Basic Algebraic Geometry
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Module 2
Lecture 5
Irreducible Closed Subsets Correspond to Ideals Whose Radicals are Prime

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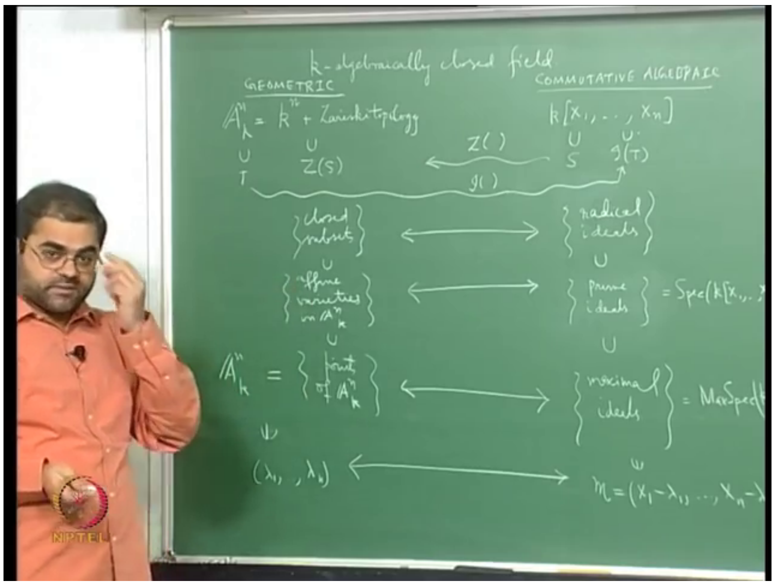

Okay, so again let us continue with our discussion on algebraic geometry, so you know let me recall the setup in which we are working k is an algebraic closed field, for example you can think of k to be complex numbers if that is convenient for you and then the whole idea is you look at kn with the Zariski topology on the one side which is called as affine n space over k, okay and this is the rather this is a geometric side and then on the other side you have the commutative algebraic side and that is supposed to be the ring of functions on that space and in this case of course is a ring of polynomial functions on that space so it is actually k of X1 through Xn so X1 through Xn are n indeterminates there n variables and if you think of this ring as a ring of functions on affine space because you give any polynomial you take any polynomial then you can think of it is a map from affine space to An to A1 which is just A1 is just k, okay so this is the functions with values in k, okay.

And therefore this is the ring of polynomial functions there functions on the space, okay. So the geometric side has got to do with the affine space the commutative algebraic side has got to do the functions, see it is a supposed to be statement of (())(3:32) who said that you know that geometry of a space is supposed to be controlled by the functions on the that you allow on this space and this is so geometry actually kind of comes into play when you have a space and you define what the functions on your space going to be, okay.

So in this case the spaces are affine space and the functions are the polynomial functions, okay. And I told you that so of course the just to recall what we have seen so far you know the given any subset S here you associate to S the common 0 locus of S which is a set of all points in affine space which at which every a polynomial in S vanishes, okay. And then there is also there is also a map that goes in this direction given a subset T of affine space then you associate to T which is an ideal in the polynomial ring and this ideal of T is just all those functions which vanish at every point of T, okay and when this correspondence goes on the one side the objects that are important here are the sub objects which corresponds to ideals and the objects that are important here are the so called the algebraic sets which are the closed sets which are 0 sets of this form, okay.

And of course you should remember that if I changed or if I replaced S with the ideal generated by S you will see that the same 0 set, okay. So basically what is that is you get on this side if you take the set of radical ideals then you get a bijection with the set of all closed subsets and this bijection is a it is inclusion reversing correspondence that which is quite obvious to see because the larger the ideal is the common 0's of all the functions and ideal will grow smaller, okay.

And the other important thing is that so if you look at radical ideals you get closed subsets, on the other hand if you look at maximal ideals, then under so you should remember so maybe I think let me write this below let me leave some space in between so there are maximal ideals, so the maximal ideals they are also radical ideals because actually you should perhaps check as an exercise probably you have already done in commutative course in commutative algebra or algebra that if an ideal is prime then it is already radical and since the maximal ideal is prime the maximal ideal is also radical but of course there are radical ideals which are not even prime, okay but in any case maximal ideals the collection of maximal is a subset of this, okay and this under this bijective correspondence goes to the smallest possible closed subsets which are actually the points of course when I say points I am thinking of a point here as a singleton subset of An, okay. So in other words you can actually write An here and of course the notation for this is the maximal spectrum of the ring of functions, okay.

