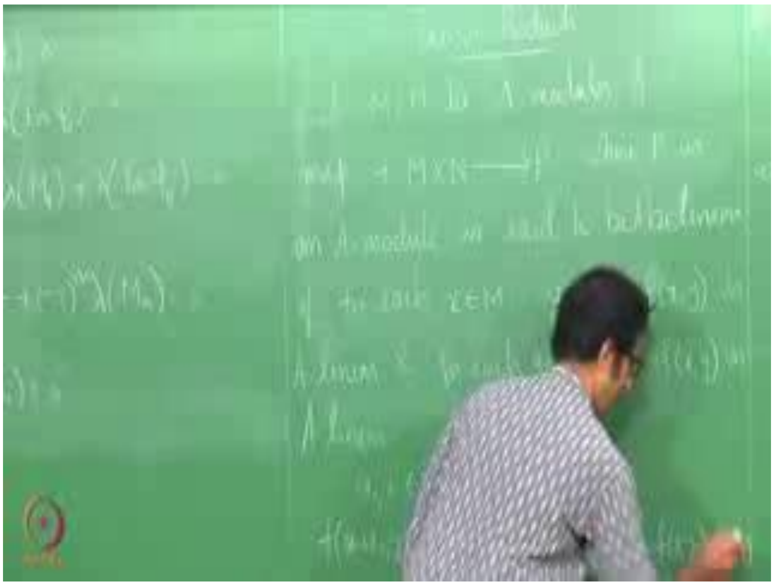

So now let us move on to study a very important classes of modules called a tensor products. Let A B; M N be finite M N be a modules, a bilinear map f from M cross N to P, where P is an a module is a map f from M N M cross N to P; where P is in a module is said to be bilinear, if for each x in M, y map in to f x y is linear and for each y in N, x going to f x y is a linear. In fact, is it be a bilinear. Or in other words for each x 1, x 2, in M; y 1, y 2 in N; a b in A; f of x 1 plus x 2 y is f x 1 y, plus f x 2 y, f x y 1 plus y 2, is f x y 1 plus f x y 2, and similarly f of a x comma a x comma y is a times f of a x comma y is a times f x y, f of x b y is b times f x y usual definition of bilinear linear in each variable.

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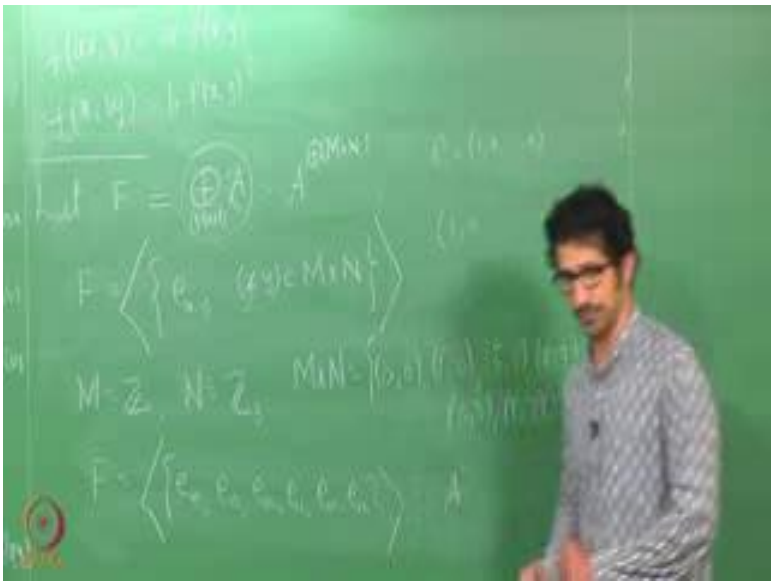

So, suppose I have a module M modules M and N, our aim is to define what is called a tensor product of M and N. This is in say product of M and N is for the tool tensor product is a very important tool in the homological algebra; homological algebra or the methods of homological algebra are applied in every part of commutative algebra algebraic geometry and so on, even in analysis and so on these are very important. Analysis the approach is slightly different, but this is a very important a module the tensor product of M and N.

