Commutative Algebra

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Lecture – 12
Nakayama's lemma and Exact Sequences

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Let us first recall the statement of Nakayama's Lemma, and then prove it. Let A be a commutative ring with identity, and M be a finitely generated A module. If I is an ideal contained and Jacobson radical of A; this is Jacobson radical of A; such that I M is M, then M is 0. Proof; this is a straightforward application of the result that we proved in the last class. Suppose I M is equal to M, then you can think of the identity map, taking phi to be the identity map. So, phi of M is M which is I M. So, phi of M is contained in I M. therefore, by the result that we proved yesterday, there exists. We obtain an element x in A such that 1 minus x or x minus 1 belongs to I and x M is 0.

Now, I is contained in the Jacobson Radical, what does that mean?

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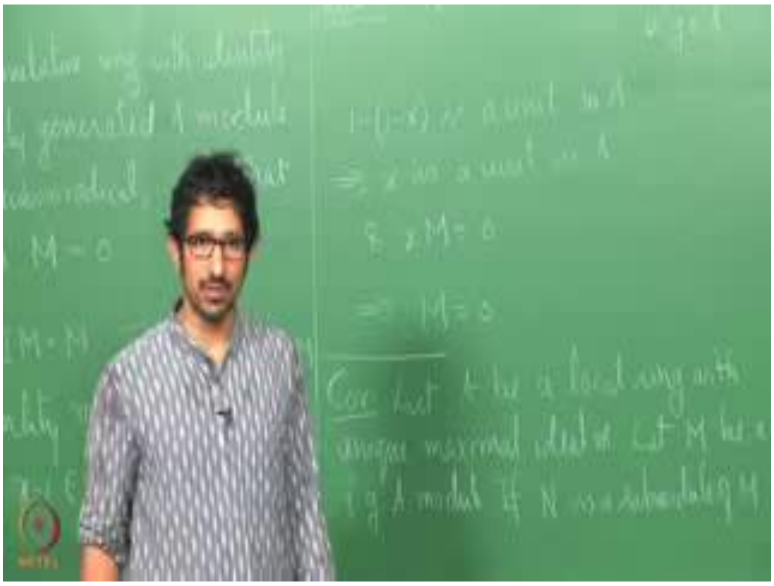

So, we proved this characterization that x is in Jacobson Radical if and only if 1 minus x y is a unit.

Student: (Refer Time: 04:02).

For all y belongs to A. This is something that we proved quite early in the course. So, using this, what we get is. therefore, 1 minus 1 minus x, taking y to be equal to 1; 1 minus 1 minus x is a unit in A, which implies x is a unit in A, and x times M is 0 and that implies M is 0.

So what Nakayama’s Lemma says is that, if you take a local ring and M to be a finitely generated A module then I M can never be M unless M is 0 for any non unit ideal. So, there are you know, as a corollary of Nakayama’s Lemma one can obtain or you know equivalent form of Nakayama’s Lemma. Let us look at 1 or 2 different versions. So, this is if I M is M then M is 0, as a corollary first of all. Let A be a local ring, unique with unique maximal ideal M. Let M be a finitely generated A module.

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If N is a sub module of M such that M is equal to N plus M M, then N is equal to M. this looks like a strange form, but this is indeed a very useful form, when we try to prove certain equalities of ideals. When we have to prove that us if I start with an ideal I and an ideal J which is contained in I, and you want to prove that I is equal to J. 1 of the methods is to prove that I is equal to J times M I. this is you know one way of applying Nakayama's Lemma and this is been useful very often, even in the research this is being used. So, this is a quick corollary proof directly follows from the statement. So, what we do is see we have to prove that M is equal to N, and here we talk about some statement implies M is equal to 0. So, how do you convert in that format?

Student: (Refer Time: 08:32).

We want to look at some module, should have a module structure.

Student: (Refer Time: 08:39).

So, what kind of module? Something this 0 will imply N is equal to M; that is what ultimately we want. So, what should be the module that we should be considering M minus N is a set, but that does not have a module structure. Take M mod N, if you take M mod N and if you prove that N is 0, then I mean M mod N is 0, then M is equal to N. So, consider the module M model N. to say that M mod N is 0 what does that, we need to prove we need to prove that for an ideal contained in the Jacobson Radical, what does Jacobson Radical in this case.

