Basic Algebraic Geometry
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Module 4
Lecture 8
Topological Dimension, Krull Dimension and Heights of Prime ideals

Okay, so what I am going to now do is to tell you about another important property that comes because of the Noetherianness, okay. So what I just explained in the previous lecture was that the Noetherianness of a topological space allows every non-empty close subset to have Noetherian decomposing that is it can be broken down into a finite union of irreducible close subsets non-empty irreducible closed subsets and if you assume that these union is not redundant it is no subset in the union is contained in some other subset in the union then this union this decomposing is unique that is the members of the union or unique, okay.

So and that is the reason why any algebraic set any closed subset of affine space for Zariski topology can be written as a finite union of affine verities. So this is one of the reasons why we call affine verities as building blocks of algebraic sets and it is general philosophy that affines are building blocks of algebraic structures in algebraic geometry. And now I am going to give you another reason for the importance of Noetherianness, okay. So that is got to do with dimension, okay so the you know the aim is somehow to try to tell you the obvious thing that you will aspect that you know the dimension of affine n space is m, okay that is what I will try to explain but.

So what I want you to begin with not get confused with is a following, see if you take affine space which is An it is just as a set it is kn, okay it is n copies of k where k is an algebraically closed field, okay and if you take kn as a vector space over k then it is very clear that it is n dimensional because it is a finite dimensional vector space and you know n copies of k as a of a field k will be dimension n as a vector space over that field, okay because it can, for example you can always write down the standard basis, okay which consist of 1 in the ith place and 0 (()) (4:05), right?

So but therefore you know if you think that I am trying to say that the dimension of kn over k as a vector space is n then you are mistaken because that is not what I am trying to say I am trying to I am trying to define dimension in completely different way I do not want to think of kn especially An affine n space I do want to think of it as a vector space, okay we are not here worried about the vector space properties, okay we are worried about the topological properties.

So the therefore what this calls for is how to define dimension of a topological space, so the answer to that is the dimension can be defined by taking the largest possible namely the supremum of lengths of strictly you know decreasing chains of closed subsets, okay and in a analogy you can compare it to the vector space situation, okay. So you know you take any finite dimensional vector space over a field there if you look at subspaces of the vector space then you know if you the if the vector space of dimension n then the largest strictly increasing or decreasing whichever way you want to see it chain of proper subspaces will be of length n plus 1 because it will start from 0 if it is increasing it will start from the 0 subspace and it will end with the full vector space and it each stage you will get a bigger subspace with dimension one more, okay.

So you start with 0 and you end with n so you will exactly get a chain of length n plus 1 of strictly increasing subspaces and that is the maximum possible, okay. So in the same way the analogy is that for a topological space you can define the dimension to be the supremum of the lengths of you know a strictly increasing chain of irreducible closed subsets, okay and this leads to a very good definition because it has got to do in the case of affine space it has got to do with competitive algebra and namely with the polynomial ring.

So in what way I will explain now so let me make this definition so what is so the aim is so my so the aim of this lecture at least the beginning is to show that the Noetherian condition is very helpful to define and show that the dimension topological dimension of affine n space is actually n, okay and it will involve commutative algebra as you will see.

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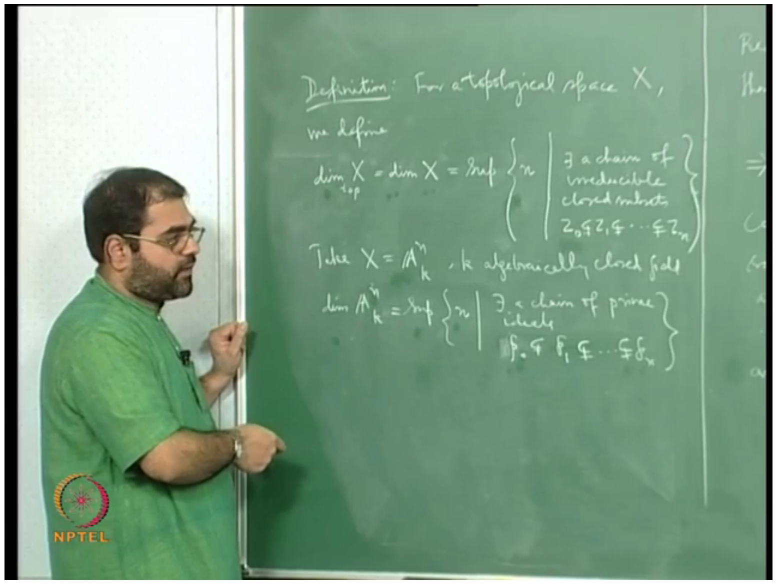

So here is my definition for a topological space X we define dimension of X, okay so this is called the topological dimension, okay some people write dimension with a subscript top, okay and sometimes we might just omit it, okay.

