Basic Algebraic Geometry
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Module 4
Lecture 9
The Ring of Polynomial Functions on an Affine Variety

Okay, so we say last lecture we were looking at the notion of dimension, okay and we noticed that there is a notion of dimension from the topological point of view that is the that definition dimension of the topological space and then there is also the notion of Krull dimension of a ring, okay and somehow these two are related in the sense that the dimension of for example the dimension of affine n space the topological dimension of affine n space is same as the Krull dimension of the ring of functions on affine n space, okay. So let me recall again to refresh your memory and also so that you feel (())(2:00) comfortable about things, let me recall what was going on.

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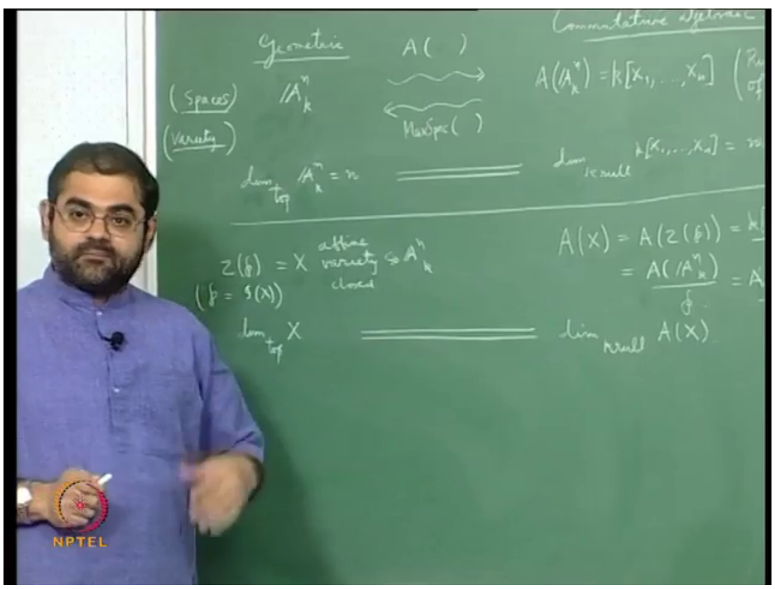

So see on the so we have as usual the geometric side and we have on this side the commutative algebraic side and here we if you take affine space over k affine n space over k this is just kn k cross k cross k n times given the Zariski topology where of course k is an algebraically closed field, okay as usual you can always think of it as complex numbers if you want to be very concrete so in that case this will be just Cn n copies of C given as the Zariski topology and then the ring of the functions so the association that I am interested in is called is given by the symbol A of variety or an algebraic set, okay and it is defined only for varieties properly this is well this should be thought of as giving you the ring of functions.

So you have the space here and here you have its ring of functions, so on this side you have the space and on this side you have rings of functions, okay. So if you take affine space of course what kind of functions are of course polynomial functions we are only interested in algebraic functions polynomial functions, so if you take this this is going to be this is k X1 etcetera Xn, okay and the fact is that you see what I want to say is that there is also an association that goes from this side to this side, okay which makes this again a bijective correspondence, okay.

So again you know this is another instance of you know the interaction between algebraic geometry and commutative algebra that there is always a dictionary which goes from the geometric side to the commutative algebraic side and goes back and forth. So see initially you know we had such a dictionary where we had taken on this side for example we have seen it in other context, for example on this side if you took you know closed subsets that is algebraic subsets, then what you can get on this side as a bijective correspondence are the radical ideals, okay.

And if you and under this correspondence the irreducible algebraic subsets irreducible closed subsets they will correspond to prime ideals, okay which are of course radical. And the of course the points the singleton sets consisting of points they will correspond to the not just ideals which are the maximal ideals largest proper ideals. So that is one correspondence we have already seen and in that correspondence what goes from this side to that side is given a set here you associate the ideal of that subset, okay and what goes from that to here as the inverse map is given an ideal you look at the 0 set of the ideal, okay and that gave a bijective you know correspondence.

Now rather than going from algebraic sets or varieties to ideals what you do is you go from algebraic sets or varieties on this side to rings of functions, okay. So that is the change in the point of view but the fact is that again you get a nice bijective correspondence and so let me explain, see what is happening is see if you why this is so important is because if I take dimension topological of An this is of course n, okay and that is just a translation of the fact that the dimension the Krull dimension of this polynomial ring is n.

