Basic Algebraic Geometry

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Module 3
Lecture 7
The Noetherian Decomposition of the Affine Algebra Subsets into Affine Varieties

Okay, so what we are going to do now is continue with our earlier discussion, so that is about the Noetherian decomposition, okay.

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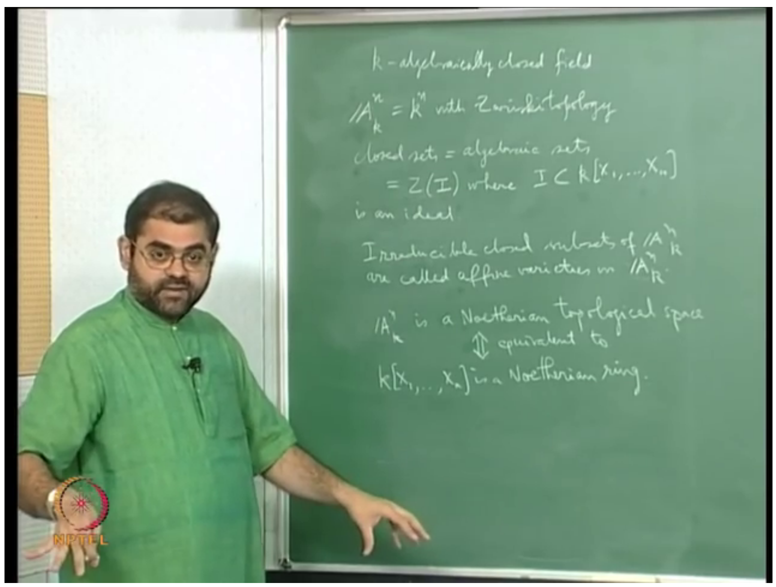

So you see let me again remind you what we are doing is we have if you remember there is the we looked at we took k to be an algebraic closed field and we took the affine space affine n space over k which is just kn with the so called Zariski topology, and what is the Zariski topology? The Zariski topology is topology for which the closed sets are given by 0 sets of ideals and the ideals should be taken in the ring of polynomial functions in n variables over k where the n is same as this n, okay.

So closed sets are also called as algebraic sets and they are just of the form Z of I where I inside the polynomial ring in n variables over k is an ideal, so this is the Zariski topology and then we have seen that there is a you know there is already a dictionary between the closed subsets of An and the ideals of the polynomial ring, in fact if you want perfect equivalence bijective correspondence then on this side you should take closed subsets of An on the other side you should take radical ideals in the polynomial ring, okay the closed subsets correspond one to one with radical ideals and this correspondence is inclusion reversing, okay.

And the larger the ideal the smaller the 0 set of the ideal, okay. So the 0 set of the ideal is simply the set of points in kn the n tuples of points which at which every function every polynomial function in this ideal vanishes, okay. And now the point is that we defined we checked that this is a topology on this space, okay that is this definition satisfies the schemes for closed sets of a topological space. So An becomes so kn becomes An with this topology which is called the Zariski topology and then I told you that the beautiful thing is that there is a algebraic geometry set there is a commutative algebra set the algebraic geometry side is the geometric properties of An and its subsets the commutative algebraic side is the commutative algebraic properties of the ideals here, okay.

And as a first result what we showed the last lecture was that you know the ideal I is prime if and only if or rather the radical of I is prime if and only if Z of I is irreducible closed subset, okay. So here when we defined irreducibility if you recall irreducibility was a strong form of connectivity, okay and whereas for a set to be connected you require that it cannot be written as a disjoint union of two proper non-empty closed subsets the condition for a set to be irreducible is much more stronger the condition is that it cannot be written as a union not necessarily disjoint but of proper non-empty closed subsets, okay.

So an irreducible set is of course connected but the converse is not true and irreducible sets satisfy all the nice properties that you similar properties as connected sets, okay. And so the point was that we called irreducible closed subsets of An as affine varieties then An, okay and so let me write that irreducible closed subsets of An are called affine varieties in An, okay and so the point is so of course the advantage of studying an irreducible closed set is that you have the additional property that it is reducible, okay.

And this irreducibility is a topologically it is very nice property it is property that is really nice in the sense that as I told you last time, for example you can define irreducibility for any subset of a topological space first of all it is not necessary that you should define a (())(6:41) for closed subsets and the point is that if you define irreducibility for any subset you will have to make the definition with respect to the induced topology, so you should say that the subset cannot be written as the union of proper non-empty closed subsets closed for the induced topology on the subset induced from the bigger topological space in which the subset is sitting, okay.

And the advantage of irreducibility is that so let me mention some of them well if a set is if a subset is irreducible then its closure is also irreducible so adding a boundary which is same as taking the closure is not going to take away the irreducibility, then if you take an non-empty irreducible set of course we by definition we always require that an irreducible set is non-empty because we declare the null set not be irreducible, okay.

So well an irreducible set if it is non-empty then it has a nice property that every open non-empty open subset is dense, okay and an every non-empty open subset is also irreducible, therefore you know in an irreducible space a non-empty open subset will is enough to test all properties of the space which are going to preserved when you take closures that is when you take limits, okay. So it is very important to be able to test on a non-empty open set, okay and non-empty open sets are irreducible for an irreducible space, okay that is one nice thing then the other nice thing is of course that any non-empty any two open sets any two non-empty open sets will intersect, okay that will tell you that the topology is not (())(8:42), okay when compared to the usual topology.

