**Commutative Algebra**
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**Lecture - 13**
# Exact sequences (Continued)

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So we were talking about exact sequences. A sequence of modules and homomorphisms, so phi n phi n minus 1 so on is said to be A modules and A homomorphisms, is said to be exact if kernel phi a minus 1 this image phi I for all I.

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We saw some examples. Let us look at some of the nice properties. So, suppose I have an exact sequences, let N be an A module, let 0 to M prime. So, did I define, what is a short exact sequence? No, an exact sequence of the form 0 to M prime to M to M double prime to 0 is called a short exact sequence.

So, this is exactly what we saw last time examples in fact, the examples were of short exact sequences. So, if you have a short exact sequence this map will be injective, this map will be surjective and kernel of this map is equal to image of this map that is precisely, what is a short exact sequence? So, let M prime to M to M double prime be an exact sequence of A modules then I have Hom M double prime N to Hom M N to Hom M prime N, I have this, given this I have this exact sequence and I have this sequence. So, what we have is this is an exact sequence. So, what this property is referred to as Hom functors left exact.

See here even if I have a short exact sequence when you apply Hom there you need not necessarily have like this, but I have I am looking at exact sequence of the form M prime to M to M double prime to 0, we have the other one as well, we have that I will come to the next corresponding. If I have 0 to N prime to N to N double prime then Hom M N double prime to that is again exact. So, the exactness even if this is even if I have this I will not necessarily have this exactness, even if I have a short exact sequence this need not be a short exact sequence, but I have exactness at the left side when you apply Hom this injectivity is always there.

So, that is what is known as a left exactness of Hom functor, Hom functor left exact. So, let us try to prove this.

Student: (Refer Time: 06:15) then we can put the (Refer Time: 06:20).

This is, if this is exact then this is exact what more are you saying.

Student: Means I am saying that may be (Refer Time: 06:32) above map.

Injectivity is there this one.

Student: means suppose M prime to (Refer Time: 06:44) here is a map a, and so if induce the map from there if star.

No, one can there are examples where this even when this is injective this map is not surjective.

Student: (Refer Time: 06:59).

Let us go through the proof.

So, see here let us call this map f and g. So, therefore, this will be g bar this is f bar right g bar. So, we want to say that g bar is injective and kernel of f bar is same as to show that 1 g bar is injective and kernel of f bar is same as image of g bar. So, how do we do that? Let us try to prove the first one suppose g bar takes a homomorphism to 0.

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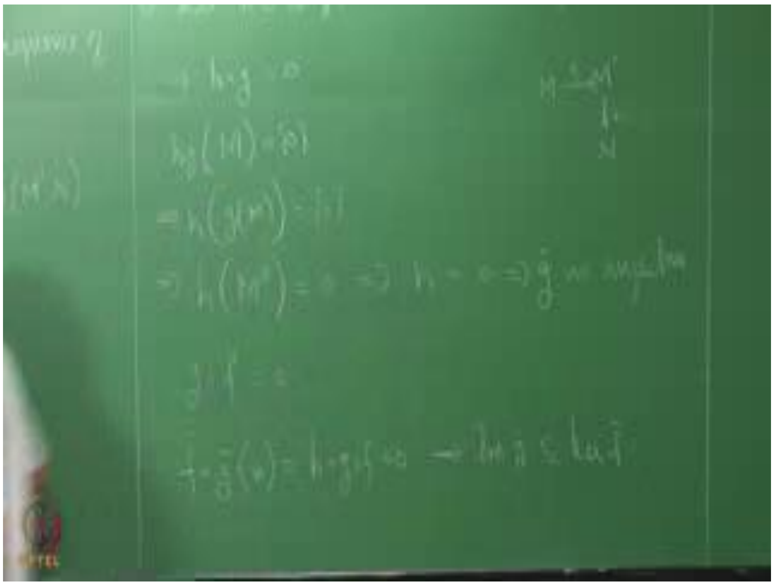

So, let h be in Hom M double prime N such that g bar of h is 0, but what is g bar of h? h is from M double prime to N g bar of h is a homomorphism from M to N obtained by composing with g, g bar of h is 0 that implies that h of h composite g is 0. We want to say that h is 0; that means, if I take any element of M prime h of that element is 0, but now if I take any element of. So, h composite g of M is 0 this implies h of g M. So, this I should write 0, but what is g M?