So max spec of a ring commutative ring with 1 means the set of all maximal ideals in the commutative ring. So what you must understand is the point here is a maximal ideal of the ring, okay and the fact is that given a maximal ideal you get a point and converse, okay. So in particular for example if you take a maximal ideal of the maximal ideal will always look in this form it will be generated by Xi minus lambda i for a n tuple lambda 1 etcetera lambda n which will be a point of An and this is the correspondence because the 0 set of this is this and the ideal of this will be that, okay.

So you should remember that in this direction the map is taking the ideal it is the I map and in this direction the map is Z map which takes which associates the 0 set the common 0 locus, okay. And in fact I told you that this is also uses in Nullstellensatz, okay in a way it is an avatar of the Nullstellensatz probably weaker or stronger probably weaker but let us look at that in exercises but the point is this is an non-trivial statement, okay what is trivial is if you give me an ideal like this it is then it is maximal is reasonably trivial to check but to converse it says that every maximal ideal is of this form which is true only when k is algebraically closed at least when k is algebraically closed that is non-trivial, okay and that uses Hilbert's Nullstellensatz.

So the what lies in between are the prime ideals, the prime ideals the collection of prime ideals they correspond to what is called the spectrum of the commutative ring and you see the spectrum or the commutative ring is supposed to be the set of all of its prime ideals and therefore you think of prime ideal here as a point in the spectrum, so a point here is a maximal ideal and the point here is a prime ideal, okay and of course this is contained as I told you prime ideals are radicals maximal ideals of prime, okay.

And what happens is that so what corresponds to prime ideals on this side are what are called as affine varieties in An, so by this I mean the so the definition is these are all algebraic these are all algebraic sets these are all closed sets which are irreducible, okay. So prime ideals correspond to irreducible subsets which are closed, okay and that is a theorem I stated in the previous lecture and I just stopped with that because the last lecture actually stopped with the definition of what an affine verity is, okay an affine verity is an irreducible closed subset of affine space, okay and why are they important? They are important because we will see later that any variety I mean that any closed subset can be broken down into a finite union of affine verities and the decomposition is unique if you make sure that there are no redundancies that is no affine variety in this is contained in no affine variety in the decomposition is contained in some other affine verity in the decomposition.

So every closed subset can be broken down into a finitely many affine verities on the in a unique way so actually unique way, I see essentially because you can always permute the pieces in the union but that should not affect the union or the decomposition and so that is one important thing about affine verities because they are like building blocks of algebraic sets every algebraic set is broken down into union of affine verities and this is very very important because later on for example I told you probably in the first lecture that there is a more advanced or I should say sophisticated language of algebraic geometry which involves what are called as schemes and schemes are some spaces with functions, okay with rings of functions and in fact what you have is not just rings of functions you have shives of rings which means that you have rings of functions on every open subset of this space so its data not only with the space with ring of functions on the whole space but it also comes with for every open set in this space you will have a ring of functions.

So your this whole data is called a shive of rings and a scheme is something to like that which consist of space and the shive of rings, okay where the shive is collection of rings for every open set in the space but the important point is the technical point is this schemes is supposed to be locally modeled in this way it is supposed to be made of affine pieces, so the for all the algebraic geometry no matter how general algebraic geometry you do these affine pieces these are the building blocks and that is the reason why these are to be first studied, okay so you should understand that affine verities are important because they are the building blocks even at the most sophisticated form of the theory, okay.

And the other important thing is of course that this sets are topologically irreducible, okay and you know irreducibility I have told you is a very strong form of connectivity, so they will have nice properties with respect to maps, so for example in topology you learn that the image of a connected set under continuous map is again connected, okay. So if you have a topological space and you have a map a continuous map from the topological space in another topological space then the image of a connected set if in the source topological space will be a subset of the target topological space which will be connected, okay.

And the same thing will happen for irreducible subsets, okay so irreducibility is a very nice thing to have on a subset, okay.