For example if you take k to be complex numbers, okay then the Anc which is c to An c cross c cross c n dimensional complex space a n dimensional taught of as a vector space cn is with Zariski topology is not (())(9:06), okay that is because of this reason because it is irreducible and the whole space is always irreducible please remember that is because the this corresponds to the 0 ideal and the 0 ideal the 0 set of the 0 ideal is a whole space and the 0 ideal is of course prime because 0 ideal is an ideal in integral domain, okay in a commutative ring with unity which is an integral domain the 0 I mean if you take a commutative ring with unity the 0 ideal is prime if and only if the ring is integral domain, okay.

So this is always reducible, okay so any two open subsets of this will always intersect so it is highly (())(9:55), okay (())(9:56) means that you give me any two points you can find open sets small enough such that they contain those two points in their own natives so this is highly (()) (10:04) but still that does not deter us from doing good geometry, okay that is the point and well so and of course there are other nice things, for example the continuous image of an irreducible set is an irreducible exactly the same as you prove but the continuous image of a connected set is connected, okay.

So having a studying irreducible closed set is of course very nice and of course from the from a geometric point of view it is very nice and from the algebraic point of from the commutative algebraic point of view also it is very nice because you are studying prime ideals, okay but then the point is how do you get to an arbitrary closed set how do you study an arbitrary algebraic set, okay. So the fact is that the answer to that is that this irreducible closed subsets which are the affine varieties they are the building blocks for the algebraic sets so there is something called the Noetherian decomposition theorem which says that any algebraic set any close set in An affine n space over k is writeable as a union a finite union of irreducible closed subsets namely affine varieties and the union is and this decomposition is unique and if you assume that there are no redundancies that means that there is no component which is containing some other component in the union, okay.

So now how do you prove that the answer to proving such a theorem is the Noetherian property, okay so last time I defined what is meant by Noetherian so Ank is a Noetherian topological space so let me quickly recall this, so again let me tell you the reason one reason for this Noetherian condition is to be able to say that any algebraic set can be broken down into a affine union of affine varieties and the finite union is decomposition is unique if assume that there are no reputations in the union, okay of course you make permutation of the pieces that occur, okay.

So if you want to study any algebraic set since you can break it down as a union of varieties it is just enough to study verity and that is why we study only varieties, okay that gives us some justification as to why to study varieties and not just algebraic sets that is part of the reason why at least in first course in algebraic geometry we study only affine varieties I mean we study affine varieties to begin with and then we probably study projective varieties but we do not study non-irreducible closed sets, okay we study only irreducible closed subsets because general non-irreducible closed set can always be broken down like this.

So the key to showing that any algebraic set can be broken down into a finite union of affine varieties in unique way is due to the fact that An is a Noetherian topological space, so if you so let me recall what I told you in the last lecture, see the Noetherianness of a topological space is the condition that the closed subsets of the topological space satisfy the so called descending chain condition that is if you give me a descending chain of close subsets each one containing the next one, okay so the closed sets are becoming smaller and smaller and smaller then such a chain has to become stationary at some point it has to become stable that means we are on the certain point all the sets occurring in that chain in the sequence they all have to be one and the same.

Another way of saying it is that if you take a descending chain of closed subsets such that every at every step it is a proper containment, okay that means it is a strictly descending chain of closed subsets then the rest we only finite you cannot have a infinitely you cannot have infinitely many you cannot have an infinite chain of closed subsets which are becoming smaller and smaller strictly smaller one after the other, okay.

So this is the descending chain condition for close subsets and this condition translates into the ascending chain condition for ideals in this polynomial ring that is become after all the closed subsets in the closed subsets in the affine space are they correspond to ideals in the polynomial ring, okay and in fact if you want exact correspondence you have to worry about radical ideals, okay but the fact is that if you give me any ascending and because the correspondence between the closed subsets of An and the ideals in the polynomial ring is a inclusion reversing correspondence the descending chain for closed subsets in the affine space will translate to ascending to an ascending chain condition on the ideals in the polynomial ring and but the polynomial ring does have the ascending chain condition on ideals because it is an Noetherian ring.

So you see one of the definition of Noetherian ring is that the standard definition is always that it satisfies the ascending chain condition for ideals that is in other words if have sequence of ideals becoming larger and larger and larger every ideal being contained in the next one after it, then that has to stop at some stage, it has to either become stationary or another which means that beyond the certain stage all the ideals should be one and the same in the sequence the other way of seeing it is at if you have strictly increasing chain of ideals then it has to be just finite you cannot find an infinite sequence of ideals which are becoming bigger and bigger strictly bigger and bigger and with this whole thing never coming to an end this cannot happen in an Noetherian ring.

So the fact is that this is equivalent to the fact that the polynomial ring in n variables is a Noetherian ring and why is the polynomial ring in n variables over k and Noetherian ring and that is just because of Hilbert Basis theorem that is Emmy Noether's theorem which says that if you start with the ring if you start with a commutative ring with 1 if that commutative ring is with 1 is Noetherian that is for example if it satisfies the ascending chain condition of ideals then so does the polynomial ring in n variables over that ring, okay now a field is always Noetherian because a field has only two ideals, one is 0 ideal the other one is a full field which is unit ideal and therefore a field is always Noetherian and therefore if you use Hilbert Basis theorem if you take a polynomial ring infinitely many variables over a field then that will also be Noetherian and that is the reason why this is a Noetherian.

