Commutative Algebra

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Lecture - 01
Review of Ring Theory

Welcome to the course in Commutative Algebra.

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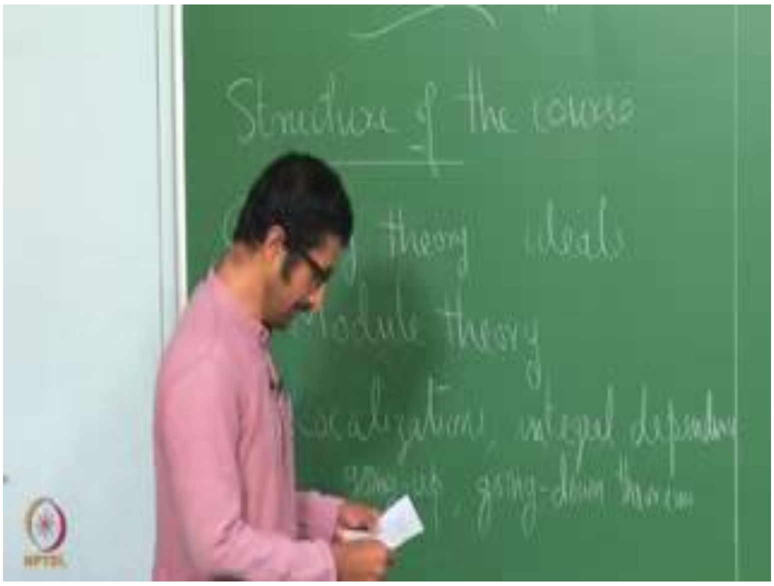

We will start with a basic Ring Theory, where we will do some basics of ideals, operations on ideals, extension contraction and so on. And then we will move on to Module Theory. Again we will begin with the basics and try to study various aspects of basic module theory. And then we will move on to localization, integral dependence, and going up going down theorems.

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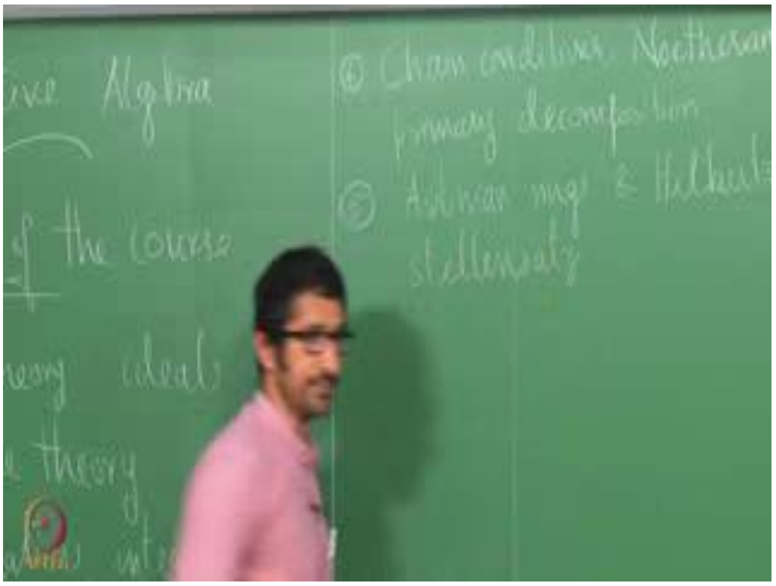

We will then move on to chain conditions, Noetherian rings. So, this is one of the most important classes of rings in commutative algebra called Noetherian rings some properties in Noetherian rings some important properties such as primary decomposition. And we will then study Artinian rings; and Hilbert's Nullstellensatz. Depending on time availability, we will do little more in this direction about Artinian rings and something about graded rings.

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So this is a rough structure of the course. I will be following the book introduction to commutative algebra by Michael Atiyah and I G McDonald. So, approximately 8 chapters of this book along with some portions probably from outside, but I will mention that when we go along. So, let us, let us begin I hope you all had a basic course in ring theory. So, I am today it will mostly be recalling the basic concepts from your algebra one first course in algebra.