So whenever I write dimension of a topological space it is always topological dimension what is it this is defined as a supremum of all n set of all n such that there exist a chain of irreducible closed subsets Z not properly contained in Z1 properly contained in Z2 and so on to Zn. So here is the definition of what dimension of a topological space means, okay. So you and mind you I am starting with Z not, alright? And I assume that just to check that I am on the right track I need to also say that all of perhaps I have to put the condition I do not have to, okay so Z not is already non-empty because see I was just worrying whether I have to put the condition Z not as non-empty but then I am saying they are all irreducible closed sets and therefore they are non-empty, okay but you I am starting with 0 this is very very important staring with 0, okay.

So now you see so you know what I want you to understand is that this is let us examine this situation when X is you know affine space, okay so put take X equal to An Ank k algebraically closed field so you take k to be an algebraically closed field and look at affine n space over k, okay. Then you know that for Zariski topology the closed sets here they correspond to the irreducible close subsets here they correspond to prime ideals in the polynomial ring in n variables which is thought of as a ring of functions on the affine space.

So what will this definition translate to here? It will translate to the following dimension of Ank is equal to the topological dimension is the supremum of all the n such that there exist a chain of prime ideals p not so let me script p p not properly containing p1 I think I will have to rather I have to number it the other way round p0 p1 pn, okay. So by this definition this is what you get, okay you have a chain of prime ideals, okay and its starts with p not and goes up to pn and you take the supremum over all n.

Now the fact is that so since you are looking at chains of prime ideals strictly increasing chains of prime ideals what this relates to in commutative algebra is called as height of a prime ideal, okay.

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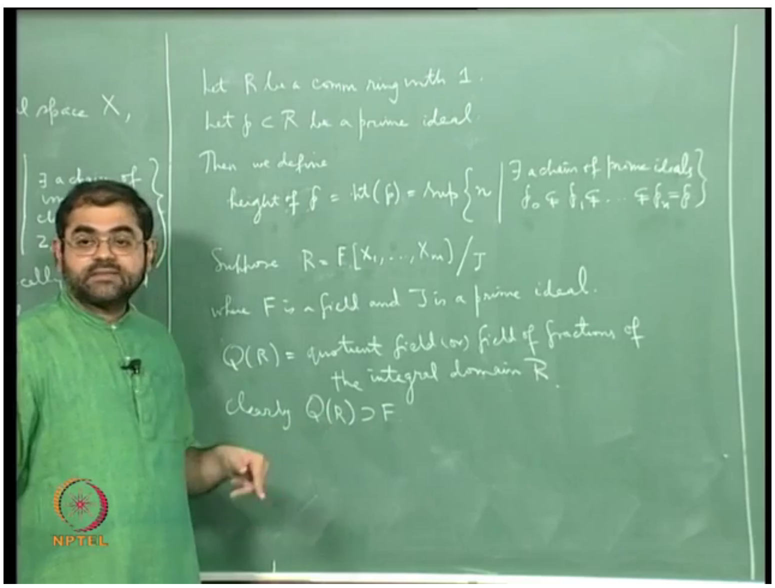

So let me recall what that is, okay so what is a commutative algebra involved let me explain that to you so what we do is let R be here commutative ring with 1, okay let p in R be a prime ideal then what you do is then we define height of p, okay height of p written as htp to be the supremum of all n's such that there exists a chain of prime ideals p not properly containing p not contained in p1 contained in and so on up to pn, okay we define the height of a prime ideal like this, alright? And with of course with pn equal to p yeah, right? So in other words you look at a chain of prime ideals which ends with p and you look at the largest possible set chain of course they it need not be finite at all, okay you may always find a chain like this for every n it might happen.