So you see the topological side on the topological side the space An is Noetherian and on the commutative that is on the algebraic geometric side at the level of topology the affine space is Noetherian for Zariski topology and in the commutative algebra side the ring of functions on affine space namely the polynomial ring that is Noetherian ring and these just correspond to each other, okay. Now I was trying to so I stated a theorem so this is the theorem that will let us to prove that any algebraic set can be decomposed into a union of affine varieties, okay.

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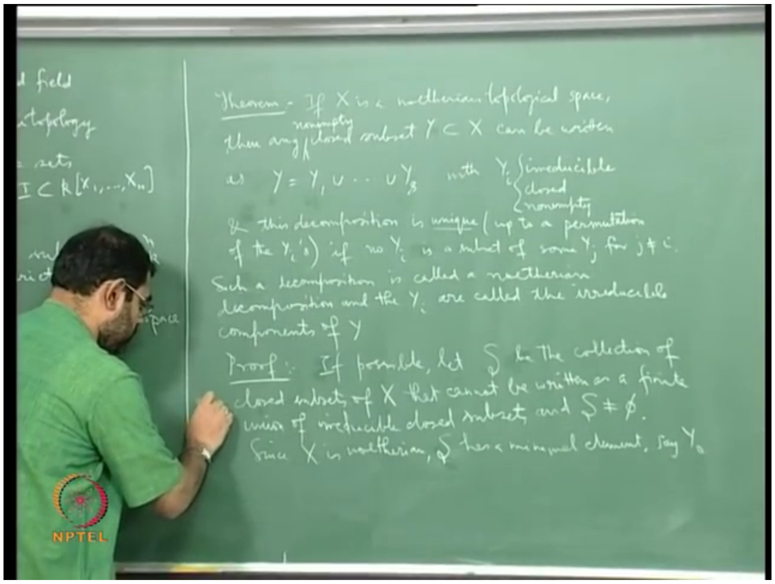

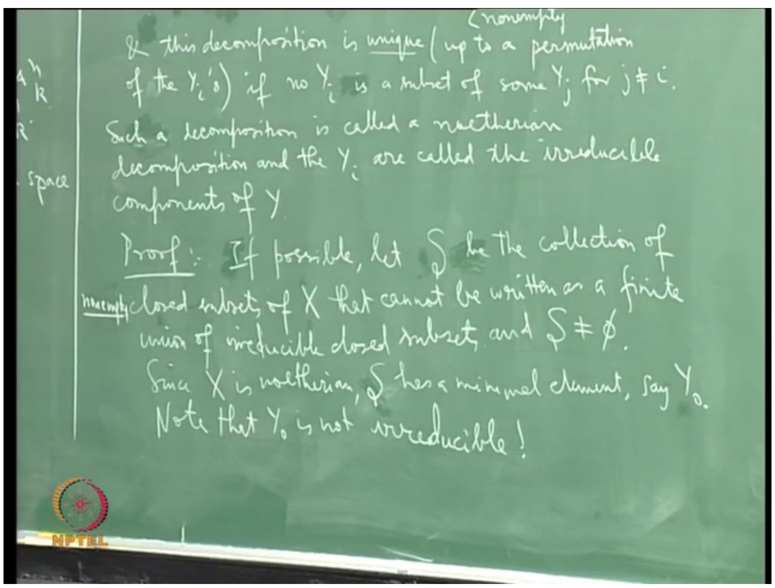

So let me recall that theorem and let me try to prove it so here is a theorem if X is a Noetherian topological space then any closed subset any non-empty closed subset Y X can be written uniquely I should say can be written let me explain what uniquely means rather let me can be written as Y equal to Y1 union Ys, okay with Yi irreducible closed non-empty and this decomposition is unique up to a permutation of the Yi's if no Yi is a subset of some Yj for j not equal to i, okay.

So any non-empty closed subset can be written as a finite union of irreducible closed non-empty subsets, okay so each of these is irreducible, it is closed, it is non-empty and this decomposition is called you can make it unique if you make sure that you do not have redundancies, okay. So for example if Y1 is if you have Y1 union Y2 and so on and if Y1 is contained in Y2 then why put Y1 you can throw out by 1 because it is already there in Y2.

So once you can write a decomposition like this you can always throw out some of them which are contained in the others and finely you can arrive at a decomposition in which none of the sets is contained in the others in any of the others in any of the others such a decomposition is unique, okay such a decomposition is unique and that decomposition is what is called as Noetherian decomposition, okay. Now and the Yi's are called the irreducible components of Y, okay so each Yi is called an irreducible component of Y, okay so the decomposition this decomposition such a decomposition is called a Noetherian decomposition and the Yi are called the irreducible components they are called the irreducible components of Y it is much similar to what you do in topology you take any in topology you define connectedness of a subset, okay and then you prove that any topological space can be written as a union of its connected components, okay.