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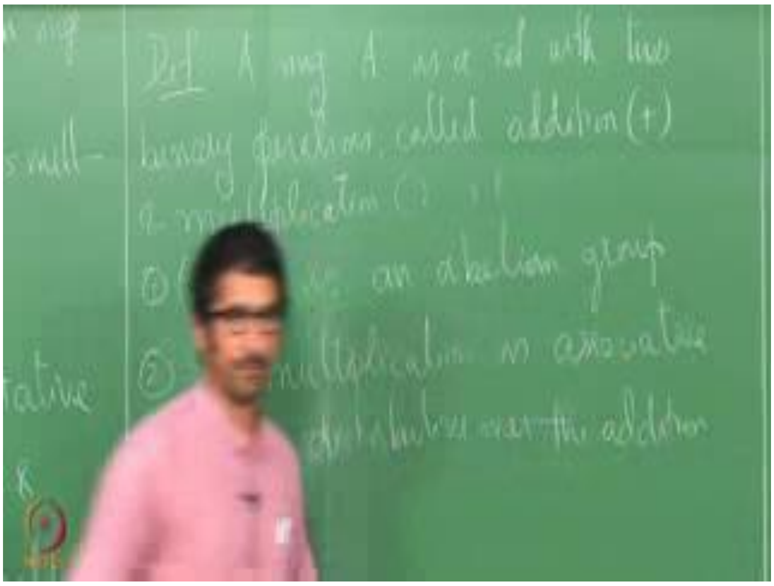

So, a ring, ring A is a set with 2 binary operations. So, called an addition and multiplication, and multiplication, such that what are the properties, the ring with addition, with addition is an abelian group. And then the multiplication is associative and is distributive over the addition.

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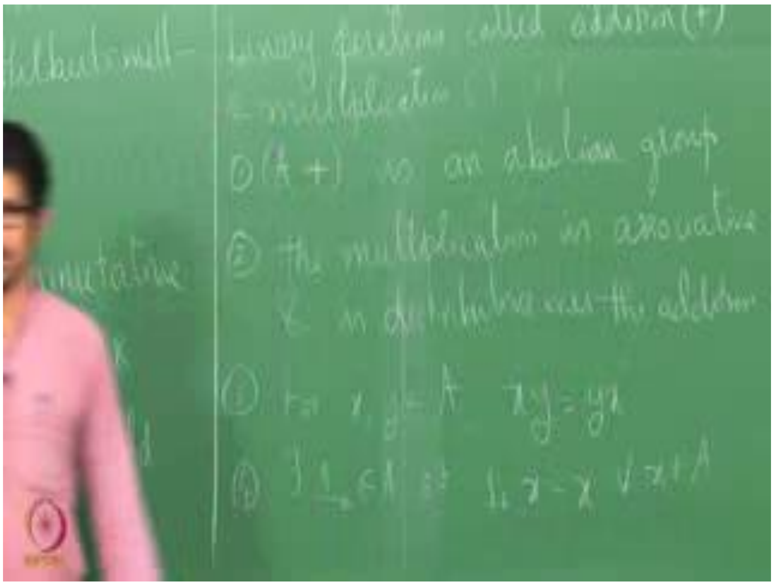

So, our courses commutative algebra, therefore we will be dealing with rings which are commutative. Or in other words we will have in for throughout this course we will have 2 more hypotheses, to the set of all these set of hypotheses for rings, that for x comma y in A xy is equal to yx. And I will also assume that our rings contain the identity element. There exists multiplicative identity, that is that x is an element called denoted by one a such that satisfied. So, our rings will contain multiplicative identity, all rings will have multiplicative identity, commutative rings with identity.

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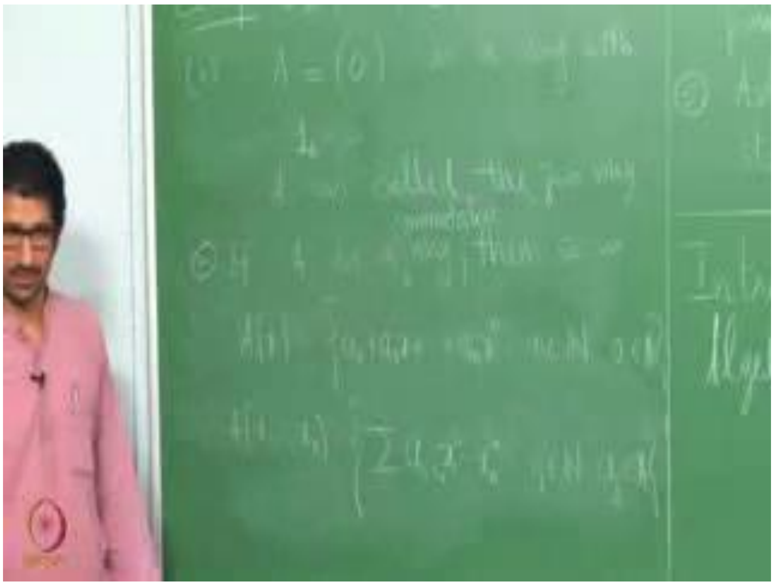

If you have doubts you can stop me and ask me let us look at them I am sure you have seen many examples. Some basic examples are telling me some basic examples Z R C. So, you missed something in between.