So the topological dimension of a space here of course when I say space here actually what I am meaning is a variety here, okay a variety means an irreducible closed subset of An in particular An itself is an irreducible closed subset of itself, okay so this is the biggest possible variety in An. And if you take any variety here then you can associate to it its a ring of functions, okay and the dimension of the variety will be the same as the Krull dimension of its ring of functions, okay.

So what I want you to understand is that if I take X variety affine variety, okay then you know X is of the form Z of some prime ideal, okay because you know if a variety I mean if an algebraic subset is always given as 0 set of an ideal and if that algebraic subset is irreducible if and only if the radical of the given ideal is prime. So in particular and of course taking the 0 set of an ideal and the 0 set of its radical does not create any changes the 0 set of an ideal is same as the 0 set of its radical.

So any irreducible closed subset of affine space is always given by the 0 set of a prime ideal, okay and of course you know this prime ideal is unique, it is unique under that bijective correspondence between irreducible closed sub-varieties I mean irreducible closed subsets in An and prime ideals of the polynomial ring. So the fact is if you take a affine variety like this then what is it that you are going to get on this side, what is it that you are going to get on this side? You are going to get the space of the ring of functions algebraic functions on X, okay and the ring of algebraic functions on X will be X is of course X of p and in fact it is actually A of An by p it is just this is just k X1 etcetera Xn by p this is what it is and, so let me draw a line like this because this statement is for An now I am making a statement for any affine variety closed inside An, okay.

Then for any variety the ring of functions is defined like this and then you again see that the dimension the topological dimension of the variety these two are equal, here also the dimension the topological dimension of the variety will be the same as the Krull dimension of its ring of functions this will happen. So this is so I want that so this is a generalization of this this is generalization of this statement which is true for all affine space and I also want to tell you what is this arrow that is coming in the other direction, okay that arrow is actually max spec that is the arrow that is going on this direction.

So let me explain that, so let me explain let us keep this diagram as the basic diagram that we want to understand and let me expand upon this, okay. So the first thing that I want to explain is well just to recall how did you get the topological dimension is equal to the Krull dimension the topological dimension of An to be equal to the Krull dimension of the polynomial ring that is because the topological dimension is supposed to be the you take chains of strictly increasing chains of irreducible closed subsets, okay and then you take the largest possible chain and subtract 1 from that, that is the topological dimension if at all it is finite, okay you have to take if it is not finite then the largest possible will be infinite and it will become infinite dimension, okay.

So how does one define the dimension of topological space? You simply write out and you try to look at existence of a chain of irreducible closed subsets, okay irreducible closed subsets which are contained each one contained in the next try to look at such a chain and try to look at the chain being strict, namely that there are no repetitions in successive (())(13:02) of a chain and amongst such chain try to take a maximum the maximal lengths ones namely the once with as many terms as possible and you if this maximum is a finite number, okay you take away one from that, that will be the topological dimension, okay.

Now why does that translate it to this side into this side two give the Krull dimension that is because any closed subset here correspond to a radical ideal any irreducible closed subsets here corresponds to a prime ideal and if you give me a strictly increasing chain of irreducible closed subsets that will correspond to a strictly decreasing chain of prime ideals, okay. So what will happen is that the fact is that the largest I mean largest in the sense the chain a strictly increasing chain of prime ideals of maximal possible length, okay is n plus 1, okay.

And of course easiest thing is you take 0 start with the 0 ideal then take the ideal generated by one variable X1 then that is contained in the ideal generated by X1 and X2 and then at the ith stage is generated by X1, X2 etcetera up to Xi and you go on up to n this is a strictly chain of prime ideals its length is n plus 1 and therefore the Krull dimension becomes n, so the Krull dimension of a commutative ring is defined to be the supremum of the heights of the prime ideals, okay and what is the height of the prime ideals? The height of the prime ideal is the maximum possible length of the maximum possible chain that you can descending chain strictly descending chain of primes that you can get starting from that prime ideal and then you have to take away one from that, okay that is the reason in all these definitions you will see that instead of taking a chain from Z1 to Zm and taking the supremum and calling this and taking the dimension to be 1 less than that we rather start with 0 we started with the index 0, okay that is the reason you start with 0, okay.