So but if it does not happen like that then you take the supremum and that will be the finite number and call that the height of the prime ideal, okay. And see here comes the following fact suppose R equal to k of so let me write this f of X1 etcetera up to Xm modulo J where F is a field and J is a prime ideal suppose R is a of this form I our question is whether this is an ascending chain or whether it is a descending chain, actually the truth is that if you put it either way anyway I am going to get the same number, okay whether I put it as if I start with Z0 and go up to Zn and it is a n ascending chain or if I start with Z0 and Zn it is an descending chain anyway it is that n that I am worried about supremum of that n that I am worried about.

So it really would not matter if I put you know the inclusion this way or the other way, okay but there is an issue when you come to the ring and I will explain that now, okay. So you see we are assuming R to be of this form that it is a polynomial ring over a field modulo or prime ideal, okay. That means that R is an integral domain which implies that R is an integral domain, okay then so now what you can do is you can look at Q of R this is the quotient field of R or it is otherwise called as field of fractions of the integral domain R this is the just the field of fractions of the integral domain now, okay mind you this is a ring and you are going modulo of the prime ideal ring modulo prime ideal is a domain, okay and if you have an integral domain you can form the field of fractions just like you form rational numbers the field of rational numbers from ring of integers which is an integral domain.

So you take the field of fractions and then what you can do is that you can look at you can see that this will contain F, okay clearly QR contains F because you say f is anyway contain in the polynomial ring as constant polynomials, okay and you are going modulo of prime ideal, okay. So in particular you are not going modulo everything, okay. So the fact is that you are certainly not since this is proper ideal you are not certainly going modulo the elements of the field, okay.

So the elements of the field still remain invertible in the field fractions of R, okay you can see this for example in commutative algebra either by looking at the universal property of a quotient field or you can use the universal property of localization, okay. For Q of R is actually the localization of R at 0 the prime ideal which means you invert everything outside 0, okay you localization with respect to the multiplicative set which is the compliment of 0, okay.

So now the point is when you so in other words what you have now is you have an extension of this field, you have the field F and you have this field extension now once you have an extension of the field in field theory you can talk about things about talk about many things about the extension first of all you can ask whether it is algebraic, if it is not algebraic you can check if it is transcendental and if it is transcendental then you can define what is meant by transcendence degree, okay. So let me quickly recall if you have a smaller field and you have a bigger field then we say that the bigger field is algebraic over the smaller field if every element in the bigger field is obtained as a 0 of a polynomial coefficients in the smaller field, okay.

And if there is an element which is not the 0 of any polynomial in the smaller field then that element is called a transcendental element, for example if you take real numbers over rational numbers, okay then the number e which is the used in defining the exponential function or the number pi which is used in trigonometry, okay they are all transcendental though of course the proves of these facts are not so easy E and pi are transcendental numbers and they are transcendental number because you cannot find them as roots of an equation in one variable polynomial equation in one variable with rational coefficients which is same as looking at with integer coefficients, okay because you always clear denominators.

So the moral of the story is that you do have fields which have a transcendental elements, so R the field of real numbers is transcendental as a field extension over the field of rational numbers and once you have a transcendental elements what you can do is you can actually define you know what is called as a transcendental version of dimension, okay. So what you can do is you can mimic what you do for a vector space situation, see in a vector space situation what you do is how do you define the dimension of vector space? The dimension of the vector space is defined as the maximal number of linearly independent vectors, okay.

So in other words what you do is you take the maximal subset of vectors which are linearly independent and take its cardinality and call that cardinality as the dimension of your vector space. So the dimension of the vector space is just the cardinality of a maximal set of linearly independent vectors, okay. Now you just mimic this in algebra and what you do is instead of linear independence which is used in the situation of vector spaces you now use algebraic independence which is the analog that you use in algebra in ring theory, okay.

So what you do is if you have a bigger filed containing a smaller field, okay and suppose the bigger field has some elements which are transcendental over the smaller field namely if it has elements which are not 0's of polynomials with coefficients of any polynomials with coefficients polynomials in one variable with coefficients in this smaller field then you can start looking at a you can start looking at a collection of transcendental elements, okay but put the condition that also put the condition that this collection of transcendental elements is algebraically independent, okay.