So I mean how do we construct this? What do we do is we first look at take the free module be the free module, be a free module or you know I will just write let f be equal to A M cross N or is also I will write in this form. What does this mean? I take as many copies of A as the number of elements of M and N. So, if M and N are infinite or even M is infinite this is direct sum of infinitely many copies of M and N infinitely copies of A. So, for each x comma y I have a e you can think of an element e in e x comma y in f; that is the element where the coordinate with respective to x comma y is 1 rest of them 0.

See there are as many copies of A as the number of elements here. So, this free module F you can think of this as generated by the set e x y, where x y is in M cross N; it is generated by this set. For example, if M is Z 2 to take a simple example N is Z 3. This F will have 6 elements, not 6 elements this will have 6 copies; your A will be I mean F will be generated by e 1 0 e 0 0, e 1 0, e 0 1, e 1 1, e 0 2, e 1 2. Now this e 0 0 is this is isomorphic to A 6 this is 1 0 0 0 0 I mean the 0. The first entry is non 0 the rest of them is 0, I consider the first copy of a corresponding to the element 0 comma 0; the second in second copy of a as element corresponding to this 1 and so on.

So this is I take this free module, now what do I do I want to see my aim is to say that every bilinear map from M cross N to a module P filters through M tensor. M tensor N am trying to construct the unique object with the universal mapping property that every bilinear map from M cross N to an a module P (Refer Time: 27:09) through or extends to M tensor N 2.

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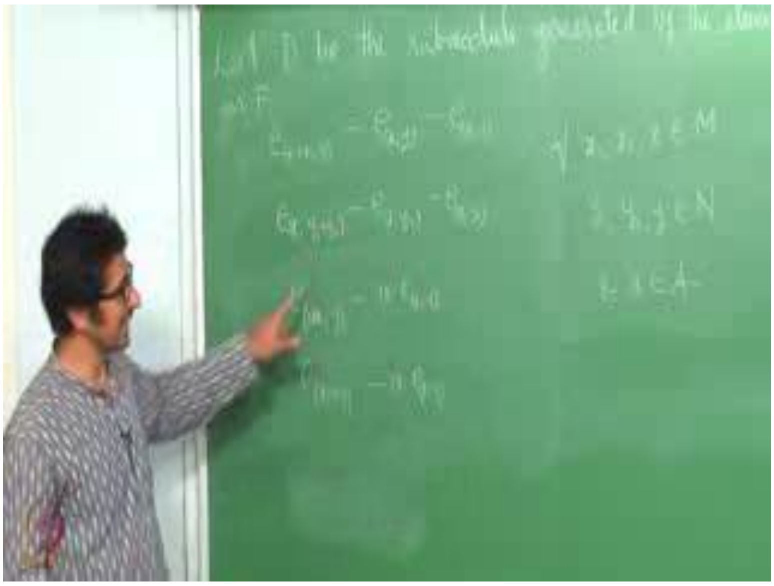

So, first what I will do is on this one I take a sub module, let D yes I am taking do you understand this definition. I am taking as many copies of A as the number of elements of M cross N. So, here I am just indicating what does e? See when I say in the standard vector space R N when we say e 1 this is 0; where this is 1 on the first entry. So, if I write this M cross N like this 0 bar, 0 bar, 1 bar, 0 bar, 0 bar, 1 bar, 1 bar, 1 bar. So, here these 2 bars are different one should be careful with that 0 bar 2 bar 1 bar 2 bar. So, for each element I have a copy A.

So this is the basis for that; now for that entry is for the first; this is the first entry let us say, then I have a corresponding copy of A here. So, I will have 1 0 this is the generator of that part that copy, that is exactly I am denoting it by e 0 0. So, here I am just following this notation here that is all. So, for each element x y I have a generator e x y; my idea is to create a module such that any bilinear map from M to N, M cross N to P filters through M tensor. So, let us look at this D be the sub module generated by the elements x 1 plus x 2 comma y, minus x 1 comma y minus x 2 comma y in F.

Student: (Refer Time: 30:08).

Sorry.

Student: (Refer Time: 30:12).

I am not this is not the only element. See looking at this immediately you will have a tendency to say that this is 0; what is our F? For each element in M cross N I have e there. So, I am in fact. So, I should probably write like this e x 1 plus x 2 comma y minus e x 1 y, minus e x 2 y; this now you understand that they are different.

Similarly I will look at e x y 1 plus y 2, minus e x y 1 minus e x y 2, similarly e a x y minus a times e x comma y, e x a y minus a times e x comma y; so for all x 1 x 2 x in M, y 1, y 2, y in N and A in a.