Student: Q.

Q then; so we should always start with the zeroth example. It should be what should be the zeroth example.

Student: 0 comma 1.

What about this simply 0. This is a ring. So, this also is a commutative ring with multiplicative identity where one of A is equal to 0 itself. So therefore, this is and this we will denote this by 0 ring, A is called the 0 rings. So, these are all standard examples that you must have seen.

Now, if this is one, I should say this is 2. If A is a ring then A x, A is a commutative ring. Then A x is you understand this notation. This is set of all a naught plus a one x plus etcetera a n x power n, n in N, and a i in R. Collection of all polynomials and this has a natural addition and multiplication that you have seen you must have seen in your earlier courses. In the in the similar manner one can define polynomial rings in polynomial ring in many variables. So, this will be collection of all I will write the compact notation, a I 1 to i n x 1 or i 1 x n power i n such that i j belongs to n. So, here my n contains 0. I assume that the natural numbers contain 0 and a i belong to A. So, this is these are some again standard examples of commutative rings.

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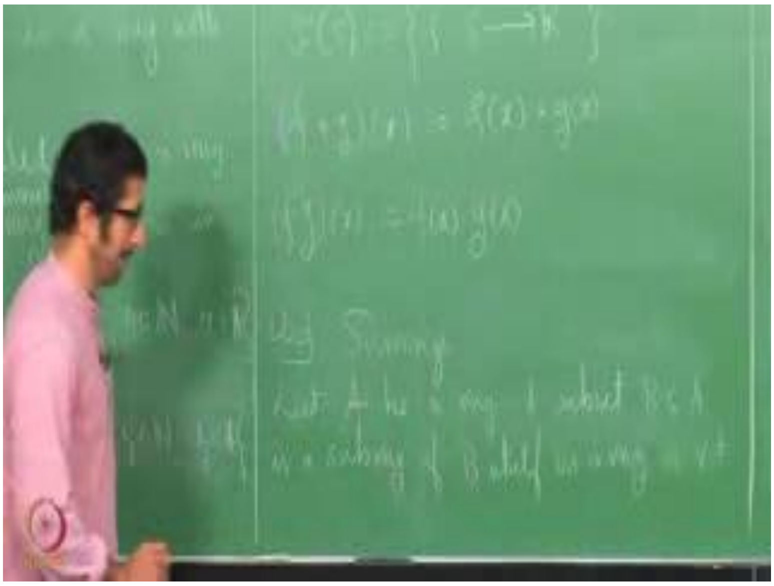

Another collection of nice examples come from set of all functions. If I have S be any set, then set of all functions on S set of all functions from S to R.

You can define an addition and multiplication on this ring. How do you define an addition? Point wise addition; so I can define f plus g acting on an element, x by definition this should be f x plus g x. And similarly f g acting on x is f x times g x, with this addition and multiplication this is a commutative ring with identity. So, once you define rings, once you think about you know some natural examples, the first thing that one does is to think about the sub structures. So, you have a ring. So, then what are the sub rings. When do you say sub set is a sub ring A? So, let A be a ring, a subset B contained in A is a sub ring, if B itself is a ring with respect to the same binary operations on A. With respect to same binary operations B is also earth then we say B is a sub ring of A.

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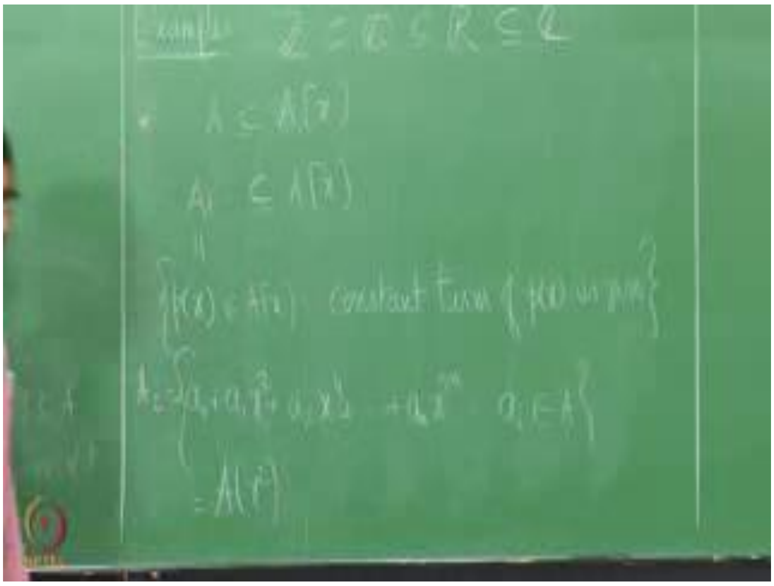