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g of M.

Student: M double prime.

M double prime, this is the g is surjective therefore; g of M is M double prime. This implies that h of M double prime is 0.

Student: (Refer Time: 10:02).

Given that this map this is exact. So, this says that h is 0 that says that g is g bar is injective. Now we want to show that the second part kernel of f prime f bar is contain is equal to image of g bar. See whenever we are talking about this exactness here, one thing can be I mean when you have to prove these 2 are equal we have to say that this is contained here and this is contained here most of the time one of the inclusions follow very quickly. See here if you look at this kernel of g is contained in image of f or in other words g composite f is 0, image of f is equal to kernel of g. So, if I take an element I take from here go here and then further go to M double prime it is going to be 0 because this image is contained in the kernel. So, this is always 0.

Now, what happens? In this case what is our f bar composite g bar of a homomorphism let us take h, what would this be? This would be h composite g composite f right that is precisely f bar composite g bar of h, but g composite f is 0. So, therefore, this is 0 which implies that image of g bar is contained in kernel f bar. Now we want to show that image of g bar contains kernel f bar. So, let us start with an element in kernel of f bar. Let h be in kernel of f bar.

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So, what is this map? H is a homomorphism from M to N. So, I have M to N this is h I have these maps M double prime and M prime this is f this is g, what is that I want to prove? I want to show that there exists some h prime in Hom M double prime N such that g bar of h prime is h, but what is g bar of h prime? This is h composite g is equal to sorry h prime composite g is equal to h. So, I am looking for a map from here to here such that this diagram commutes or in other words h is precisely this right, this is what we are trying to prove.

So, now there is a way out we want to define a map and say that this is precisely, we want to define h prime and say that this is precisely the map that works. So, I start with an element here and I want to get an element here what is that one does? You start with an element we have to make use the fact that the given sequence is exact this is surjective if I take an element I have a pre image here and what we want is if you look at that pre image you go there and come here that should be same as that element going here directly through h. So, I just define h prime, I take an element here its image should be h of the pre image, but here there is a danger there could be many pre images, so which one.

So, first let us try to define this let x be in x double prime be in M double prime that implies there exists x in M such that g of x is equal to x double prime define h prime of x double prime to be equal to h of x, with this definition at the moment this is not well defined because there are, there could be many x that satisfy this property. So, let us take let us say. So, this is first we need to show that h prime is well defined, how do I show that well defineness here? If I show that whichever pre image I choose it does not matter they will be mapped to the, if I choose 2 pre images at x 1 and x 2 if I show that h of x 1 is equal to h of x 2 we are done.

So, let us do that let x 1 x 2 be in M with g of x 1 equal to g of x 2 equal to x double prime, g of x 1 minus x 2 is 0. This implies g of x 1 minus x 2 is 0.

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Or in other words x 1 minus x 2 belongs to kernel g, but we know that sequence is exact therefore, kernel g is equal to or in other words there exists some x prime in M prime such that x 1 minus x 2 is equal to f of x prime. It is in the image of f or in other words there exists some element in M prime which maps to x 1 minus x 2. So, this implies that x 1 is equal to x 2 plus f of x prime.

Now, therefore, what is h of x 1? This will be h of x 2 plus h of f of x prime. Now we have not used the fact that h belongs to kernel of f prime, h belongs to kernel of f prime this means that f bar of h is 0 or in other words h composite f is 0 and what is this now? This is h composite f of x prime which has to be 0. Since h composite f is 0 h of f of x prime is 0 and that implies that h of x 1 is equal to h of x 2 and that implies that h prime is well defined, one needs to prove that this is an a module homomorphism.