Anyway so the fact is and Y is the Krull dimension of this equal to n because whenever you have a finitely generated whenever you have an algebra namely you have a quotient of the polynomial ring which is an integral domain, then the Krull dimension of that is actually the transcendence degree over k of its quotient field and the quotient field of this is simply the field of rational functions in variables, okay it is just consists of ratios of polynomials in n variables with the denominator polynomial not equal to 0 and that has transcendence degree n it has so transcendence degree is analogous is for algebraic independence the analog of dimension for linear independence just like for linear independence dimension is cardinality of a maximal linearly independent subset for algebraic independence transcendence is the cardinality of a maximal algebraically independent subset, okay so it is analogous to happens in linear algebra. So transcendence degree measures them largest it gives you the number of maximal I mean the maximal number of transcendental elements which are algebraically independent from one another, okay that is what it gets.

So just like dimension of vector space gives you the maximal number of linearly independent elements, okay linearly independent vectors and the transcendence degree of this the quotient field of this is n which is very clear inventively but you need to prove it, okay using something from you have to use some field theory to prove it, okay. And now you see so the first thing that I want to explain is why do we define the ring of functions on X like this, okay why do we define the ring of functions on X to be the ring of functions on the whole space divided by the ideal of X, so in fact this is I can simply write it as A of An of k divide by I of X, I of X is p where script I is the ideal of the given subset, okay so I of X is p, correct? So here also I can write that p is equal to I of X, okay.

And so in a way you know previously we were associating to X I of X which is a prime ideal, now what you are doing is you are not associating X to the prime ideal but you are associating it to the quotient by that prime ideal instead of looking at the ideals on this side you are looking at quotients by those ideals that is the only difference but looking the advantage of looking at quotients by those ideals is the fact that you get rings of functions and why is that that is what I am trying to explain it is very very easy see what you do is you see you take you look at A of X so this is you define it to be functions from so functions from X to k which are given by polynomials by polynomials.

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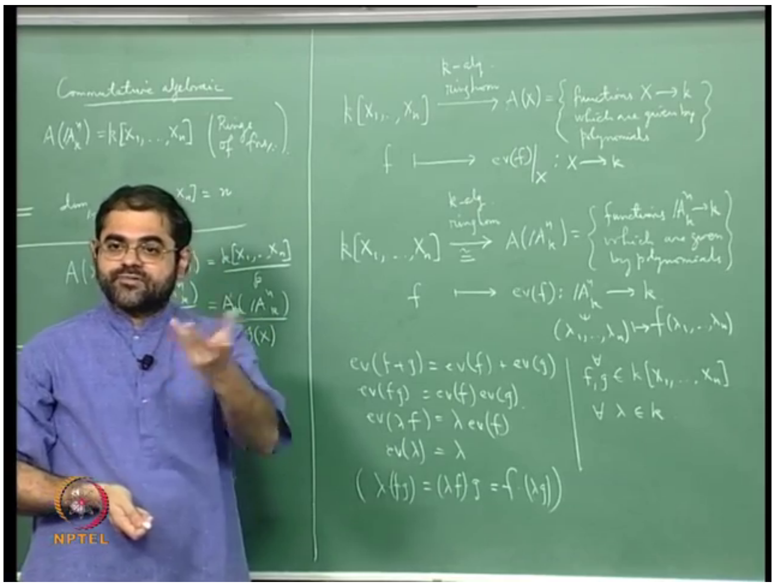

So look at this definition so A of X are just functions A maps from X to k and the point is that this map it should be an algebraic function, okay which means it should be a polynomial, okay and of course what polynomial? Polynomial in n variables because X is a subset of An and any polynomial in n variables is a function on An you can evaluate it at each point of An, okay. So actually what is happening is when you go from here to here there is something that is already happening.

So if you apply this condition you see if you apply this definition so if I write A of An this will be functions from An to k which are given by polynomials and you see if I take the ring of polynomial functions in n variables then I have a map like this that map is you start with a polynomial in n variables and you use that F to define a function think of see F is here now a polynomial define that as a if I use that F to define a function let me if I want to be very strict to distinguish the fact that here I am thinking of F as a polynomial and there instead of the polynomial I am looking at the function that it defines what I will do is I will put ev(F) this is evaluation of F from An to k is a function this is what is what F goes to, what does this do?

Give me any point in An you simply evaluate that point F at that point you evaluate the polynomial at that point this is the map. So you are just associating F to evaluation of F, okay. And what you must understand is you see if you take if you take functions like this which are given by polynomials then it is very clear that if you take two such functions there sum is also given by corresponding sums of polynomials, right?

So if a function is given by a polynomial then a sum of two such functions is also given by sum of the corresponding polynomials product of the two functions will be a function that is given by product of the corresponding polynomials and of course there are constant functions here which corresponds to the constant polynomial in other words what I am saying is that this map is actually a ring homomorphism I am saying this is a ring homomorphism this fellow on this side is actually a ring that is what you must understand, take it here also if you when I say a functions are given by polynomials if I have two functions that is given by polynomials then there sum is given by the sum of polynomials, okay.