So, here for the definition, we do not need a sub ring to contain the identity element. It is our convention that our rings contain identity element. In this course all our rings contain identity element, but in the sub ring case we do not insist that. When we are talking about the sub ring we are not insisting that sub ring contains the multiplicative identity. I will come to the reason why we are not very strict about it. I will come back to that little later. So, quickly some examples of sub rings this.

So, if I just replace this, let it be here some examples. All of them are sub rings of C right. And for a can you tell me a sub ring of R x where R x ring or let me denoted by A. A itself as a sub ring of A x. Another sub ring can you tell me another sub ring which is maybe slightly bigger than A any sub ring of A will be A sub ring of A x, but can you tell me a ring which is bigger than A and contained A x and which is a sub ring.

Student: There will constant and b constant.

Polynomials, I will denote this by let us say A 1, A 1 set of all p x in A x such that constant term of p x is 0. So, if you the natural addition and multiplication if you do that, if you take 2 constant polynomials with no constant term, you add and multiply the it is going to remain another polynomial in the same set because it cannot have a constant by the usual addition and multiplication polynomial. So we will disk, we will see more examples later.

Student: Polynomial even pause.

Even when you talk about polynomials with even coefficients you are talking about Z maybe even.

Student: Power.

Even powers, so, that is yes. So, this is are you talking about a naught plus a 1 x square a 2 x power 4 plus a n x power 2 n is the for 2 n movement.

Student: Yes.

So, A i in A. So, this ring is usually denoted by A x square. That is another subject we will see more and more examples of sub rings as we go along.

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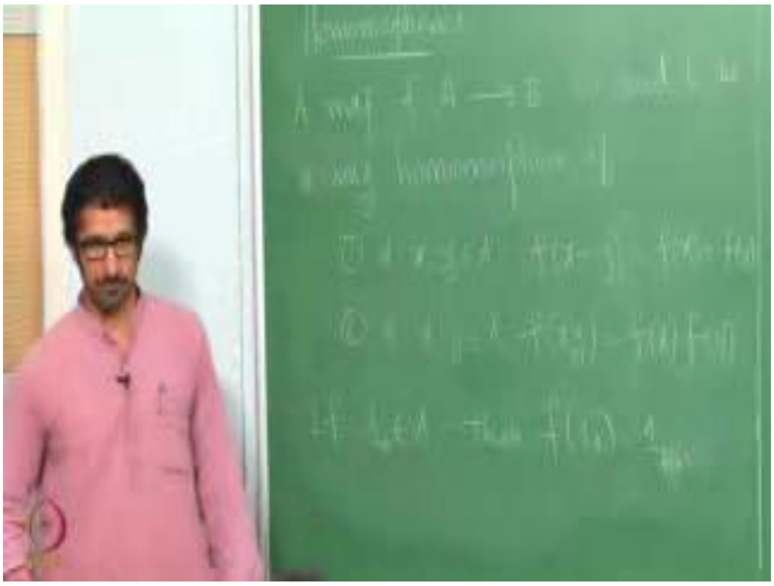

So, now let us move on to another important aspect of ring theory at homomorphism. So, A map f from A to B where A and B are rings is said to be a homomorphism to be a ring homomorphism. If what are the properties if f of x plus. So, for all x comma y in a f of x plus y should be equal to f of x plus f of y. And for all x comma y in a f of xy should be equal to f x times f y.

Then if a map satisfies these 2 properties then it is called the ring homomorphism. Now if A contains a multiplicative identity. So, in in the for the purpose of our course, all our rings are commutative and it contains the multiplicative identity. So, if 1 A is there then what can you say about f of 1 A, this will be.

Student: 1 of B.

1 of B; so this is partially true, this should be 1 of F of A. It should be a multiplicative identity of the image for that we need to say something slightly more that this is indeed a ring or a sub ring.