So, by definition h prime, by definition g h prime composite g is equal to h, by definition this is this property is true right. If I take any element here go here then this is by definition we have defined it that way. So, therefore, or by construction whatever you say h prime composite g is h or in other words g bar of h prime is h, but still we need to show that h prime is an A module homomorphism. But that is again that is straightforward. How did we define h prime? Suppose I take x 1 and some x 1 plus x 2 here, x 1 and x 2 here we want to say that h prime of x 1 plus x 2 is same as h prime of x 1 plus h prime of x 2, but if I take x 1 plus x 2 here for x 1 I have y 1 here x 2 I have y 2 here therefore, y 1 plus y 2 goes to x 1 plus x 2 therefore, h prime of x 1 plus x 2 is same as h of y 1 plus y 2, but h of y 1 plus y 2 is same as h of y 1 plus h of y 2, but that is same as h prime of x 1 plus h prime of x 2.

Similarly, you can check the scalar multiplication property aspect. So, therefore, this is easy to verify.

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So, therefore, h prime belongs to a Hom A M double prime N. So, this implies that the Hom sequence is exact, is this clear. So, this is you know what we have done here is kind of diagram chasing we have to define a map from here to here, from here we went to this one and then we looked at we know there exists we define this map here. Now similar see given we looked at this exact sequence M prime to M to M double prime to 0 and applying Hom dash comma N, instead if you have similar proposition I will leave this for you to write down a proof, let 0 to N prime to N to N double prime be an exact sequence then 0 to Hom M N prime to Hom M N to Hom M N double prime is exact.

So, if I call this f and g, this is f bar and g bar. This map takes, so if I have h here f bar of h is f composite h composite f and similarly g bar. So, in this case again one can prove that f bar is injective and kernel is equal to the image. So, here also you have to construct given an element in the kernel you have to construct an element in the image, same way diagram chasing, I will leave it you to do that and you should do that because you know just listening to me what I have done and copying down from the board that does not you know give you feeling for it. So, you write down this proof completely. You have a prototype just follow the prototype and then it write down. We will do one more property which has even better diagram chasing that is even more fun.

So, this I will leave it for you to complete. Let 0 to M prime to M to M double prime to 0 N prime to N to N double prime to 0. So, I have these maps u 1, v 1, u 2, v 2, f prime, f, f double prime. So, when you have a diagram of module homomorphism like this, you say this is a commutative diagram if at any point from here let us say for any element here; you can reach here by 2 ways - one is first come here and then come or another way is coming here and coming both are same. This is said to be commutative if f composite u 1 is same as u 2 composite f prime and v 1 sorry f double prime composite v 1 is same as v 2 composite f.

So, let this be a commutative diagram of a modules and homomorphisms. Then we have an exact sequence involving kernel and co.

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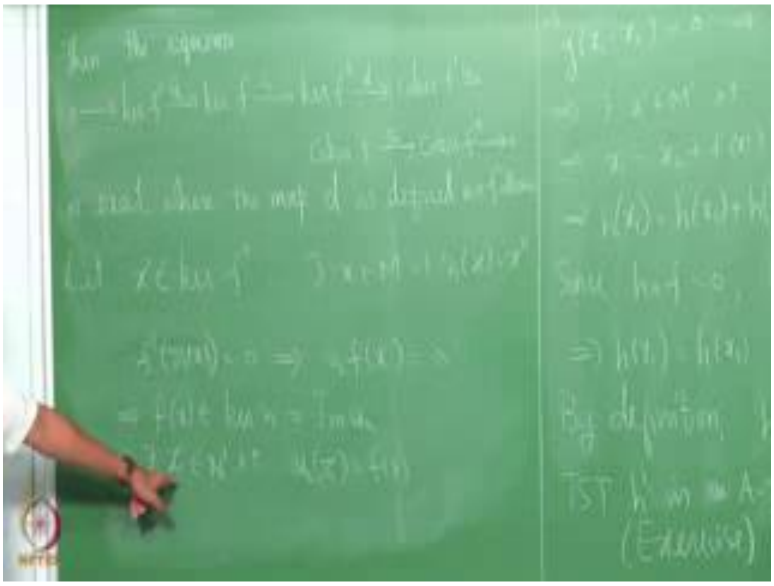

Then the sequence 0 to kernel f prime to kernel f to kernel f double prime to co kernel f prime to co kernel f to co kernel f double prime to 0.