So you know a collection of elements finitely many elements in a bigger field is said to be algebraically independent over the smaller field if these elements, okay they do not satisfy a polynomial in several in as many variables with coefficients in the smaller field, okay. So please try to understand when you do it for a vector space you will say that a bunch of vector finitely many vectors are linearly independent if they do not satisfy a linear polynomial in as many variables with coefficients in the base field.

Now what you are doing is instead of requiring a linear polynomial in so many variables, okay as many variables as the number elements you are looking at you are only saying that now you also assume that you cannot find a polynomial relation you are only saying so let me repeat that if you have finitely many elements of a bigger field we say the finitely many elements of the bigger field is they are algebraically independent if they cannot if they are not they do not have any polynomial relation between them with coefficients in this smaller field, in other words they are not 0 of a polynomial in as many variables with coefficients in this smaller filed, such a subset of elements is called an algebraically independent subset of elements, okay.

Now what you do is just like in the vector space situation you took a maximal linearly independent set and took its cardinality and called it the dimension you do the same thing here what you do, you do the analogous thing here what you do is you take a maximal set of transcendental elements which are algebraically independent, okay take a maximal set of algebraically independent elements and take its cardinality and call that as the transcendence degree of the bigger field over the small field, okay.

So the transcendence degree of the bigger field over the small field is a cardinality of the maximal number is the cardinality of a maximal set of algebraically independent elements just like in the case of vector space the dimension of vector space over the base field is the cardinality of a maximal linearly independent set of vectors. The same way the transcendence degree of a bigger field over a small field is the cardinality of an maximal algebraically independent subset of elements of the bigger field which are algebraically independent over the smaller field, okay that is called transcendence degree just mimics what we did for dimension in the linear case, okay the vector space case.

So the beautiful theorem is that if you calculate the transcendence degree of Q of R over F that turns out to be what is the Krull dimension of R, okay and the Krull dimension of R is supposed to be the supremum of the heights of its prime ideals, okay.

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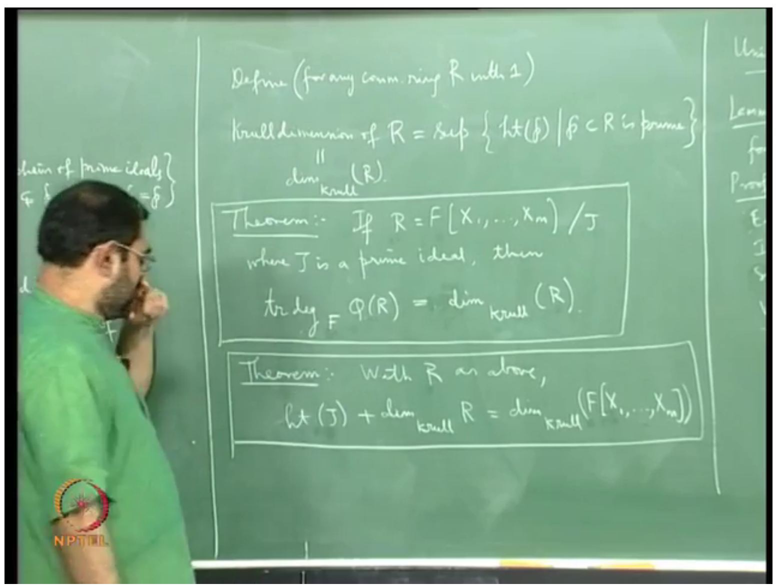

So let me write that so here so maybe so let me do so quite a few things that I have mentioned but you can kind of try to at least understand them (())(27:02) now and then do further reading so define for any commutative ring R with 1 Krull dimension of R to be equal to the supremum of height of p but p in R is prime call this the Krull dimension and the notation for that is dimension Krull R, okay then here is the theorem, so it is a theorem from commutative algebra and field theory which says the following if R is equal to F of X1 etcetera up to Xm modulo J where J is a prime ideal then transcendence degree of quotient field of R over F is equal to the Krull dimension of R.

So please understand this theorem so it is a very basic and of course a very important result what it does is it tells you what the Krull dimension of an integral domain which is a finitely generated algebra over a field measures it actually measures the number of algebraically independent elements in the fraction field of the integral domain over the base field, okay. So this is a so you know in some sense this side is analogous to what you do in linear algebra when you have a vector space over a field then the dimension of the vector space over the field is the cardinality of a maximal linearly independent subset of vectors which linearly independent over the base field in the same way when you have a field extension of a field then the transcendence degree of the field extension over the smaller field is the cardinality of a maximal set of linear of algebraically independent elements here over which are algebraically independent over the base field, okay.