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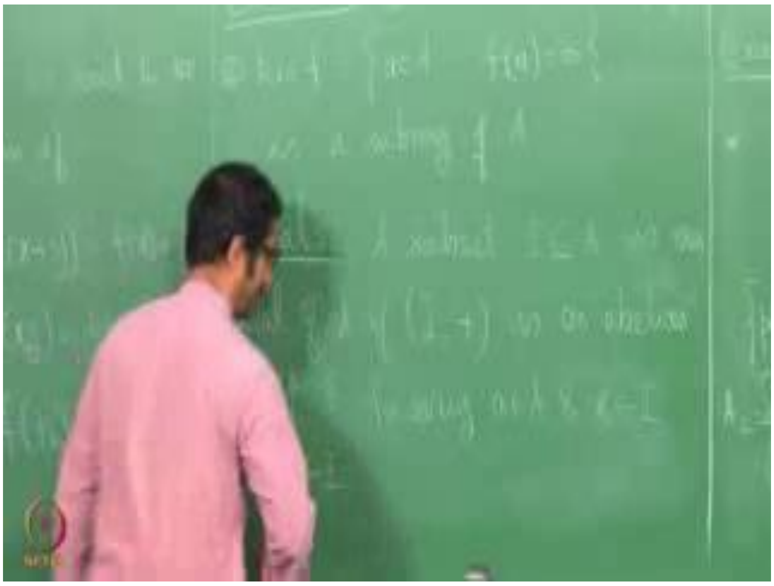

So, first observation is that, first observation is that if image of f is a sub ring of B.

And there is another object which is associated with the homomorphism which is kernel of f this is all elements in A such that f of A is 0. And observation 2 is that kernel f is a sub ring of A.

Student: (Refer Time: 21:52).

Yes, identity of sub ring need not necessarily be the identity of the bigger ring, one can you think about an example.

Student: Sir, but population for same how can be (Refer Time: 22:15).

Well first of all the sub ring need not contained the identity element. For example, in Z 6, if you look at the ideal if you look at the set 0 3. This is a sub ring of Z 6 right. Now what can you say about 3. 3 times 3 is 3 and 3 times 0 is 0. So, therefore, this is multiplicative identity of A. This is multiplicative identity of the sub ring, if I denote this by B, B is the sub ring of Z 6. And these are all you know dependent on conventions. I can even say that my sub ring contains all my sub rings, a set of sub ring if it contains 1. And I can say that f is a homomorphism if f of 1 is equal to 1, but I am not doing that in this course. It depends on the need of the development of the course one can set various conventions. So, in this I do not do that. There are books which follow this convention as well as the convention that we are using here.

So this is a kernel and images are sub rings. Now we have a slightly more general object which are called ideals. A subset i in A is an ideal of A if, if there exists if i plus is an abelian group. And for every a in A and x in i, x is in i. If i satisfy these 2 conditions, then we say that I is an ideal. So, I guess we know many examples of ideals and all these rings.

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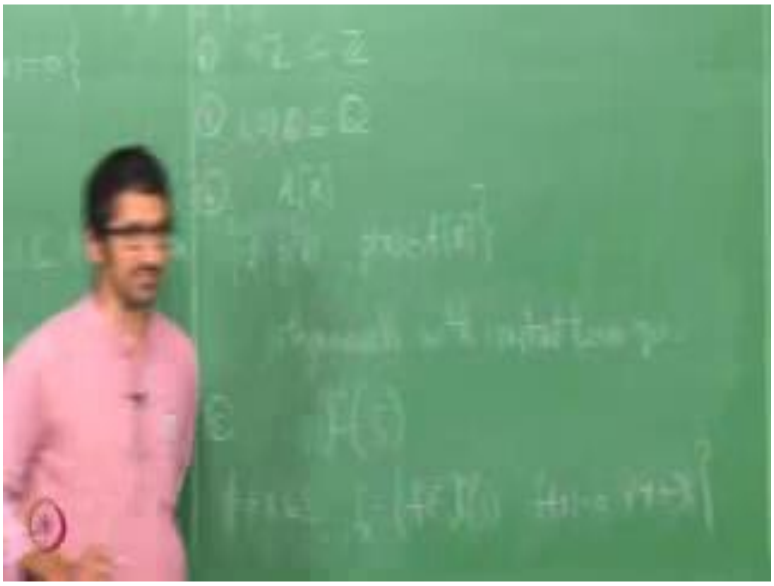

What should be the zeroth example? If I take any ring A. The set 0 this is an ideal. And in Z we have these are examples. What are the ideals of Q?

Student: 0.

0 is already there yes any other ideal.