Student: M itself.

M itself, contained an M, M times that ideal times M mod N is equal to.

Student: M mode N.

M model now let us look at. So, here it is the Jacobson Radical, let us look at what is M times M mod N.

Student: (Refer Time: 10:14).

By definition this is equal to M M plus N modulo N. The quotient is precisely equal to M M plus N mod N. So, this is a general fact, if you take for any ideal I, I times M mod N, this is same as I M plus N modulo N, this is; so how do you verify this. Look at the definition of this, what is the definition of this? Let us this is by definition set of all summation.

Student: (Refer Time: 11:29)

a i times m i bar, this is finite summation, a i coming from, coming from i, but this is same as summation a i m i whole bar, but now what is meant by. So, this one will tend to express it as I M mod N, but there is nothing like this. So, if you look at this, let me explain this, because I feel that you are not very comfortable with that. So, it is better not to deal with only with elements.

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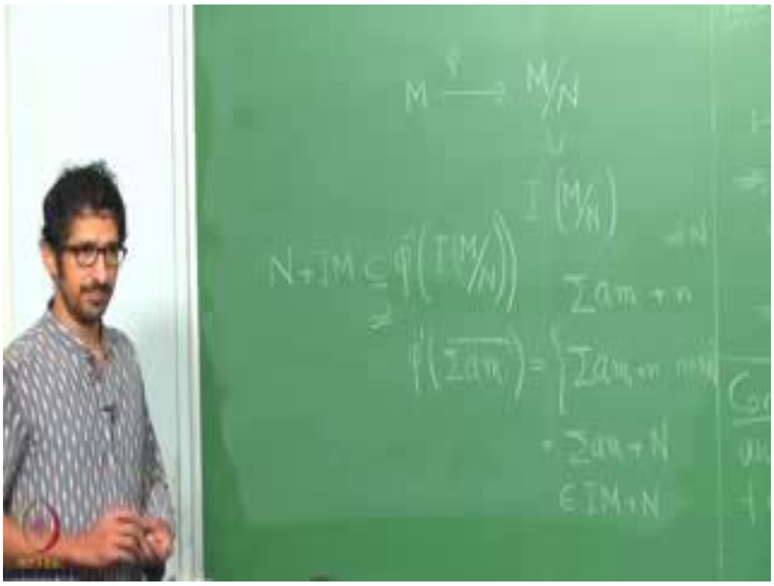

Look at this natural map from M to M mod N. So, I have this I th times M mod N, this is a sub module of M mod N. Now what would be the; so this is this is the natural map, what would be it is inverse; phi inverse of I dot M mod N.

See this is a sub module, and certainly this contains 0, so, inverse image will certainly have N there, and this will certainly contain I M, because we are taking inverse images of all elements of this form. This is also we can say this is same as the whole bar. So, I have all the elements and N also here. Now if I take any element here.

So, if I take any element; for example, if I look at any element in this a i m i bar what does it is inverse. This the inverse of this, is precisely it is not an element in here, because there are you know there could be several elements that will be mapped to this one. What are the elements? If I take any a i m i plus N where N in N, this will be here. So, this will be the set summation a i m i plus N, where N is in N, or in other words I can write it as summation a i m i plus N represented this way, and that is in I M plus N. So, what we have shown is that this is contained here. So, therefore, these 2 are equal. Is this clear? So, that is the solution for this exercise I times M mod N is nothing, but I M plus N modulo N.

Now, this is by hypothesis this is equal to M M plus N, is equal to M therefore, this is M mod N. So, by M mod N is 0 which implies M. Is this clear? We are taking M times M mod N. by exercise this is equal to M M plus N mod N, but the hypothesis is that mm plus N is equal to M. Therefore, this is equal to M mod N, but then we have M, which is contained in the Jacobson Radical which is equal to the Jacobson Radical, with the property that M times the module is equal to the module itself by Nakayama Lemma M mod N the module is 0, M mod N is 0 which means M is equal to N. is this clear?

Another nice corollary of A, now Nakayama Lemma is in talking about something similar to a basis. see in the case of vector spaces over a field, there is any minimal generating set, and maximal linearly independent set, they all have same cardinality, or any to minimal generating sets have same cardinality right, that is linearly independent we have bases answer, but we have seen that in the case of modules that is not necessarily true. There can be 2 different minimal generating sets of different cardinalities, but if we are a slightly nicer situation than just a commutative ring, then we still have something nice.