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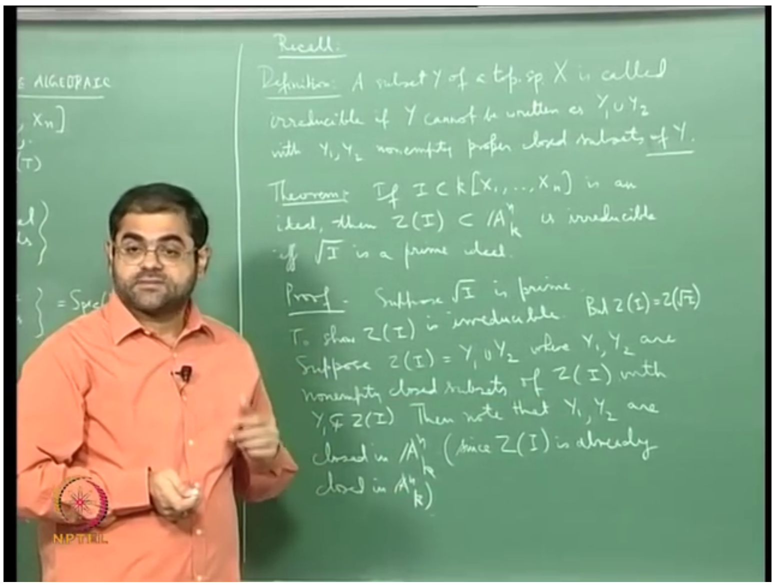

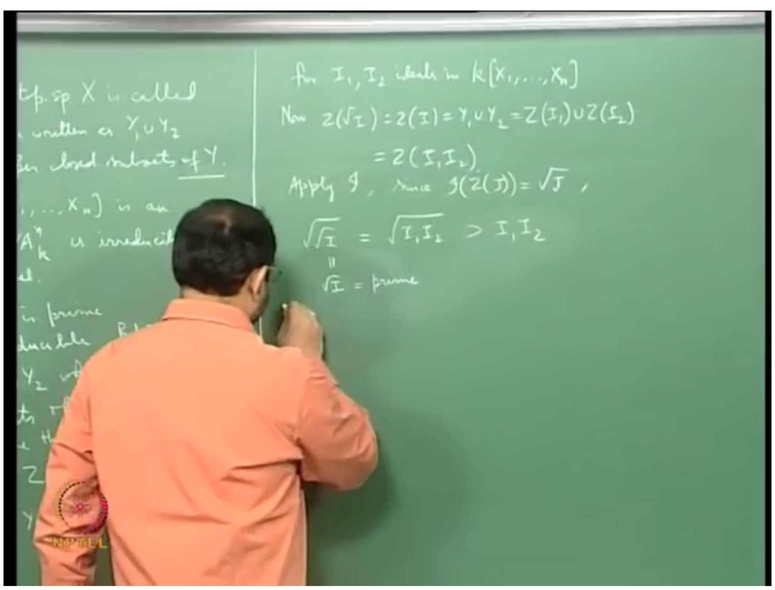

So now let me try to prove this part so let me recall definition so this is I am just recalling a topological space a subset y of a topological space so I am just abbreviating topological space top sp X is called irreducible if y cannot be written as y1 union y2 with y1, y2 non-empty proper close subset, okay. If a topological space can be written as y1 union y2 where y1 and y2 are non-empty proper close subsets then we say that the topological space is reducible, okay and the definition of irreducible is that it should not be irreducible, okay.

And what happens in the case of varieties as we will see what happens in the case of algebraic sets here namely close subsets of affine space you will see that you will get you will be able to break it down into not just the union of two we will be able to break it down in the union of finitely many subsets which are each which are themselves irreducible, okay and they will be called the irreducible components, okay but and that will be called the reducible decomposition of your given closed set, okay so that is where we are heading to, fine.

So the of course this definition as I told you implies that if y is irreducible then it is connected, okay because connected is a for it to be connected you should not be able to write it as a disjoint union of proper close subsets non-empty closed subsets, okay and that is certainly not possible if you cannot write it as a union of non-empty proper close subsets, okay. So irreducibility is a very strong form of connectedness I told you that irreducibility has lots of nice properties one thing that comes is that if space is irreducible then if a subset is irreducible then its closure is also irreducible so its irreducibility is not going to be affected if you add the boundary which is what you do when you take the closure and this is also true for connectedness if a set is connected then its closure is also connected.

And then but the other more important thing is that you see the more important thing about an irreducible space is that every open every non-empty open subset is dense and is itself irreducible that is another very important property, okay which I hope you would have tried as an exercise otherwise you should try it, it is pretty easy exercise. So what it tells you is that if you take an if you take an irreducible space and take a non-empty subset then you can test on that subset all those properties which will be preserved when you take a closure, okay that subset will because the closure of that subset will be the whole space and that is.