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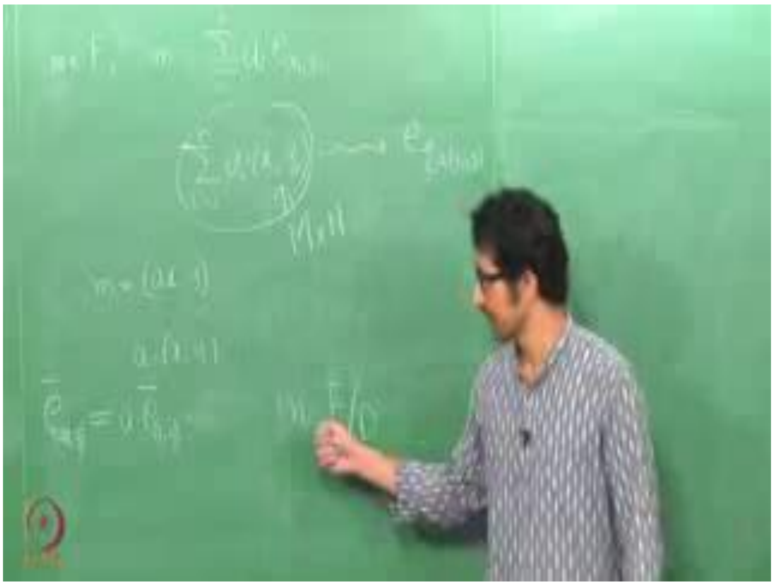

So, now let us now just said some more or notation; see how does any element of F look like? Any element of F is a finite linear combination of elements of the form e x y right. So, I can any element. So, if I take an element M in F, M will be a i, e x i y i some finite linear combination like this. Now in in the module M cross N there is an element called of the form a i x i y i. So, there is an element corresponding to this I have an element e of this summation. These 2 are not the same this is an element in F which is a linear combination of some basis elements, while this is a basis element. So, more over you can. So, that difference you should keep in mind.

Now so my D is all the elements of this form, and I take my module let T be equal to F module of D. I take T to be F module of D. This looks like to be you know highly abstract horrendous set of elements here or the module, but it is not so once we use this we will understand we can easily see.

Student: (Refer Time: 34:50).

Sorry.

Student: (Refer Time: 35:08).

Which one?

Student: (Refer Time: 35:24).

No, this is I am talking about this a i dot x i this as an element in M cross n.

Student: Sir in the first line.

This one?

Student: Yes sir (Refer Time: 35:37).

This that will be different from this 1 is that what you are saying.

Student: (Refer Time: 35:46).

No, here I am no I am not talking about elements in D or anything; I am just talking about simply representation of elements in F. See F the definition has enough things to get confused, now you cannot you can simply accept that F is this, but to understand that it takes little more effort that is why I try to explain this part. What I am trying to say here is that I have any element of F looks like this, this is different from this element; that is all I am trying to say there is a. So, there is a basis element corresponding to this element in the M cross N, each x i y i is an M cross N, but the no you take the corresponding element in F and take the linear combination there, and take the linear combination M cross N and look at the corresponding element they are different you have to, but once you come to D what do we have? When we D is sub module generated by elements of this form. So, in D modulo D these 2 are equal.

Let me just take simple example; I have my M is a x y. So, I have this. So, I have e a x y; well not this same as the other 1 I let us see what it is this is.

Student: (Refer Time: 38:15).

Sorry what I want to say is I have M is this, and I have a times x y. Now a times e x y modulo D see what are the elements in D generated by a times x y minus or e of a x y minus e, e of a x comma y minus a times C of x 1. So, if I take this is e of a x comma y minus this is 0 in F mod D right; or in other words if I take the corresponding bar this is same as in F mod D. Similarly if I take e in f these 2 are not the same, because this is a basis element this is a basis element, you cannot have 1 as a scalar multiple of the other.