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So, let us look at this; let A m be; so this is a standard way of denoting when I say A m be a local ring, it means A is commutative ring with identity and it is a local ring with M being the unique maximal ideal, and M be a finitely generated A module. Now if I look at, then if I look at this module; M is exactly the annihilator of this module. The maximal ideal M is precisely the annihilator of this module right, and therefore, this is a vector space over.

Student: (Refer Time: 20:34).

A mod M, A mod M, I mean it is a module over A mod M, which is same as same vector space over mod M, because A mod M is a field. Then this is a vector space over A mod M. can we say that it is finite dimensional, can you see that it is finite dimensional. So, this is a one remark, let M be a finitely generated A module, and I in A be an ideal, then we know that M mod I M is an A module, at least to start with. Can you say something more, is it finitely generated? Will this be a finitely generated a module?

Student: (Refer Time: 22:07).

Because M is, I am saying that M is finitely generated a model which means, see what I am I saying, M is equal to I can find a set there exists M 1 of to M N in M; such that M is equal to summation and set of all I from 1 to N; A I; M I; A I in A, M is precisely of this form. Now if I take this as an A module, what are the elements? They are equivalence classes of all elements of this form. So, in particular can you get me a generating set for this as an a module; M I bar, like M 1 bar M 2 bar to M N bar will be generating set for M mod I M as an A mod. Can you say? So, this is a finitely generated a module, can you say something more. M mod I M is an A mod I module. Can you say this is a finitely generated A mod I module will the same generating set.

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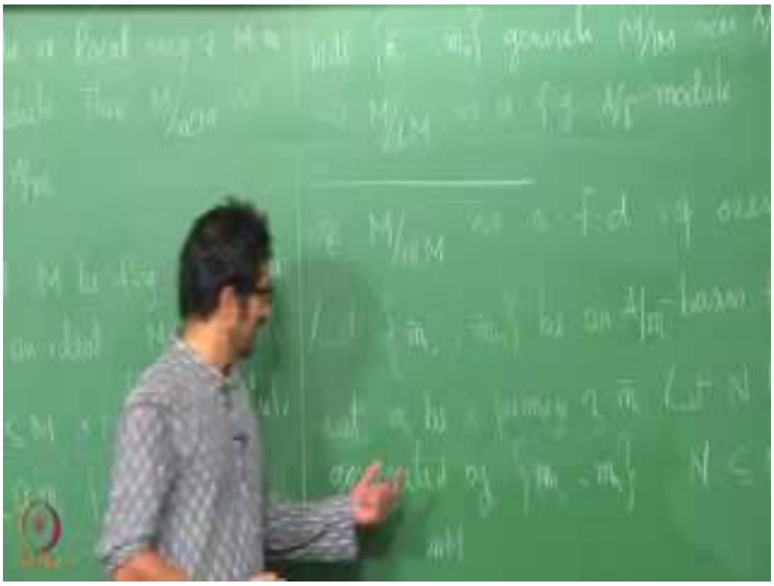

Will M 1 bar up to M N bar generate M mod I M over M mod I.

See if you look at the multiplication here. We are taking linear combinations with elements in a. now I is contained in the, I mean I is equal to the annihilator of M mode, I M I is contained in the annihilator, because you know there could be other elements that annihilates M. So, I plus annihilator of M will be equal to the annihilator of M M mod I M, so, I is contained in the annihilator; therefore, if I take any element from I, those linear combinations will certainly become 0, it will vanish. So, if you take 2 elements which are same modulo I, their product and that summation etcetera will give you the same element, which means the same set will generate M mod I M over A mod I. So, the answer here is yes. So, what have we proved? We have proved that M mod I M is a finitely generated A mod I module.

So, if M is finitely generated a module, then M mod I M is a finitely generated a module. Not only that, M mod I M is a finitely generated A mod I module. and we have not only proved that, we have proved that if I take a generating set of M, the same set, from the same set I can get a generating set for M mod I M, as an a module as well as, as an A mod I module, the same set will generate. So, now, let us return to our situation. So, M is a finitely generated a module; therefore, M mod M M is a finitely generated A mod M module, but A mod M is a field therefore, M mod M M is A.