And in the case of linear independence the condition is that those elements those finitely many elements do not satisfy a linear polynomial in as many variables with coefficients in the base field and in the case of algebraic independence the condition is that those finitely many elements do not satisfy a polynomial of higher degree in as many variables with coefficients in the base field that is the analogy, okay and that these two are equal is the theorem is the theorem from commutative algebra, okay I need also another theorem I think this stating this theorem will in retrospect check whether I have modeled with the inclusions in this definition or in this definition, okay.

So what is the theorem? So what this theorem says is that this theorem actually you know connects the height of the ideal with the dimension of the quotient, okay. So the theorem is so with R as above that is R is of polynomial ring in finitely many variables over field modulo prime ideal, okay height of J plus dimension Krull dimension of R is equal to the Krull dimension of this, yeah so what I want to basically say is that if you are looking at J equal to 0, okay if you are looking at J equal to 0 and you look at the height of the 0 prime ideal then the height of the 0 prime ideal is just 0, okay because I can start with 0 and that is it I cannot make it larger, so the height of the 0 prime is just 0, okay plus the Krull dimension of R will just give me again the Krull dimension of R because in this case if I put J equal to 0 then R is actually F of X1 etcetera up to Xm.

And the Krull dimension so of course you may put J equal to 0 I will get I do not get anything but the point is that the Krull dimension of this is actually n because the Krull dimension of a ring is actually the transcendence degree of the quotient field of that ring over the quotient field of that integral domain over the base field. So if you take a polynomial ring in m variables and look at its quotient field you will get the field of quotients in n variables, okay. So the quotient field of this will be F round bracket X1 through X this is the set of all quotients of polynomials in m variables with of course the denominator being non-zero and as you can easily see you have to check that this if you take the quotient field of this the number of linearly I mean algebraically independent variables will be m it will be these X1 through Xm, okay you can cannot have more than m algebraically independent elements over F, okay.

So what this will tell you is probably I do not need this now maybe I will need it later when I look at general case of an affine variety but for the moment what you get immediately from all this is that the dimension of the your affine space, okay the dimension of your affine space will be by definition it be the supremum of all these things, okay and you can see that this is the same as the Krull dimension of the ring of polynomial functions on affine space which is equal to n, okay.

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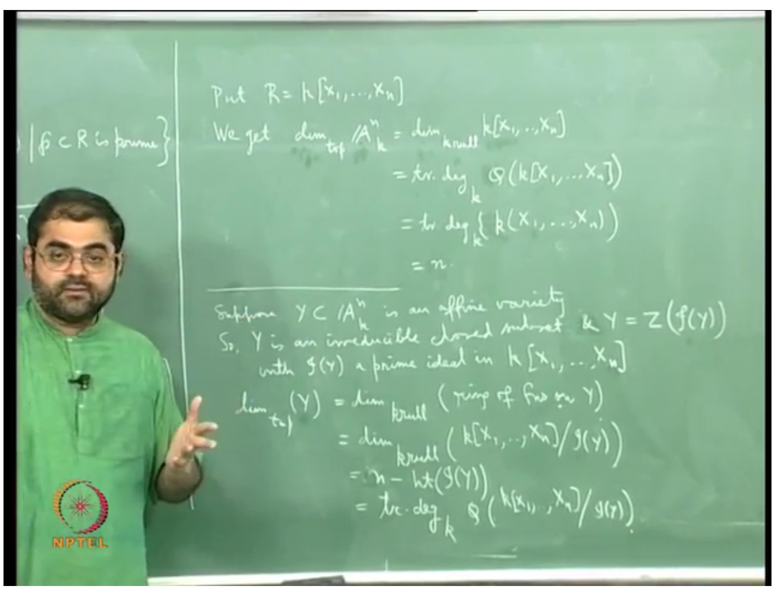

So what I want you to understand probably I do not need this now this so maybe I will do the following thing at the moment let me so let me so let us not worry about this, okay let us not worry about this this is not immediately relevant for the discussion but we are this is what is important, put R equal to k X1 etcetera up to Xn we get dimension topological dimension of An is equal to dimension Krull of k X1 etcetera Xn which is by defining equal to I mean which is by this theorem equal to is the transcendence degree over k of the quotient field of k X1 etcetera Xn and that is of course equal to transcendence degree over k of Q of X1 etcetera Xn this is the notation for the quotient field of, sorry k round bracket X1 through Xn is the quotient field of k square bracket X1 through Xn and k round bracket X1 through Xn consist of ratios of polynomials from this ring polynomials in this n variables with of course the denominator polynomial being non-zero, okay.