In the same why here you are saying that any topological if you want any topological space to be written as a union of its irreducible components and you want that to be a finite even the condition you have to put on the topological space is that it should be Noetherian, okay that is a nice condition and that condition is there for us for the affine space as we have already seen. So if you believe this theorem you will immediately get that any algebraic set can be uniquely decomposed into affine varieties in this sense, okay.

So that proves the statement that we want, alright? So you have to prove we want to prove this it is pretty easy to prove so the point I want to say is that here we are going to use the various other definitions of Noetherianness of a topological space, so you know the original, for example if you take a starting point the definition of a Noetherian topological space that there is DCC for closed subsets that is there is you cannot have a strictly decreasing sequence of closed subsets one containing the next which is infinite, okay then this is equivalent to also saying that any non-empty collection of close subsets has a minimal element, okay and that is also equivalent to saying that any non-empty collection of open subsets has a maximal element because a open subset is just you know complements of closed subsets in any topological space.

So you know so DCC for closed sets is same as ACC for open subsets and DCC for closed subset is same as saying that any family of any non-empty family of closed subset has a minimal element and that is equivalent to saying that any non-empty family of open subsets has a maximal element, so these are all various avatars of the definition of the Noetherianness of a topological space, okay and it is not surprising because if you have seen this in commutative algebra or in algebra you know that there are also several ways of defining a ring to be Noetherian one of course a standard way is to say that you know it has ascending chain condition for ideals, okay but then there are other equivalent conditions the other useful equivalent conditions are that every ideal is finitely generated, okay which is also a very very important condition and they get another condition is that given any non-empty collection of ideals there is always a maximal element maximal with respect to inclusion, okay.

So whenever we say maximal it is always nothing is mentioned maximality is always with respect to inclusion subsets, okay. And of course you know each one has its own each of these schemes has its own importance, see for example the while the ascending chain condition ideals gives rise to the descending chain condition for closed sets the fact that every ideal is finitely generated it is also very important because that is what tells you that if you take any ideal, okay any ideal mind if you take any ideal and look at the 0 sets, okay if you use the fact that Noetherianness means that every ideal is finitely generated it will tell you that whenever you are looking at the 0 set of an ideal you are just looking at the common 0's of bunch of finitely many polynomials.

So are not an ideal an non-zero ideal is always going to be infinite, okay there are going to be infinitely many elements in it, okay. So the point is that it looks like when I take the 0 set of an ideal namely all the points in the affine space which at which every one of the functions the ideal manage it looks like I am solving too many equations, okay but it is not true in fact you are trying to find common 0's of infinitely many equations, okay but that is not really true what is really happening is that you are only finding set of common 0's of only finitely many equations, why finitely many because though simply those finitely many equations for example which generates this ideal and that is true because the ideal is an ideal of a Noetherian ring and it finitely generated.

So you are always looking at only finite common 0's of finitely many polynomials and this finiteness is very very important because it allows you to do for example calculations of the computer so if you have a if you want to look at the 0 set of an ideal then you know I look at 0 sets I take the generate of the ideal and I first look at the 0 sets of the first generator and then intersect it with 0 of second generator and go on and I can have to do this process only finitely many times, okay.

So that is the reason why you can do computational commutative algebra which will help in algebraic geometry, okay. So anyway so let me comeback so the point is that the property for the Noetherianness the Noetherian hypothesis on the topological space I am going to use is not the is not ACC on closed subsets that I am going to use not DCC on closed subsets that I am going to use but what I am going to use, I am going to use the other equivalent definition that any non-empty family of closed subsets has a minimal element, okay so that is what I am going to use, okay.

So let us see how to use so it is very simple argument so what you do is basically you assume that there are you try to contradict this statement, okay so what you try to say is you assume that there is a subset which cannot be written as a finite union of irreducible closed subsets, okay then that is a specimens because that is something that contradicts this theorem, okay the point is that you should not show that that you should show that there is not a even a single one like that, okay. Now the point is what you do is you put all these specimens together in a subset and apply the existence of a minimal element for that subset, okay that is the whole point. So what you do is so let me write this let if possible let S scripts S in be the collection of closed subsets of X that cannot be written as a finite union of irreducible closed subsets, okay if possible let us script S be the collection of closed subsets of X that cannot be written as a finite union of irreducible closed subsets and S non-empty.

So that means you have to show that this collection scripts S is empty mainly you should show that there is no closed subsets that cannot be written as a finite union of irreducible closed subsets so you go by contradiction what you do is you take script S to be a non-empty collection of closed subsets which have a property which contradicts the property of the theorem not even the stronger uniqueness of decomposition it contradicts even existence of decomposition, okay.

So S is non-empty, now after all S is just a collection of closed subsets of the topological space X and X is Noetherian but you know Noetherian one equivalent definition of the Noetherian condition is that given any non-empty collection of closed subsets it will always have a minimal element, okay. So since X is Noetherian S has a minimal element say Y not, so there is so Y not is a kind of smallest kind of subset irreducible I mean smallest kind of it is a closed of course non-empty closed subsets I you will have to I mean you will have to be careful about this should be non-empty closed, okay because of course decomposition is only being made for a non-empty closed subsets, okay.