Student: Q 0 Q itself.

0 and Q itself; so another example: here is A is an ideal of A itself. The whole ring is an ideal, because it obviously, satisfies us both the conditions. So therefore, this and Q now, in A x can you give me an example of a an ideal. So, ideal generated by x, what does that mean which means. I say I look at all the collections x dot p x where p x is in A x. So, this is something that we have seen just a few minutes back. What is this ideal or what is this set? We did we saw a different representation.

Student: Constant term 0.

Polynomials with constant term 0. Now, we had another example F of s. Can you give an example of an ideal in F of s?

Student: The single constant well be cannot.

Set of constant functions. Will it be an ideal? It should satisfy I take any element in this ring multiply the set A multiplies the element it should again beings this.

Student: Take a substrate finite substrate other substrate.

So, take a nonempty subset let us say x non-empty subset of x and look at this collection I of x to be set of all f in F s such that F of x is 0 for every x in X. So this is I want you to check that this is indeed an ideal now.

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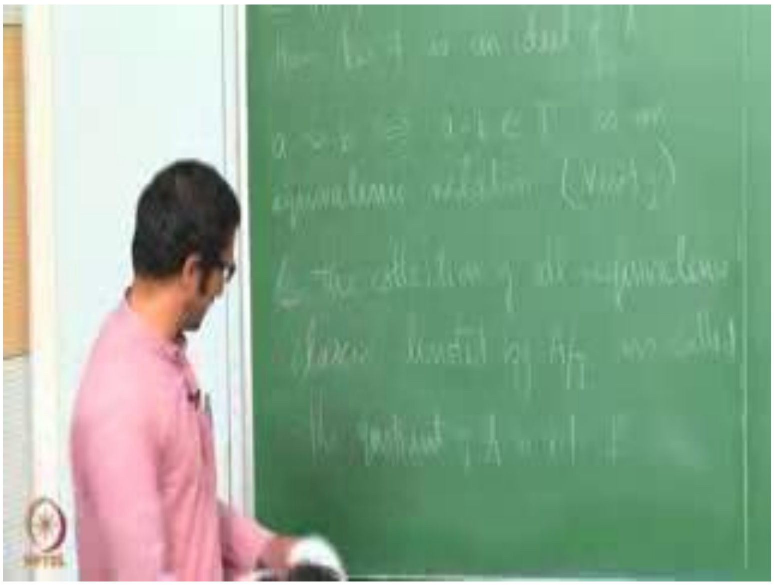

Let us go back to what we wrote here. Kernel is a sub ring. This is again another easy exercise if f from A to B is a ring homomorphism. Then kernel of f is an ideal of A. Again this is something that you must have done in your first course. Now once we have the concept of ideal that gives us an equivalence relation that is I can define an equivalence relation a related to b if and only if a minus b is in I. The collection of all equivalence classes of this relation this is an equivalence relation. And the collection of all equivalence relation equivalence classes is denoted by A mod I is called the quotient of A with respect to I.

So, in our situation I is an ideal in commutative ring. So, this there is a natural addition and multiplication defined on A which is inherited from natural addition and multiplication defined on A mod I which is inherited from A.

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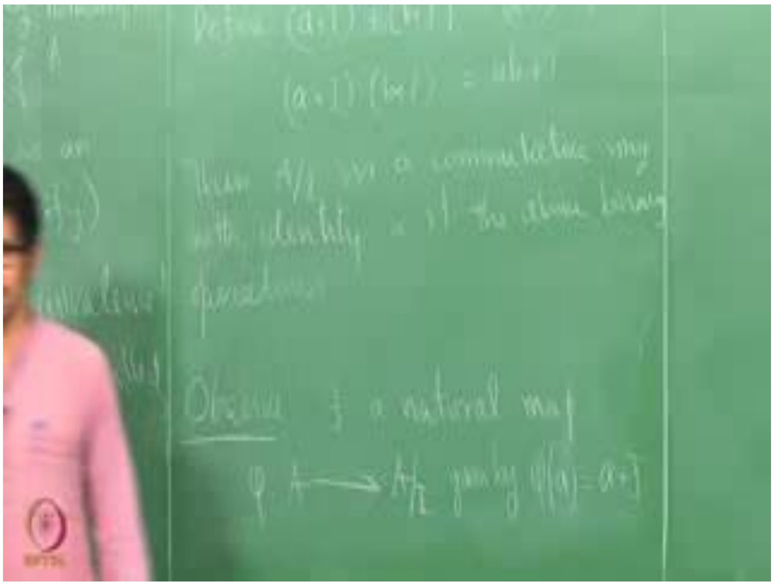

Which is a plus I plus b plus I to be equal to a plus b plus I and a plus I times b plus I defined to be a b plus I. This natural definition of addition and multiplication gives a ring structure to A mod I is a commutative ring with identity with respect to the above binary operations. First of all, we have a natural map from A to A mod I there is a natural map there exist a natural map phi from A to m mod i. What should be the natural map phi of A equal to.