So, I will call this map u 1 bar, this is v 1 bar, this is u 2 bar and this is v 2 bar, and this is exact where the map d is defined as follows do you understand the sequence. I have, if I have a commutative diagram like this then from kernel, see kernel f prime is a map sorry a sub module of M prime this kernel f is a sub module of M and kernel f double prime is a sub module of M double prime. So, first what it says is that you restrict u 1 to kernel of f prime it will be mapped to kernel of f, u 1 of kernel of f prime will be contained in kernel of f that is easy to check and similarly here and on the other side this is u 2 bar and v 2 bar are the induce maps.

So, what is co kernel of f prime? It is a quotient module of n prime co kernel of f is a quotient module of n what this says that there exists a map, u bar is a map induced by this homomorphism u, u 2 bar. So, u 2 bar of some element x prime is u 2 x bar and that will be well defined and you know it will be homomorphism from here to here, A module homomorphism and similarly v 2 bar.

So, these are to start with the existence of those maps. Now what it says is that u 1 bar is injective which is pretty much obvious because u 1 is injective. If I have that map u 1 bar which maps from kernel f prime to kernel f once I show that u 1 bar maps from here to here this injectivity of this is pretty obvious because of the injectivity of u 1. Next thing you need to show is that there is a map from v 1 bar from here to here, here it is exact that is what we need to prove, I will come back to that. So, I will just, first I will show that you know this kernel I mean u 1 bar of kernel f prime goes inside kernel of f.

I will come back, come to the definition of I will do the definition of d first. So, I want to define a map from kernel of f double prime to the co kernel of f prime. So, I want to define a map from a sub module of this to a quotient module of N prime. So, what do I do? This is yeah we have to keep chasing the diagram I start with an element in the kernel of f double prime. So, let x double prime be in kernel of f double prime I want to define, I mean I am trying to define what is d from here to here. So, I start with an element in the kernel of f double prime.

Now, this is a sub module of M double prime therefore, there exists some x in M, there exists x in M such that v 1 of x is equal to x double prime, surjectivity of v I mean v 1. The first row, I did not say this, commutative diagram of A module homomorphisms with the rows being exact. The first and second rows are exact. So, therefore, u 1 that v 1 is surjective and hence given this x double prime I have an element in x in M such that v 1 of x is x double prime.

Now, what can you say about v 1 of f of x sorry f double prime of v 1 of x, v 1 of x is x double prime and x prime x double prime is in the kernel of f prime therefore, this is 0 and that implies diagram is commutative. So, you go to M double prime and come to n prime or travel the other way you are going to get the same element or in other words we have v 2 of f of x is 0; that means, f of x belongs to kernel v 2, but what is kernel v 2 this is image v 1 therefore, there exists some Z prime in N prime such that v 1 of z 1 is, sorry this is a yeah image of u 2. So, u 2 of z 1 is f of x.

So, what have we obtained? We have started with an element x double prime here, we got a pre image here and said that image of that under f is in the kernel of v 2 therefore, I have an image in N prime and we are indeed trying to define a map from kernel of this to a quotient of this. So, we have now got hold of an element. So, immediately what we should do is define d of M x double prime to be the image of, I mean see the image of we want to define 2 quotient of n prime. So, define it to be.

Student: Z 1 bar.

Z 1 bar, Z prime bar whatever. Now we have to tackle, the first thing that we need to check is this is well defined or not and we have traveled a lot from x double prime to reach Z 1 prime, how do we know whether we travel a different path we arrive at a different element.

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So, define d of x double prime, d from kernel f double prime to co kernel of f prime by d of x double prime equal to. So, first to show that d is well defined. So, here how do we show that this is well defined? I mean, where are the issues? To start with I am taking an element here and going back here, here there could be many pre images.