So there is a minimal element which means that Y not is a non-empty closed subset Y not cannot be written as a union of proper closed subsets and it is the smallest in the collection S, smallest with respect to what? With respect to inclusion of subsets which means that if there is an element of S which contains Y not I mean if there is an element of S which is contained in Y not then it has to be equal to Y not that is what minimality means minimality means that if there is some other thing which is smaller than this then this has to be this, okay fine.

Now you watch carefully since Y not is in S the first observation is Y not is not irreducible, okay see because if Y not is irreducible then Y not can be written as Y not the fact that it cannot be written as a finite union of irreducible closed sets tell you that it cannot itself be irreducible because if it is irreducible then it is itself when I say union this union can be just 1, okay. So what you must understand is the element of this script S by definition they are all not irreducible.

So what happens is Y not is not irreducible, okay note that Y not is not irreducible Y not is not irreducible, okay. Now but then what does it means Y not can be broken down into two as a union of two proper non-empty closed subsets, okay.

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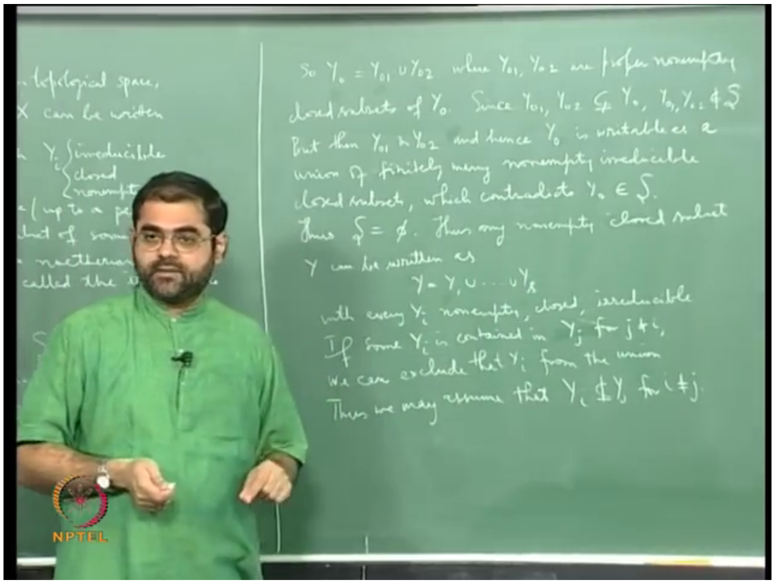

So Y not so let me continue here so Y not is equal to Y01 union Y02 where Y01, Y02 are proper non-empty closed subsets of Y not, okay Y not is not irreducible so it is reducible so it can be broken down as a union of two proper non-empty closed subsets.

But now watch Y01 is a close set which is smaller than Y not and Y02 is also smaller than Y not so Y01 and Y02 cannot belong to script S, see Y01 is a proper subset of Y not and it is a non-empty closed set and Y02 is also similarly a non-empty proper close subset of Y not. So these two closed subsets cannot belong to scripts S because if they belong to script S they will contradict the minimality of Y not, Y not is supposed to be the smallest, okay.

So the fact that Y01 and Y02 do not belong to script S means that they can be written as a finite union of irreducible closed sets and if that is true then Y not also can be written as a finite union of irreducible closed sets namely you just take the union of the irreducible closed sets the finite union of irreducible closed sets that gives Y01 and take also the finite union of irreducible closed sets that gives Y02 put them together that will tell you that Y not becomes writeable as a finite union of irreducible closed subsets and but that is not possible because Y not is supposed to be in script S, so this contradiction tells you that script S has to be empty and this argument finally tells you that every closed subsets not-empty closed subset is necessarily writeable it can be necessarily be broken down into finitely many irreducible closed subsets, okay.

So this gives you the existence of a decomposition not without the uniqueness you will get the uniqueness next, okay so let me write that down. Since Y01, Y02 are proper closed subsets of Y not Y01, Y02 do not belong to the collection but then Y01 and Y02 and hence Y0 is writeable as a union of non-empty or finitely many non-empty irreducible closed subsets which contradicts Y0 belongs to the family, does the family is empty in other words every non-empty closed subset is certainly writeable as a union like this of non-empty proper of non-empty irreducible closed subsets, okay.

So does any non-empty irreducible, sorry non-empty closed subset Y can be written as Y equal to Y1 Ys with every Yi non-empty closed irreducible, of course whenever I say irreducible it is automatically it is automatic it is non-empty because we have bark the null set from being taken as irreducible, okay. So now of course if some Yi is contained in some other Yj you can throw the Yi out, so you can assume without loss of generality that no Yi is contained in any other Yj, okay and then what the theorem says is that with that assumption this decomposition is unique that is what we are going to prove next, okay.

So if some Yi is contained in another in Yj for J not equal to i we can exclude that Yi from the union does we may assume that Yi is not contained in Yj for i not equal to j that is the union is not redundant, okay this you can assume this is obvious. The point is that once you assume that this decomposition becomes unique, okay that is what we are going to prove next.