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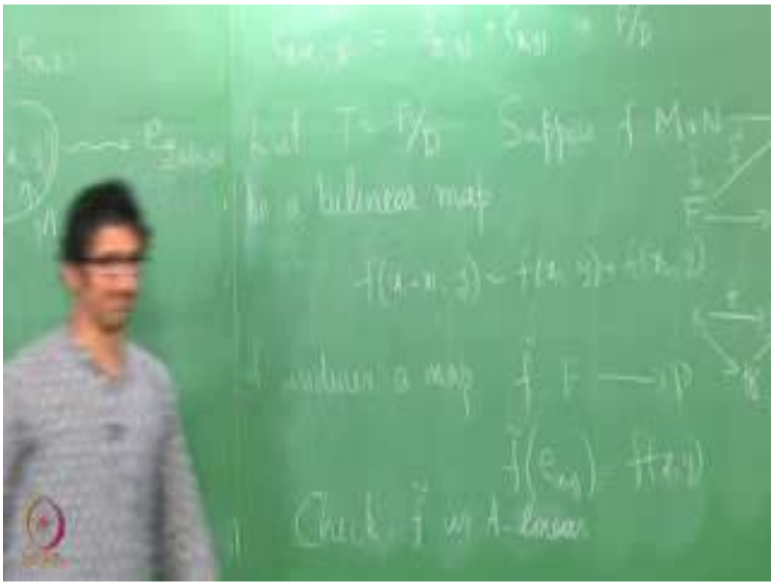

But then modulo D these 2 are same that is so we have constructed. Similarly what can you say about e x 1 plus x 2 minus sorry comma y, this is same as e x 1 comma y minus e x 2 comma y.

Student: (Refer Time: 40:10).

Sorry plus in F mod D.

So now let us let me define T and we have correspondingly you know using these properties I have I can extend this further in a mod F mod D this is same as this one. So, let T be equal to F mod D; suppose I have a bilinear map f from M cross N to P be a bilinear map what does that say? It says f of x 1 plus x to y is same as f of x 1 comma y plus f of x 2 comma y and f of and so on. You know I would not write down all the conditions f of x comma y 1 plus y 2 is correspondingly and so. Once I have this map, this induces a map let us say f tilde from f to P. What is my f is free module on M cross

N. So, what I do is f tilde of e x comma y is f of x y; e x y is you know generating set for f it is a basis for f. So, I define f tilde of this or if you want to define not a general element summation a i e x i y i same as summation a i f of x i y at. This induces map from f induces map from f to p. Now our aim is to say that every bilinear map filters through the tensor product or in other words I can extend it to a map from M tensor N to P.

So here I have. So, I have this map M N to P, now I have a natural inclusion here to F and this I have map from F to P. I am looking for something here T to P. So, I have this this is F tilde, and I am looking for some f prime which again you know comes from f. See T is a quotient of f, suppose I have a module homomorphism let us call it K 1 to K 2 and let us say L is a sub module of K 1 when can I say that it will you know it will induce a map from here to here.

Student: (Refer Time: 44:20).

If L is contained in the kernel of f see then L is naturally killed by f, then you can always you know extend this map or you know this f will induce a map from here to here. So, therefore, now I have map from F to P to say that there is a map from T to P if I say something I am (Refer Time: 44:55) what should I say.

Student: (Refer Time: 44:59).

That D is contained in the kernel, but is that so, f is a bilinear map.

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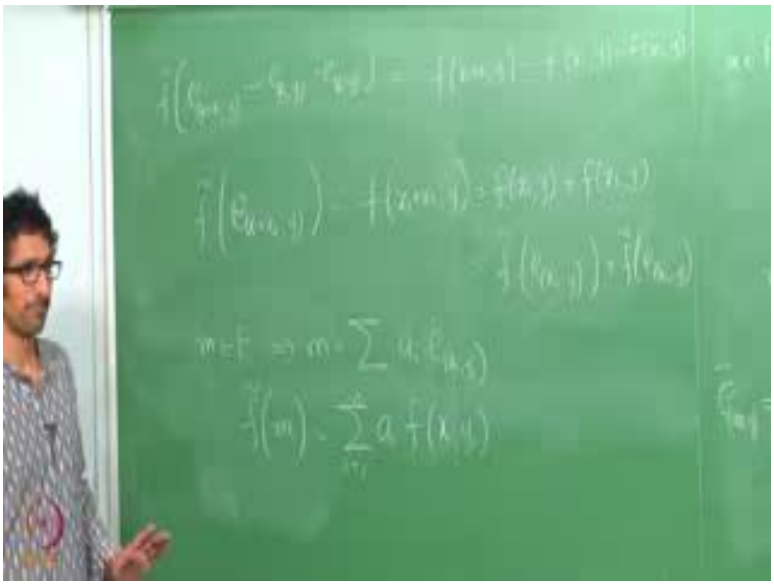

So, if I take contained in the kernel of f tilde, what does f tilde of e x 1 plus x 2 comma y minus e x 1 comma y minus x 2 comma. If I take an element in D, this will be 0 because this is by definition what is this? This is f of x 1 plus x 2 comma y minus f of x 1 comma y minus f of x 2 comma y right.