So, first if I choose 2 elements here, 2 different elements here whether that would make a difference or not. Once we see here once we reach this stage, there exists when we say there exists Z prime in N prime such that this is true this Z prime is unique because u 2 is injective the bottom row is exact; that means, u 2 is injective. So, if I know that there exists an image that is unique there is no confusion here. So, we only need to deal with, if I choose 2 different elements that go to x double prime that would not create any trouble. So, let us try that.

Let x 1 x 2 be in M such that v 1 of x 1 equal to v 1 of x 2 equal to x double. Once we reach here, what is the next step? Once you see this equation what should come to your mind next - v 1 of x 1 minus x 2 is 0; that means, x 1 minus x 2 is in kernel of v 1, but we know what is kernel of v 1, what is kernel of v 1? This is image of u 1, what does that say? That says there exists some y prime in M prime such that x 1 minus x 2 is equal to u 1 of y prime or in other words x 1 is equal to x 2 plus u 1 of y prime.

Now, let us, see once I prove that, see we need to prove that pre image of f x 1 I mean f x 1 minus f x 2 that is 0 in the co kernel what we need to show is that f of x 1 minus x 2 is 0 in the co kernel that is our map is defined from here to here not into N, in N it need not be 0, but in the co kernel if that is 0 then the corresponding pre images will give you 0, I mean the difference between the corresponding pre images.

So, f of x 1, x 1 is x 2 plus something. Now let us look at what is f of x 1 is equal to f of x 2 plus f of u 1 of y prime right. Now, see if I take f of x is anyway in the kernel of v 2 because f of x 1 and f of x 2 both are in the kernel of and this is also in the kernel see what is f of x 1? What is f of x 1? Sorry and what we want to show and I am saying how did we arrive at this see f of x 1 is x 1 is the pre image of x double prime, x 1 is here f of x 1 is here, now you go on v 2 of f x 1 is same as f double prime of x double prime but that is 0, see x 1 is here this is the pre image of x double prime x double prime is mapped to 0 here therefore, this f x 1 this is map 2 0 under v 2.

So, therefore, f of x 1 similarly f of x 2 both of them are 0.

Student: (Refer Time: 44:05).

That is how we have chosen, where is x double prime? What is x double prime? We are defining a map from right, we have we have started with an element in kernel of f double prime. So, therefore, this is f x 1 is 0 and v 2 of f x 1 is 0, v 2 of this is 0 therefore, v 2 of this is 0. So, if I show that you know modulo the co kernel these two are same I mean modulo the image of f these two are same we are through, but is this not what we have shown now f of if you now take.

So, let us. So, this now f x 1 there exists z 1 such that, what was that? u 2 of z 1 is equal to x 1 oh sorry f of x 1 yeah f of x 1 and there exists u 2 such that sorry z 1 there exist z 2 such that u 2 of z 2 is f of x 2 plus f of u 1 of y prime because this is also in the kernel of u 2 which is in the image kernel of v 2 which is in the image of u 2. Now what is see u 2 of z 1 this is not what I want to say yeah sorry. So, this is f of x 1 minus x 2 is f of u 1 of y prime, but f of u 1 is u 1 of y prime is same as, see f of u 1 of y prime is same as u 2 of f prime u 2 of f prime of y prime.

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Now, if I apply. So, for x 2 I have a z 1 for x 1 there exists z 1 in N prime and for x 2 there exists z 2 in N prime, what would be u 2 of z 1 minus z 2? This is equal to f of x 1 minus x 2 which is equal to u 2 of f prime of y.

But now u 2 is injective. So, that that z 1 minus z 2 is equal to f prime sorry f prime of y prime there is a y prime that says z 1 is same as z 2 plus f prime of y, but what is co kernel of f prime? Co kernel of f prime is this is n prime modulo image of f prime. So, modulo image of f prime z 1 bar is same as z 2 bar. So, this implies z 1 bar equal to z 2 bar in N prime mod image of f prime which is co kernel of f prime. So, what we have shown here is that the map d is well defined.

So, in the next class, I will probably take 1 or 2 of them and show that they are you know this indeed maps from; well defined map from here to here and I will leave the rest of those easy checkings to you to complete.