And that is this is equal to n transcendence degree of this is n, okay that is again a fact that we will accept from field theory from commutative algebra that polynomial ring what you have done to the polynomial ring is what you have done in constructing this polynomial ring is that you started with the field k and you added n in determinants and these n elements are algebraically independent by definition because any try to understand that if you look at X1 through Xn they are elements of this ring and this ring sits inside its quotient field say they are also elements of the of this quotient you can think of each Xi as Xi by 1 divided by 1 just as you think of an integer as a rational number given by the integer divided by 1, okay.

And therefore these Xi's are all elements here, okay and the fact that they are all algebraically independent is the fact that if you write a polynomial in the Xi's with coefficients from k and if it is equated to 0 then all the coefficients have to be 0 that is what it means to say that the Xi's are you know indeterminates the fact that Xi's are indeterminants says that they are transcendental over k and any polynomial relation amongst them is 0 if and only if all the coefficients are 0, okay that is in other words they are all algebraically independent. So it is very clear that X1 though Xn are algebraically independent and therefore the transcendence degree has to be at least n and then if you do some field theory you can check that the transcendence degree is exactly n.

So by this definition you will get that the dimension of An the topological dimension of An is the same as Krull dimension of this ring and that is equal to n, okay. Now I will let me come back to let me come back to this this statement here and that is got to do with trying to do it to all these dimension count even for an affine variety, okay. So let me do this for any affine variety and use this, so you see suppose y inside An is an affine variety suppose Yn says An is an affine variety, okay so a Y is an irreducible closed subset, okay and of course Y is well 0 set of the ideal of the Y with of course ideal of Y a prime ideal in the polynomial ring Xn, okay.

Now you see so what I want to say is that a statement similar to this also can be made what for affine varieties, so what is happening here is the topological dimension of affine space is the Krull dimension of the ring of functions on affine space see that is what the first statement says the topological dimension of affine space with Zariski topology is the Krull dimension of the ring of functions on the affine space and the ring and the Krull dimension of the ring of functions on the affine space is the transcendence degree of its quotient field over the base field, okay which is essentially this theorem, okay.

A similar statement holds for any irreducible closed subset, so what will be what you aspect the theorem to be the theorem the fact will be that if you take the topological dimension of Y, okay notice that y is a subset of affine space and affine has Zariski topology so Y has also the induced topology and in fact Y is itself a close subset so therefore any close subset the close subsets of Y are precisely the closed subsets of affine space which are contained in Y there is no difference, okay.

And so Yi if you take the Zariski topology induced on Y and you look at the topological dimension of that that will turn out to be equal to the dimension the Krull dimension of the ring of functions on Y, see look at this statement the topological dimension of the space is the Krull dimension of the ring of functions on this space and the ring of functions on this space is the all the polynomials, so here I should write ring of functions on Y, okay but is a ring of functions on Y? The ring of functions on Y to get the ring of functions on Y you will of course all polynomials which are functions on the whole affine space are also going to be functions on Y because after all you take a polynomial I can evaluate it on affine space I can also avail evaluate it on a subset of the affine space so I can take all the polynomials and restrict it to Y.

But the point is two such polynomials two different polynomials there may still define the same function on Y that is because there difference maybe a function which vanishes on Y what this tells you is that the functions on Y are the same as the functions here on affine space up to translation by elements of the ideal (())(42:59). In other words what you are doing is you are looking at co-sets of iY in the polynomial ring which means you are actually looking at the quotient ring.