So you can test on any non-empty open subset an any non-empty open subset will be dense and that also tells you that if you take (())(20:40) non empty open subsets they will intersect, okay they cannot be disjoint from each other. So these are of the nice properties of irreducibility and later on it will come we will again look at it probably it is not so hard you can even check it of hand I think that the image of an irreducible set continues to be irreducible under a continuous map, okay which is a same kind of statement that you get for a connected set, okay.

Now I go to this theorem which I stated last time so the theorem is a following if I in the polynomial ring in n variables over k is an ideal then Z of I the 0 set of I the (())(21:39) points in An which are common 0’s of all the polynomials in I is irreducible this is a subset of this topological space you see this topological space is just kn given the Zariski topology, okay. So since is a subset of a topological space this definition applies and you can put the condition that this subset is irreducible and the theorem says that this is irreducible if and only if the radical of I is a prime ideal.

So what this means is that if I already started with radical ideal I mean if I already started with prime ideal then Z of I will be irreducible and conversely if I already started with the radical ideal then saying Z of I is irreducible is same as saying that the ideal itself is prime, okay and that essentially what gives you this correspondence in the middle that the affine sub the affine varieties in An they correspond to prime ideals, okay.

So well so the proof is quite straight forward so let us do both ways, so let us begin with let us assume a radical of I is prime suppose radical of I is prime, what do I have to prove? I have to prove Z of I is irreducible to show Z of I is irreducible, okay but actually you see Z of I is same as Z of radii, okay this is something that I told you last time two ideals J1 and J2 have the same set of common 0's if and only if the radical of J1 is equal to radical of J2, so it is since I and radii have the same radical namely which is radii they both have the same 0 set, okay and but even otherwise this is quite trivial to see directly, okay because radii mind you is defined to be all those polynomials some positive integral power of which lies in I.

So it is like taking the radical of an ideal is like expanding that ideal to include nth roots of its members positive nth roots of its members, okay that is nth roots for positive n, right? And so anyway so how do I check a set is irreducible so the (())(24:59) is I have to check that it is not reducible I have to check that it is not reducible so I have to check that if it can be written in this form with y1 and y2 as close subsets and if I assume that y1 and y2 are both non-empty and also that y1 and y2 are both proper that should not happen that is what I have to check.

So what I will do is I will assume that it can be written as in this form with y1 and y2 non-empty but I will assume that I will assume further that y1 is proper close subset and I will try to prove that y2 is not proper namely that y2 is everything if I do that then I am done, okay. So that proves that it is not reducible in other words that it is irreducible, so what I will do is suppose that Z of I is y1 union y2 where y1, y2 are non-empty closed subsets.

So here I have to go back to the definition and stress on something which I have not written there with y1, y2 non empty proper close subsets mind you of y, okay that is something that I had not written but I did say that in my last lecture so let me stress it when I say so I told you y is just a subset of a topological space what are the meaning of saying that subset of y is closed in y that you have closed subset of y this is the language of induced topology a subset of y is said to be close subset of y if it is gotten by intersecting y with this close subset of the ambient space the larger space X in which y sits, okay.

So when I write Z of I is y1 union y2 where y1, y2 are non-empty close subsets of Z of I what you must understand is that since Z of I is already closed in X it follows that y1 y2 are not just closed subsets of Z of I but they are actually close subset of X itself because the close subset of a close subset will continue to be a close subset, okay. See in other words if when I say y1 is close subset of Z of I it means y1 is Z of I in (())(27:43) with close subset of X, okay but then you see Z of I itself is close in X as if I intersect with another close subset of X the intersection of finitely many closed subsets is again a close subset for in the topology because the schemes for a topology if you take the schemes for close sets tells that you take any finite number of close sets and you take the intersection the result is again a close set in the whole space.

So what this tells you is that y1 which is supposed to be intersection of Z of I with a close subset of X is itself a close subset of X, so let me stress that suppose Z of I is y1 union y2 where y1, y2 are non-empty closed subsets of Z of I with y1 a proper subset of Z of I of course I will then try to prove that y2 is equal to Z of I that in other words that y2 is not proper, okay. Then note that y1, y2 are closed in the larger space X which is actually in our case An because by definition as I just told you y1 has to be Z of I intersection with a close subset of An but Z of I is already closed in An and the intersection of two close subsets of topological space is again a close subset.