And the product is given by product of polynomials, so these are all rings, okay they are commutative rings and they contain constants the constant functions are always there because constant functions come from evaluation of constant polynomials, so constant functions are there so this is these are k algebras a k algebra is nothing but a ring a commutative ring with one with k which is also a vector space over k, okay. And in such a way that the multiplication in the ring is compatible with the scalar multiplication lambda times Fg is same as F into lambda times g, okay.

So the vector space structure and the ring structure they are compatible that is what an algebra is, okay. So these two are both k algebras, okay so they are rings and the fact is here also I can start with take a polynomial in n variables I can have this map, what is this map? This is just F going to evaluation of F restricted to X from X to k is the same map but you restrict it to the subset X after all because X is a subset of An you can always restrict the map, that is what I mean to say that these are functions which are given by polynomials when you say they are functions given by polynomial they are actually given by restrictions of polynomials that is what this map is, okay.

And what you must again understand in this case also is the ring homomorphism because f plus g evaluation of f plus g will be evaluation of f plus evaluation of g evaluation of f into g will be evaluation of f into evaluation of g and evaluation of lambda will be just lambda if lambda is a constant polynomial, alright? So you can see that this is a this is not just the ring homomorphism it also preserves the vector space structure, mind you what is there on the left side is also a k algebra this is a commutative ring with 1 which is also vector space over k and the scalar multiplication is compatible with the ring multiplication if you take a scalar lambda and you if you take two ring elements two polynomials f and g then lambda times f g is a same as lambda f times g that is a same as f times lambda g it really does not matter to which factor of a product you multiply a constant you are going to get the same polynomial, right? In a product of polynomials.

So these are actually k algebras, so what is happening is that you are getting ring homomorphism which are actually homomorphism of k algebras these homomorphism also satisfy they also they are linear they also respect the vector space structure. So evaluation of f plus g is evaluation of f plus evaluation of g it respects addition it respects multiplication and it respects scalar multiplication, okay and evaluation of lambda is lambda this all this is true and this is for f g in polynomial ring and every lambda in the every constant polynomial, okay.

So this is so in fact instead of writing ring homomorphism let me write k algebra ring homomorphism k algebra means it is a ring homomorphism which is also k vector space map and the of course the algebra conditions is on the rings that the scalar multiplication and the ring multiplication are compatible that is you know you also have lambda times f g is equal to lambda f times g is f times lambda g this is the algebra condition that multiplication by a scalar is well behaved with respect to multiplication between two ring elements not of its vectors, okay.

Fine, so now what I want you to understand is that this map is an isomorphism this map is actually an isomorphism because you see if first of all if you have two polynomials if so you know to it is ring homomorphism to show that it is an isomorphism I have to show its surjective I have to show its injective to show its injective I have to show its (())(28:25) if you want because it is a ring homomorphism it is surjective by defining because my definition of ring of functions is that they are given by polynomial so it is automatically surjective both of these maps are surjective because by definition a function is supposed to be coming from a polynomial so it is so both maps are surjective mind it, the only thing you will have to worry about is injectivity, okay.

So if a function f is such that evaluation f is 0, okay for every point of An then that function has to be the 0 polynomial that is because if it is not the 0 polynomial, okay then you should be able to find some point where if you evaluate it you will get a non-zero element of k, okay so that tells you that this is injective and it is already surjective so it is an isomorphism and this is what tells you that this definition is correct that to think of the ring of functions on An as the polynomial ring is correct because of this argument.

Now in the same way if you look at this argument what you will get is you will get this actually, okay what will happen is that this ring homomorphism is surjective what is the kernel of this ring homomorphism, this ring homomorphism is not injective it has a kernel, what is a kernel? It is all those functions whose evaluations when restricted to X becomes 0 but that is precisely all those functions which vanish on X but what are the functions which vanish on X they are precisely the functions in this they are the precisely the polynomials in the ideals of X therefore the kernel of this map is actually the kernel of this map is actually IX and is surjective, therefore source modulo the kernel is isomorphic to the target so you get that k X1 etcetera Xn modulo IX is isomorphic to Ax by this map, okay.

So these two statements justify that these definitions are correct, okay so they give you the naturalness you know behind these two definitions, okay.