So the moral of the story is that the ring of functions on Y has to be the ring of functions on the bigger space which is affine space modulo the ideal of Y, okay so this is the Krull dimension of the ring of functions on the affine space which is the bigger space k X1 etcetera Xn modulo the ideal of Y, okay notice that the ideal of Y is prime, okay and the ideal of Y is prime and therefore this quotient is an integral domain, okay and we are in this situation of this theorem you are having a polynomial ring over a field you are going modulo over prime ideal and then the Krull dimension is actually the transcendence degree of the quotient field of the integral domain over (())(44:07) field.

So if you use that theorem you will get that this is transcendence degree so this is by definition transcendence degree, yeah but before I do that let me use this theorem, okay let me use this theorem, alright? So what this theorem will tell you is that it is the Krull dimension of R is the Krull dimension of the bigger ring minus the height of the ideal by which you are going to get R. So this will tell you that this will be just n minus height of the ideal of Y and of course this will be equal to transcendence degree of the over k of the quotient field of this quotient ring k X1 which is an integral domain, okay.

So the moral of the story is that you can calculate dimensions for you have a formula for dimensions for affine close affine sub verities of affine space, okay.

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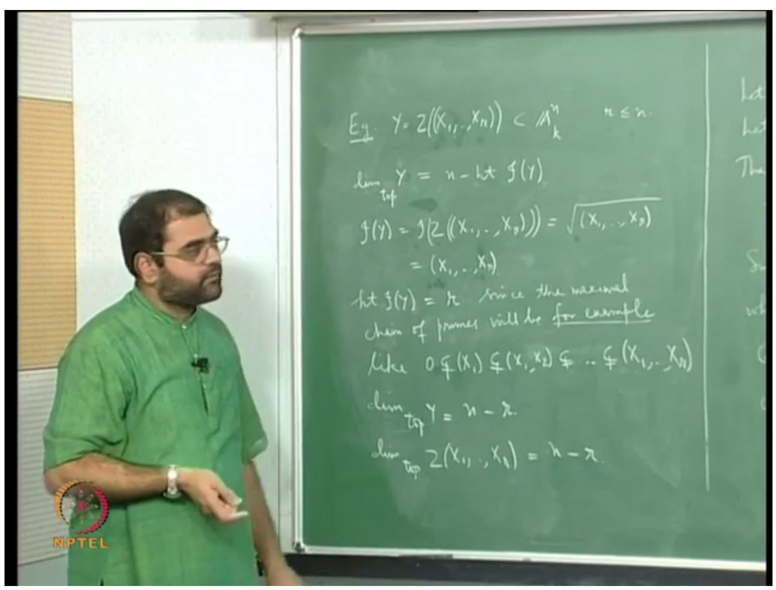

So as an example you can look at Y equal to say Z of X1, okay more generally suppose I take the ideal generated by X1 through Xr, okay where inside affine space where r is less than or equal to n, okay then you can see that so then dimension of Y as a topological space is by definition going to be n minus height ideal of Y, okay and the fact is that the ideal see the ideal of Y is just ideal of Z of X1 etcetera up to Xr, okay and you know if you take I of Z of some ideal you get its radical we just I know one of the important consequences in Nullstellensatz.

So what you will get here is the radical of X1 through Xr and you can check that it is its own radical that is because it is prime, why is it prime? Because if you take the polynomial ring modulo this ideal you will get the polynomial ring in the other variables that is a very easy check and since when you go modulo this you get a polynomial ring in some variables which is an integral domain this has to be prime and since this is prime it is already radical, okay.

So you will get this and therefore and you know if you take height of I(Y) you will get r, okay because you are going to look at a maximal chain like this which starts with something smaller and goes up to the ideal and you know you can see that since the maximal chain of primes will be for example like 0 properly contained in X1 properly contained X1 comma X2 properly contained in and so on X1 to Xr, okay so you will see that the height is r, okay and but I am saying for example because one has to prove it, okay the fact that you have a chain like this tells you that the height of this ideal is at least r, okay because height is supposed to be supremum the fact is that you cannot get a chain of bigger length that is the fact that needs to be proved, okay.

But if you assume that you can if you believe that then height of I(Y) is r and you will get dimension of the topological dimension of Y is equal to n minus r, okay and this is so in other words what you are saying is the topological dimension of the 0 set of X1 through Xr is n minus r of course you know if I put n equal to r then X1 through Xn will be a maximal ideal that will correspond to the origin so the 0 set will be a single point and the dimension will become 0 n minus r the point will have 0 dimension, okay.