So the reason is since Z of I is already closed in An by definition because you know that is how the Zariski topology was defined the Zariski topology was defined just by taking for the close sets subsets of the form Z of I, okay. So but what does that mean saying that y1, y2 are close in An means that y1 and y2 are the 0 sets of some ideals, okay. So this implies so y1 is Z of I1 y2 is Z of I2 for I1, I2 ideals in kX1 etcetera Xn, okay this is what you get but then what is now y1 union so if you take Z of radii which is Z of I which is y1 union y2 is actually Z of I1 union Z of I2, okay and you know but this is a same as Z of I1, I2 because this is something that we this is how we proved that sets of the form Z of I they form a topology by declaring such sets as such subsets as close subsets in fact what we proved is if you take Z of S1 union Z of S2 union etcetera up to Z of Sm where Si is our subsets not even ideals.

Then the union is just Z of the product S1 times S2 times etcetera Sm we proved that, okay. So this is Z of I1, I2, okay and you see it is (())(31:48) that you know I will use so I will now use the Nullstellensatz, okay. So what will happen is you see apply I if you apply I the Nullstellensatz tells you that since I of Z of J is a rad J, okay mind you this is the statement that involves the Nullstellensatz and that is valid for any ideal J, actually I of Z of J always contain rad J that is very easy to see, the non-trivial thing is to say that I of Z of J is contained in rad J namely it is a statement that if F if a polynomial is in I F Z of J namely if a polynomial vanishes on Z of J then some power of the polynomial is in J you cannot have a polynomial some power of which is not in J to vanish on all the 0's of J that cannot have, okay.

So this is this statement uses Nullstellensatz and if I apply this on both sides what I will get is I will get radical of radii is equal to radical of I1, I2, okay I just want to say that this contains I1, I2, okay this is the way I have to go, okay. So you see radical of an ideal always contains the ideal, okay because the radical is supposed to be all those elements some positive power of which is in the ideal and the first positive power of every element of the ideal is in the ideal so the ideal itself contain its radical ideals and see the fact is that this is radii, okay because taking rad more than once is not going to change anything and this is prime.

So what you are getting is you are getting a prime ideal contains a product of ideals now you see we use this following fact for commutative algebra it is a very simple fact that if a prime ideal contains a finite product of ideals then it has to contain at least one of them, okay.

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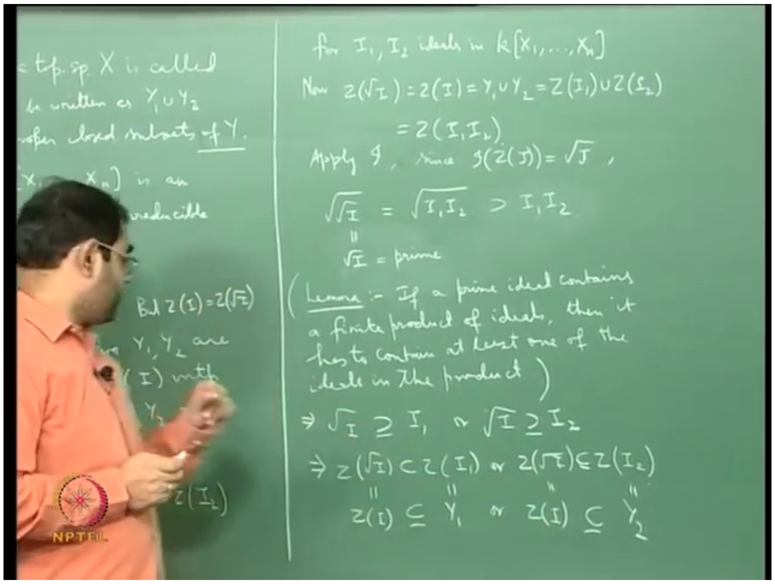

So here is a lemma is a very simple lemma from commutative algebra if a prime ideal contains a finite product of ideals then it has to contain at least one of those ideals one of the ideals in the property, of course you know of course all these is in this statement the background I must assuming is that you are working in a commutative ring with 1 and you are having finitely many ideals and if you have finitely many ideals J1, J2 through Jm then there product is J1 dot J2 dot etcetera Jm which consist of just finite sums of products of m tuples taken from the Cartesian product of all the J's, okay and if a prime ideal contains the product J1, J2, Jm then it has to contain some Ji and this is just this is very easy to see because is just a definition of prime ideal that if a prime ideal contains the product then it has to contain one of the finite product then it has to contain one of the factors of the product it is just a restatement of that if you try to work it out.