Student: (Refer Time: 27:15).

Vector space over A mod M, not only that it is a finite dimensional vector space, so this and M mod M M is A if dimensional vector space over A mod M. So, it has a basis. So, let us take M 1 bar up to M N bar B and A mod M basis for M mod M M. So, what would, though you know one cannot. Can we expect this will generate M, a priori either is no no way one can expect this you have a generating set for a quotient, how do you expect that to generate the whole module, it need. So, let us take let us let N I be, maybe I will just write this M I be a pre image of M I bar. So, when I say pre image in the natural map. Let N be the sub module generated by M 1 to M N. So, my question is, whether N is equal to M or not. I mean to start with we cannot really believe that it is. So, let us see. So, I have this sub mod, N is a sub module of M, if M M is:

Student: Generated by M I.

Yeah.

Student: Because any generator, it is not M I bar we have to be in M M.

This will not contain all this M I.

Student: yeah, yeah.

Student: no I am saying that is M I M into M M, yes not a changing to same, then implement the any generating (Refer Time: 30:18) is M M. So, that the M becomes is 0 M M. So, that are all (Refer Time: 30:23).

There will exist some element, no. see we might have something in between N and M. So, anything here will become 0 when you take quotient.

Student: yeah.

So, this will not matter at all, I did not understand what.

Student: I was saying that that is the generating set for M mod N.

Yes.

Student: if set is some other element (Refer Time: 30:53) M N M I.

Ok.

Student: other generating set then should belong to.

Yes it should belong to.

Student: it M I to elements.

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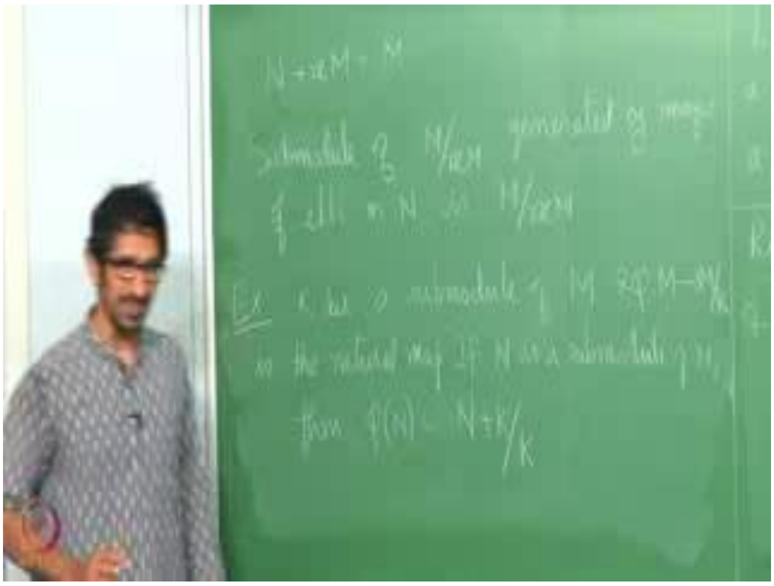

So, or in other words see what you are trying to say is that, if I show that this is M right, that is precisely what you are saying, and this is what you wanted to show I mean if we show this then we are done, because of the corollary of Nakayama Lemma very good. Now is this true?

Let us look at M mod, this is N this is a sub module of M, what is the image of, what is the sub module of, the images of elements of N in M mod M N; sub module of M mod M M generated by images of elements in N. what is this? What is N? N is the module sub module generated by M 1 M 2 up to.

Student: (Refer Time: 32:44).

This will be equal to s M mod M M, because N is generated by M 1 M 2 up to M N. So, if you take it is images here, it is images precisely the module or the vector space M mod M M. Now if I look at the natural map, what does phi of N. What does phi of N here, if I take any sub module.

Student: (Refer Time: 33:35).

No, how do you; so, this is a general question, suppose I have K be a sub module of be a sub module of M. So, this is exercise, K be a sub module of M, and phi from M 2 M mod K is the natural map. If N is a sub module of M, then phi of N is equal to. Can you give me a precise representation for this in M mod k. it is the exercises very similar to the way we proved M mod I M.

Student: (Refer Time: 34:54).