Student: A plus.

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A plus I the equivalence classes which a belongs to. And then phi is not simply a map, but it is a ring homomorphism. And slightly more this is a yes phi is a surjective ring homomorphism.

If I have a f from A to B is a ring homomorphism. So, this is called isomorphism theorem. This says that if this is A to B is a ring homomorphism then, image of f is isomorphic to a mod kernel of f, this is something that you have seen in linear algebra group theory and possible in your ring theory course as well. Linear algebra this theorem has a nice name what is that?

Student: Ragionality.

Ragionality theory; the ragionality theorem is precisely this it is the first as small some theorem of vector space vector spaces.

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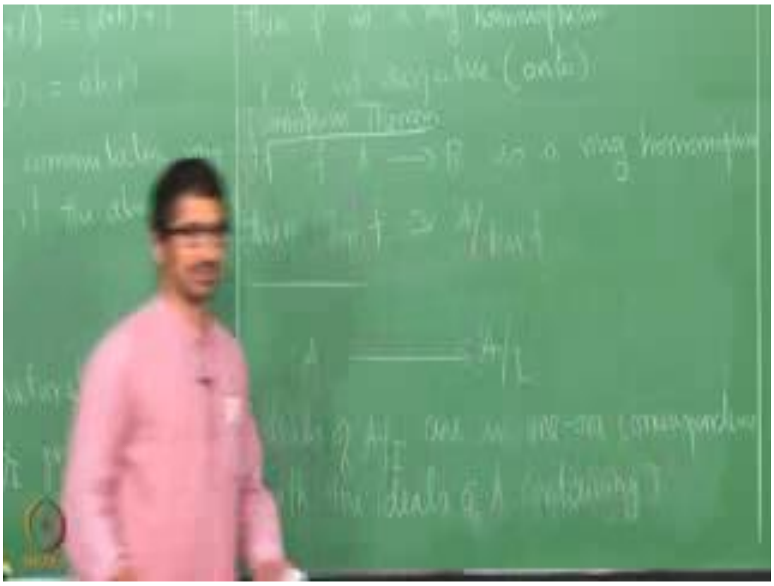

So let us look at the questions. We have we have a ring obtained from A by doing an operation we are going modulo. So, I have a ring A I have ring A mod I. One of the natural questions to think about is what are the ideals you know in A mod I. So, this is again explained by this looking at the ring homomorphism natural in a ring homomorphism, one can immediately obtain an explicit expression of explicit description of all the ideals of A mod I. So, what are the ideals of A mod I, are precisely are in one-one correspondence with the ideals of A containing I.

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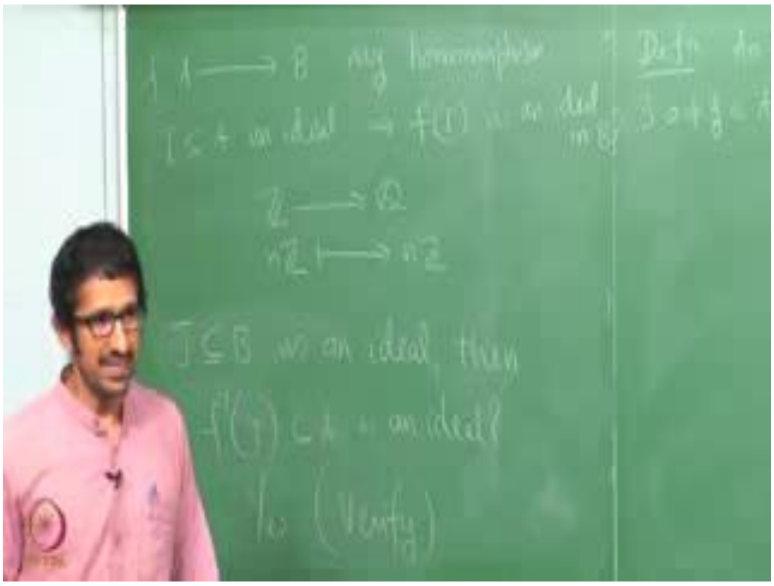

So, if you take any ideal here, it is inverse image will contain I why is it so? Why is the image?