So what this lemma will tell you is that radii has to contain I1 or radii has to contain I2 but then this now you apply Z, okay you apply Z to take the 0 locus and remember that when you apply Z the inclusion is revised, so what you will get is Z of rad of I the Z of radii is contained in Z of I1 or Z of rad of I is contained in Z of I2 this is what you get and mind you but Z of rad of I is mind you is just same as Z of I and Z of I1 is y1 and Z of rad I is again is Z of I here and this is y2.

So what you are saying is Z of I is contained in y1 or Z of I is contained in y2, okay but what we starting with was that y1 and y2 are contained in Z of I, so what this means is that either Z of I is equal to y1 or Z of I is equal to y2 but then you are assuming that Z of I is not equal to y1 so what this will tell you is that Z of I has to be equal to y2 and that tells you that you cannot reduce Z of I, okay.

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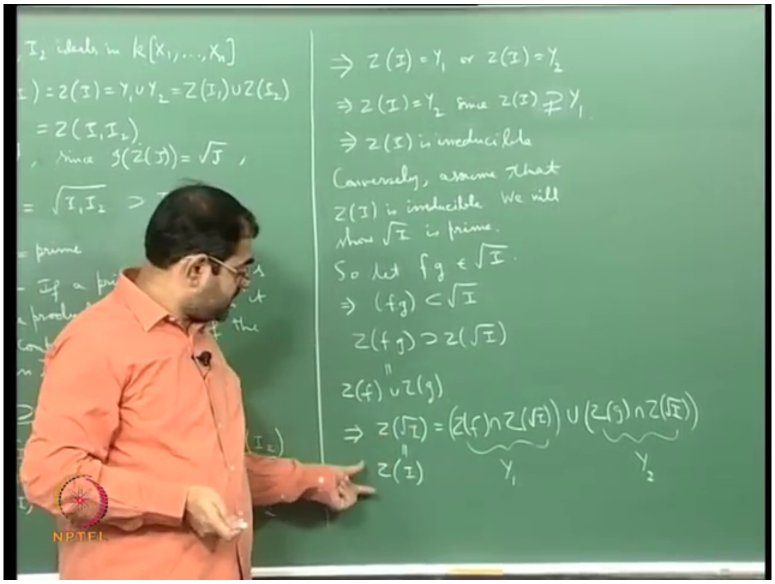

So let me write that down this implies that Z of I is equal to y1 or Z of I is equal to y2 this implies that Z of I is equal to y2 since Z of I is supposed to properly contain a proper subset of, sorry supposed to properly contain y1, okay and this implies that Z of I is irreducible, okay.

So we have started with radii prime, okay and we are able to reduce that Z of I (())(38:07), now we will do the other way now we will assume Z of I is irreducible and show that radii is prime, okay. So conversely assume that Z of I is irreducible will show radii is prime, okay. So how do you so is again a translation you just have to check the condition for a prime ideal you have to take a product, how do check something is a prime ideal? How do you check an ideal is a prime ideal you take a product of two elements of the ring as belonging to the prime ideal and demonstrate that one of the two factors of the product is that ideal, okay.

And so what we will do is so let f times g belong to radii let the product be in radii, okay. So what this will tell you? This will tell you that the ideal generated by fg f into g is a subset of radii, okay because if an element belongs to an ideal and the ideal generated by that element is also in the radial because the ideal generated by an element is just simply multiples of that element by ring elements, okay. So if I now you apply Z if you apply Z I will get Z of fg contains

Z of radii, okay and but know Z of fg is just Z of F union Z of g, okay this is exactly the same statement that Z of I1 union Z of I2 Z of I1, I2, okay so Z of fg is Z of f union Z of g, okay.

And of course when I write Z of F for a single element F by that I mean Z of a single element F is same as Z of the subset consisting of the single element F and this is also the same as Z of the ideal generated by F they are all one and the same, okay. So what happens is that so you know so now can so what this tells you is that you see Z of radii has been written as Zf intersection Z of radii union Zg intersection Z of radii, okay you see Z of f is a close set this union contains this so you intersect this with the smaller subset you will get back the smaller subset.