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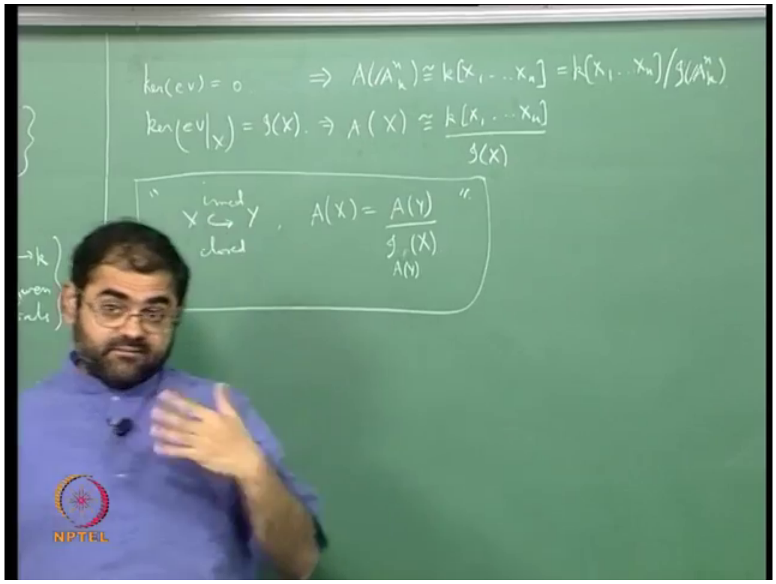

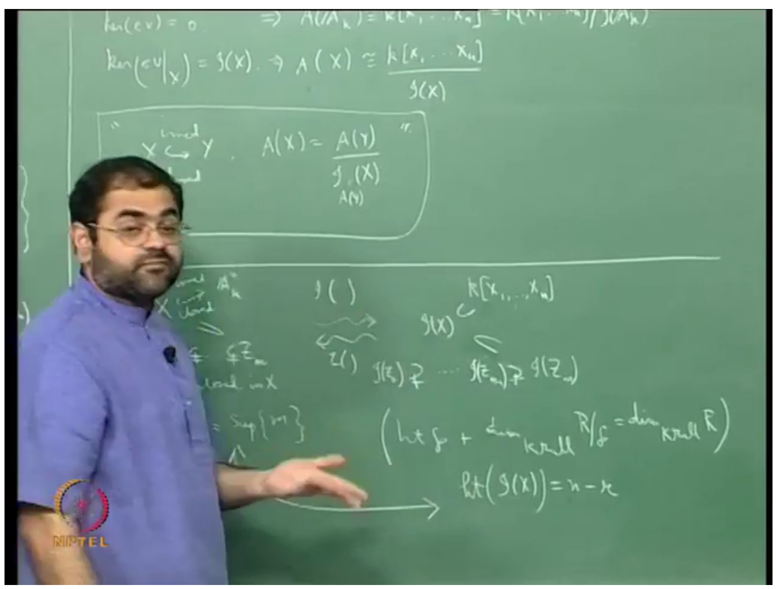

So let me write that down let me write here kernel of evaluation is 0 this is so you know this ring homomorphism is just ev because it is f going to ev and this ring homomorphism is ev restricted to X you evaluate and then restrict it to X or you do the evaluation on X. So kernel of evaluation is 0 kernel of evaluation restricted to X is IX and so it tells you so this tells you that A of An is isomorphic to k X1 etcetera Xn and A of X is isomorphic to k X1 etcetera Xn modulo ideal of X, okay.

So what you are doing is that see here by brute force we write like this that is we (())(32:18) took think of the functions as I have to think of the polynomials as functions, okay but if you want to be more strict you should do it like this there is a stricter way to do it, okay. So in which you distinguish between the polynomial and the function that it defines by evaluation, okay so when you make this distinction you get these isomorphisms not equal to you get isomorphisms in a very nice way and why is that nice because this can also be written mind you this is also equal to k of X1 etcetera Xn modulo ideal of An because ideal of An is 0, right?

So what it tells you is that this formula AX is k X1 etcetera Xn by IX works even for X equal to An, okay. So the so you get this you know you get this kind of you get a philosophy that if X is embedded in as an irreducible closed subset of a bigger space, okay this is irreducible closed in a bigger space then A of X has to be A of Y divided by the ideal of X in Y in A of Y you get something like this, okay. So what this tells you is that it tells you that you know the notion of so the point that you must appreciate here is that this quotient ring homomorphism this is a quotient homomorphism because it surjective homomorphism is always a quotient, okay because a target is a quotient of a source by the kernel so it is a quotient and the quotient is actually taught of as restriction of functions, okay.