Student: 0 is in ideal.

0 is there in any ideal. So, if you take the inverse image, phi inverse of 0 is always there in the inverse image. Now phi inverse of 0 is nothing, but I. So, therefore, I is always contained in inverse images of ideals here. And you can see that if you take any ideal here it is image will be an ideal in A mod I. So, that you know poses a question, what can you say about ideals if I have a ring homomorphism from A to B. And I have I is an ideal, is it through that F of I is an ideal in B, is it true no. Can you give me a quick example?

Student: Z to Q.

Z to Q identity map nZ a snap to n Z, but nZ is not an ideal in Q the only ideals of Q are either 0 or the whole of Q, but now let us ask another question; if J and B is an ideal, then what about f inverse of J.

This is indeed an ideal. So, the answer is yes we have not done this verification one can do.

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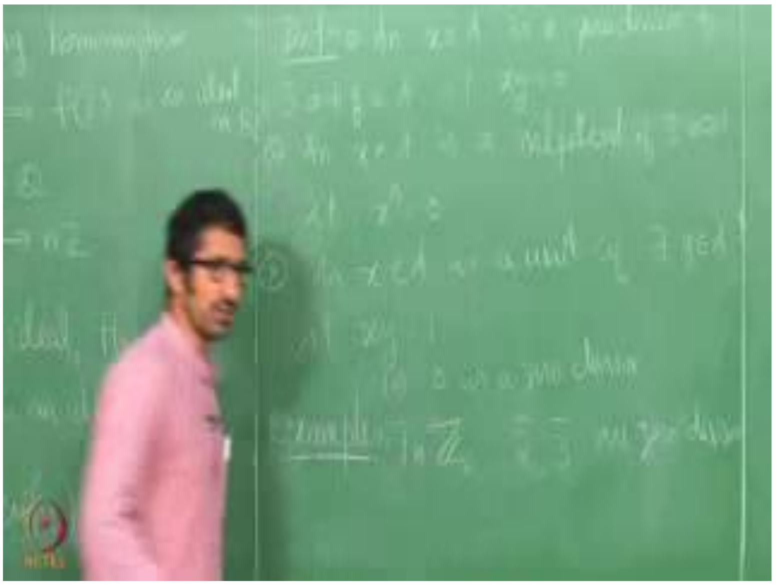

Now, let us a look at another set of definitions. An element x in a is a 0 divisor, if there exists a non 0 element B in A such that xy is 0. Or in other words x divides 0. So, there exists an element y with the property that xy is 0. So, let me just give 3 definitions. An element in an x in a is a nilpotent if there exist n bigger than to 1, such that x power n is 0. And an element x in a is a unit, if there exists y in a such that xy is 1; these 2 3 classes of elements. Well there are elements which are neither a non 0 divisor nor nilpotent nor a unit, but these are some important classes of a elements. So, some quick examples, have you seen examples of 0 divisors.

Student: Z 6.

In Z 6, what are the elements? 2 3 R 0 divisors. One more important example of a 0 divisor in any ring 0 itself. So, that that should be the zeroth example, 0 is a 0 divisor..

So, by convention we do not talk about 0 divisors in the non 0 in a 0 ring. In a 0 ring 0 is not really 0 divisor. Because this is not satisfied you know that does not exist a non 0 element. So, it cannot be 0 divisor.

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Some examples of nilpotent elements; can you give me an example of nilpotent element in some ring, something similar can you cook up an example of nilpotent element?

Student: Z mod 2.

In Z mod Z 4 and 2; 2 square is 4. So, this is in Z 4 this is nilpotent. Of course, you have seen unit is any you take any n non 0 element in a field it is a unit, non 0 elements of a field non 0 element of a field. Now do you see some relation between these 3?

Student: Nilpotent is always a 0 divisor.

Nilpotent is always a 0 divisor. So, is a 0 divisor, but the converse us not true right here, 2 and 3 both of them 3, 3 square is 3, 3 cube is 3 3 power 4 is 3. So, therefore, 3 power n is never 0. So, this is not a nilpotent and can you see some relation between 3 and these 2?

Student: One unit 10 inverse 0 by 0.

Sorry.

Student: 10 unit never be 10 unit 0.

So, if you take a collection of all 0 divisor. This is contained in this set this and this sets are?

Student: Disjoint.

Disjoint right collection of all unit is and collection of all 0 divisors are disjoint sets.