So if I intersect this with Z of radii I should get Z of radii and that is and intersection as you know distributes over the union by simple set theory, okay so you get this but what you must realize is that this is if I call this as y1 and if I call this as y2 what you will notice is that y1 is a close set it is a close set because it is the intersection of two close sets so it is the close set, similarly y2 is a close set and you have written Z of radii as a union of two close sets but mind you Z of radii is same as Z of I but what is assumption on Z of I is the assumption of Z of I is that it is irreducible. So the moral of the story is that either one of these is empty, okay and if both are non-empty then one of them has to be the whole Z I itself, okay.

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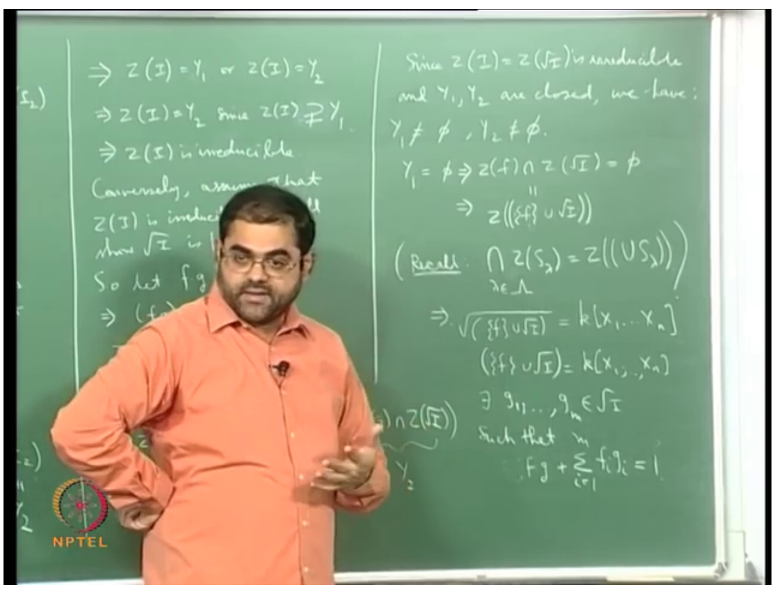

So let us write that out let me write that out here since Z of I could Z of radii is irreducible we have and y1, y2 are closed we have the following possibilities so I want to say y1 is non-empty y2 is non-empty, okay so let us write down y1 if y1 is empty this will tell you that that means so that will imply that Z of F intersection Z of radii is empty then this intersection is supposed to be Z of F union Z of the ideal generated by f union radii, right? This is what it is supposed to do by definition because what is how do you show that the close sets form a topology, how do you show the algebraic sets form a topology.

So what you if you recall you can recall that set of if you take intersection over alpha or lambda in capital lambda some indexing set of Z of S lambda this is just Z of the ideal generated by the union of S lambdas, okay if S lambdas are subsets of the polynomial ring, okay and you take the EZ S lambdas this are collection of close sets how do you show that the intersection of see how do you show that the intersection of an arbitrary collection of closed sets is closed it follows from this calculation, okay.

So Z of f intersection Z of radii will be Z of f union radii ideal generated by f union radii and so what that will imply is if I apply I to both sides if you apply I to both sides and use again use this Z of I of Z of J so rad J so what I will get is if I apply I to both sides you will get radical of the ideal generated by f union radii will be if I apply I to the null set, okay then I get the whole ring, okay because what is I of a subset it is all those polynomials which vanish on the subset, okay I of a subset of affine space is all those polynomials in the polynomial ring which will vanish on that subset but if that subset is empty there is nothing to test every polynomial will satisfy this condition.

Therefore I of null set will be just a whole polynomial ring, okay and you know if this happen I mean I essentially have to show that if this happens I am done otherwise I have to proceed further, okay. So what does this means? This means that f the ideal generated by f union radii is itself the polynomial ring see so this again a fact I am using that you know if an ideal if the radical of an ideal contains the unit then the ideal itself contains a unit because saying that the radical of an ideal contains a unit say 1 tells you that there is some power of this which is equal to 1 there is some power of this the ideal generated by this union which is equal to 1 but then that if the some power of an element is equal to 1 then that element itself is a unit, okay.

That means that the ideal generated by this itself is the whole polynomial ring, okay and what this will tell you is that so you know there is some so what this will tell you is the following that so there exist a g1 etcetera gm in radii such that sigma fg plus fg plus sigma over I, I equal to 1 to m fi gi is 1 this is what it is, okay I mean an element in the ideal generated by the union like this will look like this you will have to pick actually you have to pick finitely many elements from this subset and then take ring linear combinations of that and such a ring linear combination is equal to 1 because 1 is there on the right side.