N plus K mod K, it is precisely N plus K mod K. So, therefore, let us apply that here, what is the image of N here. The image of N is.

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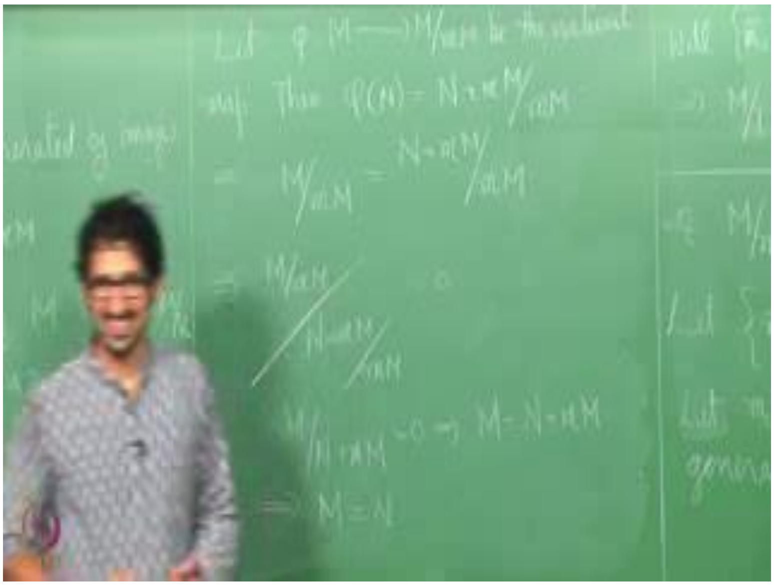

So, now let phi from M to M mod M M be the natural map. Then phi of N is equal to N plus M M mod M M; therefore, but image of this is equal to. We know that this phi of N is equal to M mod M M, because images of M generates M mod M M this is equal to this implies M mod M M is same as N plus M M mod M M. or in other words this mod this is 0, but what is this mod this, which is isomorphic to. This implies M mod M M modulo N plus M M mod M M is 0, but what is this?

Student: M mod M M.

This is isomorphic to M mod N plus M M is 0, which implies M is equal N plus M, by the corollary of Nakayama Lemma this says M is equal to N. So, what have we shown here? We have shown that if I take an a modern basis of M mod M M, this number, this set will generate M 1, M 2, M N, sorry this will generate the module M. the corresponding inverse images will generate M. note that during the whole proof, we did not use that this is a basis, we only use the property that it generates right. So, now, suppose this and we take, suppose this is a basis, basis means it is a minimal generating set right. Now what can we expect. if this is a generating set for M mod M M as A mod M vector space, then we have proved that this is a generating set for M as an A module.

Now, I am saying this is a minimal generating set or it is a basis.

Student: (Refer Time: 38:45).

Can we expect this to be minimal? Let us go through the proof, let us see whether we can. Suppose this is not a minimal generating set, what does that mean?

Student: (Refer time: 39:04).

It means that there exists one of the M I is;

Student: (Refer Time: 39:12)

Which can be written as, you know which can be obtained from others. This is when do you say yes set is not a minimal generating set. Minimality is with respect to the inclusion. So, if this is not a minimal generating set; that means, there exists a subset which is some minimal generating set, which is a generating set. So, I can find at least 1 M I which is written as a combination of others. So, let us say M N is written as linear combination of M 1 up to M N. Now take it is image into M mod M M. The coefficient of M N will remain non 0 at least that will remain non 0. So, you have a linear combination of basic elements that leads to 0, which is a contradiction.

So, we have proved that, if this is a basis for M mod M M over the field A mod M, then this is a minimal generating set. We do not use the terminology basis, because we do not have linear independence here, what we have shown is, that this is a minimal generating set. Now over a local ring this uniquely determines the number of elements in a minimal generating set. So, any minimal generating set will have this dimension of this over this, number of elements. So, therefore, one can define the term. We do not have dimension in the case of this, we do not have the definition of linear independence, but we do have minimum and a minimal general cardinality of a minimal generating set.