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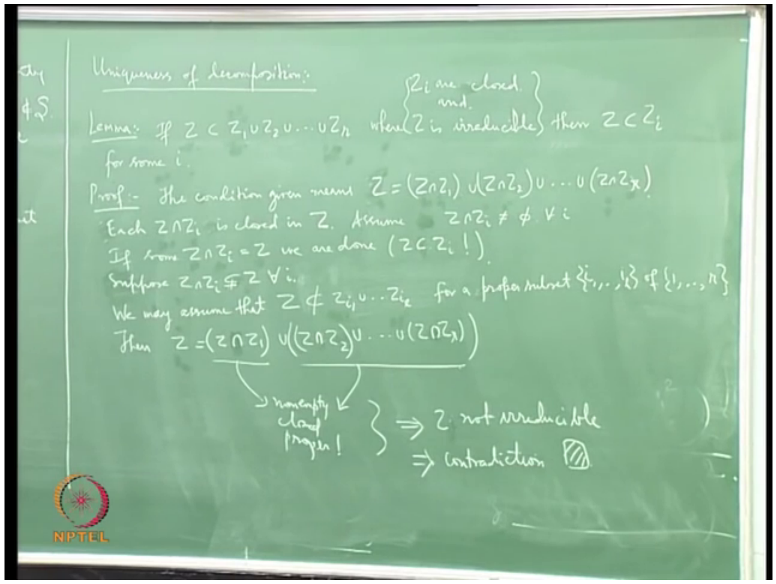

So let me do that so let me write that down uniqueness of decomposition so let me write a lemma I will use small lemma if is it is contained in Z1 union Z2 union Zr where Z is irreducible then Z is contained in Zi for some i, just a moment also need that all the Zi's are let me assume that Z is (())(40:59) and all the Zi's are also closed where Zi are closed and Z is irreducible then Z is contained in Zi for some i that is correct yeah.

So if you have a union of closed sets finite union of closed sets and if there is a set we just contained in irreducible set which is contained in finite union of closed sets, okay then if it is irreducible it has to go into one of them, okay so the proof is well you see I mean you just have to use the see the condition given means that Z is actually you know Z intersection Z1 union Z intersection Z2 union Z intersection Zr, okay because if you intersect Z is contained in this union so Z intersect that union is Z but Z intersect that union is just Z intersect each member of the union and then you take the union so because the intersection distributes over the union, okay so that is what it means.

And notice that each Z intersection see since Zi are closed in the ambient topological space the big topological space therefore Z intersection Zi is closed in Z, okay. So the each Z intersection Zi is closed in Z, okay that is very clear because that is what induced topology means induce topology means induce topology on a subset is that subset of that subset is closed if that subset which you are claiming to be closed is gotten by intersecting with a closed set in the bigger space, okay.

So Z1 any Zi is closed in the big topological space therefore it intersection with a subset is closed in that subset, okay fine. Then the other thing is that of course you know I can in this union I can simply forget if some Z intersection Zi is empty, okay. So if some Z intersection Zi is empty I do not write it at all so I can assume no Z intersection Zi is empty or rather is non-empty for every i, okay I mean if it is empty just forget that index and re-index we will get a lesser number of indices, okay.

So after this assumption maybe this r will come down it might come down to a smaller number, okay but let us not it is not really one is really not worried about what that smaller number is one is only worried about the fact that it is finite, okay. Then the other thing that you must understand if some Z intersection so you see if some Z intersection Zi is equal to Z then we are done because if some Z intersection Zi is Z then you are just saying that Z is contained in Zi, alright? So we are done, okay Z is contained in Zi, alright? So if some Z intersection Zi is equal to Z then you are done, okay if not if suppose Z intersection Zi is properly contained in Z for all i this is the other possibility you have to show that this possibility does not occur, okay you have to show that this possibility does not occur and the only possibility that occurs is Z intersection Zi is Z for i and therefore Z contained in Zi for some i that is the essentially the lemma, okay.

Now suppose Z intersection Zi is proper subset of Z so what it will tell you is that you have broken down Z into a finite union of non-empty proper closed subsets now that is not possible because Z is irreducible, okay the irreducibility says that you cannot break it down into a union of two proper non-empty closed subsets but you can extend to it finitely many the reason is because you see suppose I have this condition what I can assume is that I can also assume that Z is not contained in any union of a sub collection of the Zi's, okay.

So in other words I am saying that you can also assume without loss of generality that you know Z is not in the union of say for example you throughout Z1 it is not in this union because if it is in this union I would work with this so try to make sure that Z is not contained in any smaller union a union of subset proper subset of the collection of all these Zs you can assume that, okay assume that Z we can in fact we can assume so let me rewrite this you may assume that Z is not contained in Zi1 union Zil for a subset a proper subset I1 etcetera Il of 1, 2 etcetera r you assume that, okay.

So you can assume that otherwise if there is something like this then work with this call this subset as 1 through replace this collection with that subset and what you will do is you keep on reducing to a stage until you will come to a point where you get a collection like this, okay that is Z is contained in the union, okay Z intersection Zi is proper subset of Z for every i and Z is not contained in any smaller union we will come to a stage like that, okay now you have to say that is not possible you see then you see you will have Z is equal to Z union Z1 union Z union Z2 eZ union Zr I will break it down like this.

See this is non-empty closed topper, okay and this guy, oh sorry these are all intersections I been careless, yeah. So this is if you look at it both of these guys here are non-empty closed proper, so what I have done is I have broken an irreducible set into two pieces which each of which is non-empty closed proper that is a contradiction to the irreducibility of Z so this can never happen therefore the only thing that can happen is that Z has to be in Zi for some (())(50:05), okay.