And now what I want to say is that from this we will have to say that so you know if this happens that is if y1 is empty it should more or less follow that if y1 is say that again yes, y1 is empty, yes.

Student: radii is equal to Z of g intersection Z of radii

Professor: Z of,

Student: (())(49:01) is equal to the remain thing on the (())(49:04)

Professor: Oh, I see you will just get Z of radii is equal to Z of I g intersection Z of radii and then you will get therefore,

Student: We get Z of radii and then Z of g.

Professor: okay.

Student: So if we apply I on both sides you will get a g belongs to radii.

Professor: You will get g belongs to radii, right? You will get g belongs to radii, okay.

So this is not required you are right, okay so let me get rid of this, good let me get back to the easier part of the argument but I would strongly encourage you to think about that whatever I wrote down, okay. So and the fact is that that also will lead to something, okay that will also lead to what you want but you have to so the point is you have to keep translating back to the geometric side if you are on the geometric side you should translate to the ideal side if you are on the ideal side you should translate to the geometric side by applying this I and Z appropriately, okay.

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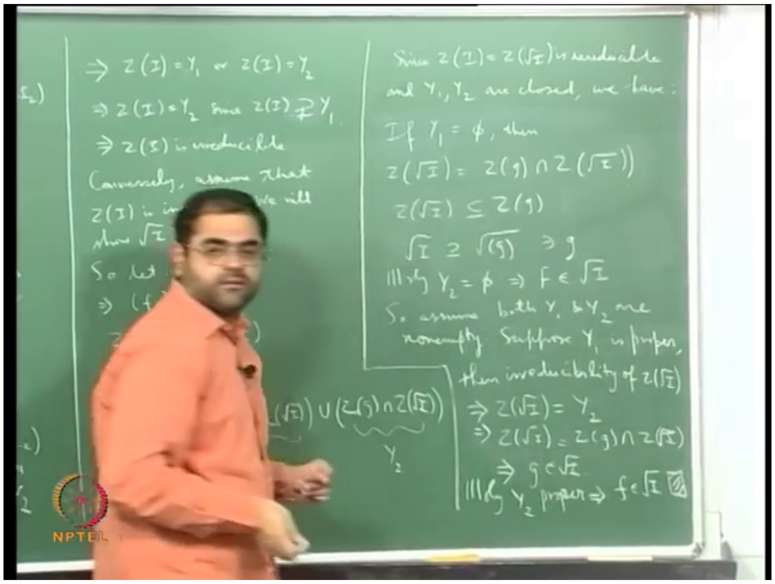

So as you as one of you has rightly pointed out what you should what one does is that if y1 is empty then I mean Z of it is obvious that Z of radii is just Z of g intersection Z of radii and so which means that Z of radii is contained in Z of g, right? And because the right side is contained in Z of g and now you apply I you will get I of Z of radii which is just rad of radii which is radii containing I of Z of g will be just radical of the ideal generated by g, okay and to which g belongs, okay.

So if y1 is empty you get g is in radii, okay alright. So you assume y1 is not empty, okay if y2 similarly if y2 is empty then you will get f in radii mind you f times g is in radii so you have to prove either f is in radii or you have to prove g is in radii, okay and y1 equal to empty directly gives you g is in radii, similarly y2 is empty implies f is in radii, okay and you are done. So assume both are not true, okay so assume both y1 and y2 are non-empty, okay so you will have to trash out all the possibilities, okay suppose y1 is proper then irreducibility of you are right Z of radii implies that if this is proper then that cannot be proper if y1 is proper then y2 cannot be proper so y2 has to be everything, okay and in that case you see g belongs to radii, right?

So you are done essentially so let me write that out suppose y1 is proper then irreducibility of Z of radii implies that Z of radii is y2, okay and this implies that, again the same argument literally Z of radii is equal to Z of g intersection Z of radii and this will imply that g I guess this will imply g is in radii, okay so if y1 is proper you will get g is in radii, similarly if y2 is proper, okay you will get f is in radii, okay. Similarly y2 proper will imply f is in radii and that completes the proof, okay, right?

So it is very clear that an ideal here is prime if and only if the radical of an ideal here is prime if and only if the corresponding 0 locus here is irreducible, okay that proves it, okay. So I will stop here and then we will continue in the next part with some examples, okay.