Student: (Refer Time: 45:56).

F tilde well; so this is check f tilde is A bilinear a linear sorry that is f tilde is linear by definition; in the sense that see it is like I am defining this is the exactly like the way I define linear transformation from a vector space with a given basis to you know another vector space with a given set of elements. So, here I am defining this form a basis for F. So, I am defining it on the basis or in other words to make a precise definition it is f tilde of summation a i e x i y i is summation a i f x i y.

Student: So, sir (Refer Time: 46:59) f tilde (Refer Time: 47:03).

What are you saying f tilde of.

Student: (Refer Time: 47:21).

That is exactly what we are saying in the on elements of see this is this is equal to.

Student: (Refer Time: 47:35).

F of?

Student: (Refer Time: 47:40).

F x 1 y plus f x 2 y.

Student: Which is equal to (Refer Time: 47:52).

No that that is I mean there is no problem this is exactly what we are saying, this is equal to f tilde of e x 1 y plus f tilde of e x 2 y right? Or in other words f tilde of e x 1 plus x 2 y, minus x 1 y minus x 2 y this is 0.

Student: No sir why are we taking (Refer Time: 48:41).

It is not I am not saying this is this this need not be 0.

Student: (Refer Time: 48:51).

F tilde is I mean by definition f tilde is linear that is how I am defining f tilde. So, let me write down the definition of f tilde here, how do I how does any element of f look like? So, let us take m to be in f, m is of the form summation a i e x i y I, I define f tilde of M to be equal to summation a i f x i y i this is. So, f tilde is there I am linear by definition this is exactly how we define the in the vector spaces also, like we defined on a basis elements and extended linearly to the whole vector space this is exactly what we are doing here.

So therefore, this is 0, so what we have shown is that f tilde of a is 0 for all a in D; this implies f tilde induces a map f prime from F mod D to P.

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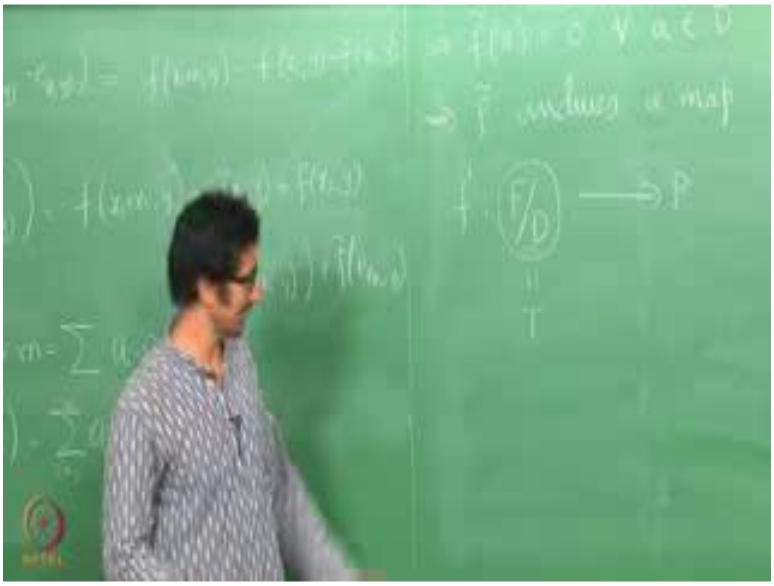

Now, let me denote this by. So, this is; see here f tilde is only a linear, f prime we do not have to we have in defined what is called bilinearity here, but f prime of x 1 plus e by definition itself p of x 1 let us let me just say that this is what we call it as T, I have already defined this, we will prove that this is this T is uniquely determined up to isomorphism in the sense that if I have this universal mapping property that any bilinear map from M cross N to a module P it filters through another module T prime, then T prime has to be isomorphic to T.

We will complete the properties of tensor product in the next class.