So this is a so what this demonstrates to you is that you get the most natural thing namely if you go if you take if you cut down by r equations then your dimension also cuts down by that many equations roughly this is what a (())(50:20) but then it is technical to check that you know this the height of this is exactly r, okay but what I am trying to demonstrate to you is so of course here I have used this result I mean this allows you to do dimension calculations for sub varieties of affine space closed sub varieties of affine space and here is a standard example.

So the when you take Z of X1 through Xr you are taking the locus given by the intersection of all the hyper surfaces when you take 0 set of any X1 you are looking at the equation X1 equal to 0 that is called the hyper surface because it is cutting by 1 equation and now what you are doing is now you are successively cutting by r of these equations and obviously you should expect the dimensions should also go down by r and that is exactly what this computation says.

And the fact is that this is exactly what you would expect this is exactly what happens but the commutative algebra that intervenes is locked in these two theorems and that involves the definition of transcendence degree, it involves definition of Krull dimension and which in turn depends on dimension I mean the definition of height of a prime ideal, okay. So these are the I mean this is the commutative algebra that comes in.

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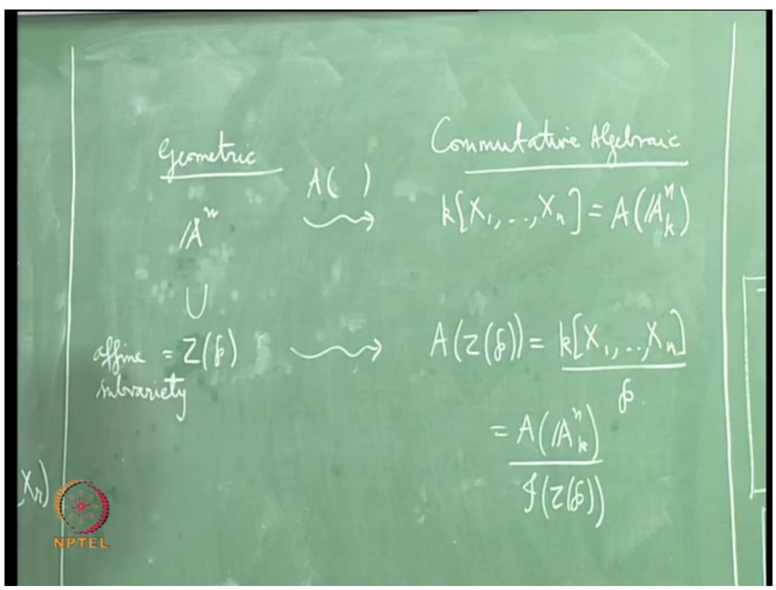

So the moral of the story is the following, the moral of the story is that if I draw a diagram with you know so maybe I can do that if a draw a diagram with a geometric side here and the commutative algebra at side here if I start with An, okay then what you go to is k of X1 etcetera up to Xn which is the ring of functions on An, so now you see now this dictionary what I am writing down is not what I is different from what I wrote down earlier, earlier I was looking at subsets of An say close subsets of An here and I was looking at ideals here, okay but I am not doing that now, what I am doing is I am defining a function which to every set gives it set of functions.

So this is A of so this is symbol standard symbol so A of Ank is set of functions on An if you give me a Z affine verity Z of p affine variety affine sub-variety which is a closed subset irreducible closed subset of An, okay then and if you apply this A you will get A of Z of p is just k of X1 through Xn mod p the affine the ring of functions on a affine variety is just the quotient of ring of function the ambient space modulo the prime ideal who's 0 is define that particular irreducible closed subset so can also be written as A of An by I of Z of p if you want, okay.

So you see now you get a picture, the picture here on this side you have the affine variety An mind you An itself is an affine variety because it is an irreducible closed subset, okay and you have its irreducible closed subsets which are proper affine varieties and there are the corresponding rings of functions. So the moral of the story is that we have this correspondence between space the geometric spaces on this side and the rings of functions so in all of algebraic geometry the point is that everything that is geometric here gets translated into commutative algebra and vice versa, okay.

So for example the topological dimension on this side is the Krull dimension on the other side that is what the theorem says, okay. So the notion of topological dimension here corresponds to the notion of Krull dimension there that is the translational, okay that is what you must understand, so I will stop here.