So this implies eZi eZ not irreducible and that implies that implies a contradiction which proves the lemma, so your lemma is proved. So the moral of the lemma is that if an irreducible set is contained in a union of closed finite union of closed sets it has to go into one of them, in other words it cannot fall into pieces in a union it has to go exactly into one of them, okay. Now let us apply that lets apply that to the uniqueness of decomposition so let me go back here, okay.

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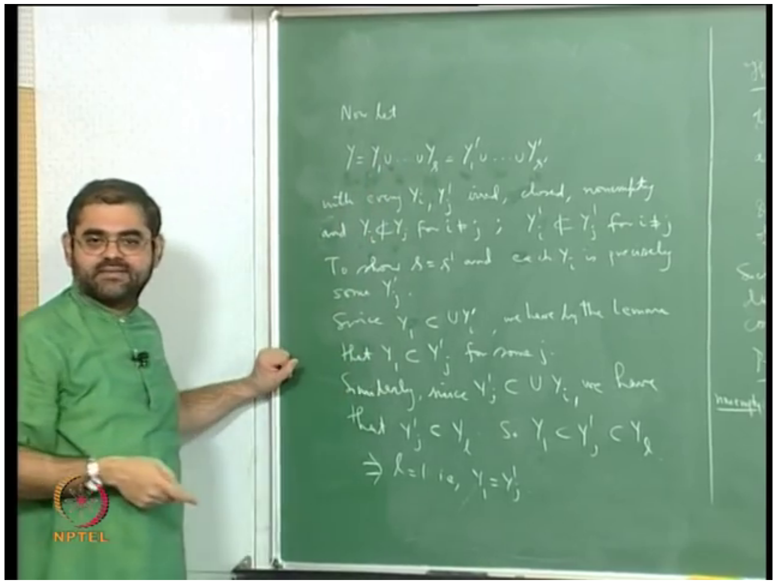

So let me go back here now assume now let Y equal to Y1 union etcetera Ys and let that also equal to Y1 prime union Y prime s prime with every Yi Y prime j irreducible closed of course non-empty, okay what you have to prove? You have to prove that S is equal to S prime and you have to prove that every Yi is some Y prime j for a unique j, okay.

So of course you know and of course you are assuming that there are no redundancies Yi is not contained in Yj for i not equal to j Y prime, Y prime i is not contained in Y prime j for i not equal to j where of course here ij's are indices from 1 to S prime and here the ij's are indices from 1 to S, okay that is to show S equal to S prime and each yi is precisely some Y prime j, okay that is what you have to show so and how do you that it is very very simple you see so let me first state it inwards I am going to use that lemma take Y1, now Y1 is a sub of this union so it is a sub of this union, okay.

Now the lemma says that the set Y1 I do not need the fact that the Y1 is closed I only need the fact that Y1 is irreducible, Y1 is contained in the union finite union of closed sets therefore it has to be in exactly one of them it is contained in at least one of them not exactly it is contained in at least one of them. So if I apply the lemma I will get that Y1 is contained in some Y prime chain, okay now again apply the lemma to Y prime chain that Y prime j is contained in all the union of Yi's and that Y prime j is irreducible, therefore Y prime j is contained in some Y j prime, okay.

So if you write this down you will get that Y1 is equal to Y prime j you will get Y1 is equal to Y prime j.

So since so let me write that, since Y1 is contained in the union of Yi primes we have by the lemma just using the fact that Y1 is irreducible that Y1 is contained in Y prime j for some j, okay. Similarly since Y prime j is contained in the union of Yi's we have since Y prime j is irreducible that Y prime j is contained in some Y I do not want to use k but let me use l if you want, okay so you will get Y1 is contained in Y prime j which is contained in Yl, okay so finally you get Y1 is contained in Yl but you are not supposed to have that because the Yi's are all they are non-redundant none of them is contained in the others.

So what this will tell you is that l equal to 1 l has to be 1 and that will tell you that Y1 is equal to Y prime j this implies l equal to 1 that is Y1 is equal to Y prime j, okay. Now the point is you re-number all these, okay you re-number all these guys so that you call Y that Y prime j call that as Y prime 1 and the fact is that you can now strike off from the union the Y1 and the Y prime 1 the fact that you can strike off (())(55:46) this topological fact that any non-empty subset of irreducible set is dense, okay.

So I mean the fact is that if you take this union, okay and you throughout Y1, okay usually in open set because what you have thrown out is the closed subsets, okay because you are taking a complement removing Y1 from this union is by taking the complement of Y1 in Y that is a complement of a closed set, okay so it is open but if it is non-empty then it is dense so if I take Y1 out and take the closure I will get the union of the other pieces because what I have taken out when I take Y1 out from Y I am taking out also Y1 from Y2 through Ys, okay.

So what I will get is essentially a subset of Y through to Ys, okay and what I have thrown out from each of the Y2 to Ys is a close subset, it is a proper closed subset, okay. Therefore whatever that is left out if I close it up I will get back Y2 through Ys. So in other words from this union if I take out Y1 and if I close it up what I will get is Y through to Ys, so this is you can think of this as a cancelation property, okay. So well so let me write that or rather that leads to a cancelation property so let me write that just a couple of more lines then we can wind up, so let me write that down.