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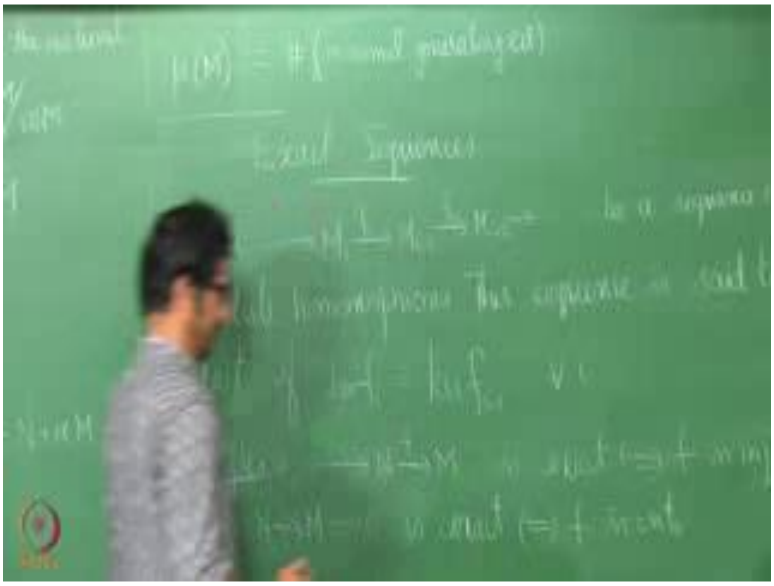

So, therefore, this is mu M; this is cardinality of a minimal generating set. This is a well defined terminology when you have a finitely generated module or a local ring.

This is one of the; you know most important applications of a Nakayama's Lemma. So, now, let us a move on to study exact sequences. So, we saw and we have been dealing always with a homomorphism. Suppose I have a sequence of homomorphism M I to M I minus 1 to M I minus 2 be a sequence of homomorphism, a module homomorphisms. So, this is, I will denote this by f I f I minus 1 and so on; each of them is an A module homomorphism. This sequence is said to be exact, if image of f I is equal to kernel of f I minus 1 for all. So, I can have A sequences of, long sequences of modules in homomorphisms. Let us look at one or 2 simple examples. What is meant by this is exact. Kernel f is image which means, which is 0, for any module a modules N and M this is exact if and only if f is injective right. This is exact if and only f is injective. Now can you formulate? Kernel should be image, what is the kernel of this map.

Student: whole of M.

Whole of M, this:

Student: (Refer Time: 45:29).

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This is on to, this is exact if and only if f is onto; let us look at some precise examples. So, this is one of the most basic important examples in the area of commutative algebra. So, I take my ring to be K x y, and look at this exact sequence 0 to A to a direct sum A to A to 0. Now what is this map, this is y minus x, and this is x y, I am writing this the matrix. So, or in other words, here a is mapped to a y minus a x, and any r s is mapped to; so, here I am representing as in the case of vector spaces and so on we are raised whenever we have to write this as left multiplication, we write this as the vectors as column vectors, we are following the same convention.

Student: (Refer Time: 47:30).

This should be r x plus s bar.

Student: sir.

Yes.

Student: (Refer Time: 47:40).

Sorry.

Student: this basic called as a equation (Refer Time: 47:46) comma 0 and there is position (Refer Time: 47:49).

Yes. So, correct. So, if I look at this A to a direct sum A to A to 0, x mapping to x 0 and A B mapping to B, kernel should be the image.

Student: (Refer Time: 48:18).

Sorry.

Student: that is multiplication of this.

Exactly, so here what is y and what is x in this case?

Student: y is 0.

But here this is the ring a. well these 2 are slightly different here I am talking exactly about the variables, not really y equal to answer. Well this is one of the natural examples I could have mentioned early before this, but this is you know this example has much more, if you study commutative algebra further, this is one of the most basic example of a resolution of these 2 elements, the ideal generated by x comma y, this is something that you will learn, if you do an advanced course in commutative algebra, but this is you know interesting example by itself.

Student: why there is not a (Refer Time: 49:40).

This case, these are variables y and x are simply variables.

Student: (Refer Time: 49:49).

No if you take x equal to 0 the ring changes, ring is no more A. This is for any commutative ring, you take any commutative ring this, but in this case if you put x is equal to 0 the ring is different thing. So, here one can easily see that this is, I mean this is an exact sequence. So, this is, this is injective and this is subjective, and kernel of this is same as image of this. So, this is I will leave it as an exercise. We will continue.