So the moral of the story is that the in whenever you see a quotient in commutative algebra algebraic geometrically it means that you are restricting functions to a subset closed subsets, okay that is the geometric meaning of what a quotient is that is what this is, okay so the quotient homomorphism is just gotten by restricting functions literally, right? So that tells you why you know it is correct to define the ring of functions for a affine variety like this, okay of course we will really not worry about all the time writing isomorphic to isomorphic to, okay so by abusive of notation we will simply put equal to, okay but what that equality is to be very precise it is an isomorphism because when you simply write the write the ring they are actually polynomials but you want to think of them as functions you must evaluate those functions so there is a difference and it is this isomorphism is to signify that difference.

But then one forgets this and by abusive of notation keeps writing equal to, okay. Fine, so this much is clear then I want to explain why this and this are one and the same why are this and this one and the same, see that is quite that is again quite clear because if you look at the way it goes you see what is see if you take X, okay and you in X if you give me a sequence of irreducible closed subsets, okay so these are all irreducible closed in X, okay then you see what this will correspond to is see mind you X is sitting inside X itself is an irreducible closed inside An X itself is an irreducible closed inside An, okay. So if you go to the if you go to the polynomial ring the ring of functions on An which is k X1 etcetera Xn, okay then you see this X as a closed subset of An on this side will correspond to the ideal of X which is prime, okay mind you that there is an inclusion like this because these are all closed in X and that is for the closed irreducible close in An, okay.

So what will happen is you will get something like this you will get ideal of Zm so I should go the other way so ideal of Z not will contain so it will be like this ideal of so let me write it proper containment ideal of Zm minus 1 and so on proper containment ideal of Z not. So this diagram on this side translates to this diagram on the other side, okay and of course you know why this because we have a on this side to that side if you take the if you apply this I you know that is a bijection because its inverse is given by Z you know this bijection, okay this is a bijection between radical ideals and closed subsets and if you are only looking at irreducible closed subsets then it is a bijection between irreducible closed subsets and prime ideals, okay.

And of course it is inclusion reversing, alright? So see what this will tell you is that in the earlier lecture I define what is meant by height by height of a prime ideal, what is a height of a prime ideal? You start from that prime ideal and then you take a descending sequence of strictly descending sequence of prime ideals, okay and take the maximal possible length that is the height of the prime ideal, okay. So you know if I start with IX and if I take a strictly descending sequence of prime ideals starting from IX and going down all the way to the smallest prime ideal which is 0 mind you 0 is a prime ideal therefore because these are all integral domains 0 is a prime ideal polynomial ring is an integral domain.

So then what this will tell you is the height of that plus this m, okay and if I take the see here if I take m to be maximum if I take maximum such m I am going to get dimension of X, okay dimension of X is actually supremum for such m, okay and suppose as in this case if what this will tell you is that if the dimension of X is say r, okay then you have a maximal change here which is length r, okay that will correspond to that will tell you that this has length r, okay I mean this will be r, okay plus you know if I further continue because of maximality this will be if I further continue it to go all the way down to 0 which will give me the height then if I add both I should get the dimension of the polynomial ring, okay.

If I take this, okay with n equal to r which is the maximal possible and then if I further go down, okay from I of X to 0 that part will give me the height that plus this r will give me the dimension of the polynomial ring this is just reflection of this fact that you know height of a prime ideal plus dimension Krull dimension of r mod p is equal to Krull dimension of r, okay I wrote down a formula like this last time that is if r is an integral domain which is finitely generated as a k algebra then and p is a prime ideal of r then the height of the prime ideal plus the Krull dimension of the quotient is equal to the Krull dimension of the whole ring that is the just the reflection of this fact that I explained just now.

So moral of the story is that the dimension of X will be actually if the dimension of X is r then the height of IX will be n minus r, so the dimension of X is r here this will correspond to height of IX is equal to n minus 1 because r plus n minus r adds up to n which is the dimension the Krull dimension of the whole polynomial ring. So the moral of the story is the but then what is n minus r? n minus r is the dimension of the quotient but what is the quotient? The quotient is the ring of functions that is why the dimension of the topological dimension of X here corresponds to the Krull dimension of the ring of functions, okay.

That explains why these two are equal, okay you understand that so I am just trying to give an explanation why these two are one and the same, right? So I will stop here and then we will continue in the next talk.