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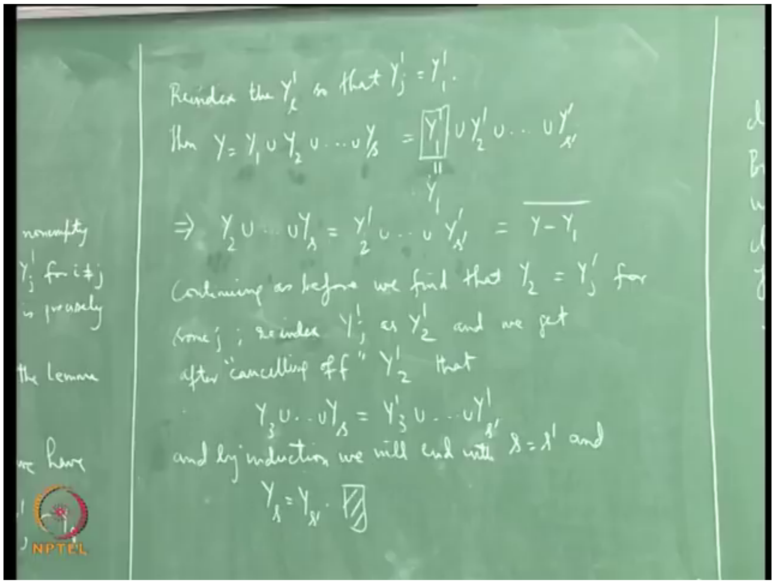

A re-number re-label or re-index the Y prime j the Y prime l so that a Y prime j is Y prime 1, okay that Y prime j which we got to be equal to Y1 call that as Y prime 1 re-number it, okay then Y is equal to Y1 union Y2 union Ys that is also equal to Y1 prime union Y2 prime union Y prime s prime, okay and you know the point is that this is the same as Y1 is equal to Y1 this is Y1, okay.

And now what I want to say is that if you I want to say that this implies that Y2 union Ys is actually equal to Y1 I mean Y2 prime union Y prime s prime which is like you cancel off the Y1 that is on both sides of the union and this cancelation property is actually because of the irreducibility property, okay that any non-empty open subset of an irreducible space is dense and in fact actually this is equal to Y minus Y1 closure in fact you take this union from that union you remove Y1, okay what you get will be an open subset of Y through to Ys and it is a non-empty open subset of Y through to Ys therefore its closure will be Y through to Ys, okay because in each piece the elements for example in Y2 that you have removed which are common with Y1 you will get an open subset of Y2 if I close that up I will get my Y2 back that is because this open subset that I have removed I mean this closed subset that I have removed is the common elements between Y1 and Y2 and that is not the whole of Y2 because no Yi is contained in some other Yj.

So if I remove Y1 from Y2 and close it up I will get back Y2. So in the same way from this whole thing if I remove Y1 which is equivalent to removing the intersection of Y1 with each of these pieces and then close it up I will just get the union of these the other pieces and now what you can do is you can continue the induction you can next strike off Y2 and Y2 prime, okay I mean you can strike off Y2 and some other Y prime j, okay because they will be equal by the same argument and that Y prime j can be re-indexed so that you get that Y prime there becomes Y prime 2, okay then you can strike that off and you can keep on doing this and this process is has to end finitely and when it ends you cannot this S has to be equal to S prime because if it is not if S is less than S prime then at some point you will get a null set here equal to a set there which is not empty and if S is greater than S prime at some point you will get a null set on the right side which is equal to non-empty set of left side.

So this argument can go on you can go on only finitely many times and the fact that you cannot have a non-empty set equal to empty set it will force that S has to be equal to S prime and every Yi is unique Y prime j, okay. So continuing as before we find that Y2 is equal to Y prime j for some j renumber re-index Y prime j as Y prime 2 and we get after canceling off Y prime 2 that you will get Y3 union Ys is equal to Y prime 3 union Y prime S, Y prime S prime and by induction we will end with S equal to S prime and Ys equal to Y (())(63:04) that will be the end of the proof, okay.

So finally you get this uniqueness of decomposition, okay. So the moral of the story is that uniqueness of decomposition and Noetherian decomposition itself holds for a Noetherian topological space and the nice thing is that it holds for affine space and therefore any irreducible any closed subset of affine space namely any algebraic set is a finite union of affine varieties and these affine varieties are unique if you assume that the union is non-redundant that is none of them is contained any of the other and this is called the Noetherian decomposition for closed subset, okay.

So this tells you partly why it is worthwhile to study affine varieties rather than just studying algebraic sets because the affine varieties are building blocks for any algebraic any algebraic set can be decomposed into affine varieties, so the affine varieties are the building blocks for our algebraic sets and in fact it is true also in the widest generality of algebraic geometry in the most sophisticated language of algebraic geometry it is the affine pieces that are the building blocks the most general object in algebraic geometry the most sophisticated object is called a scheme and the definition is that it is build up by the building blocks which are called affine schemes and this is the philosophy the affine are like the bricks that make up the whole building, okay they are the building blocks, okay. So I will stop here.