Basic Algebraic Geometry
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Lecture 42
Any Variety is a Smooth Hypersurface On an Open Dense Subset

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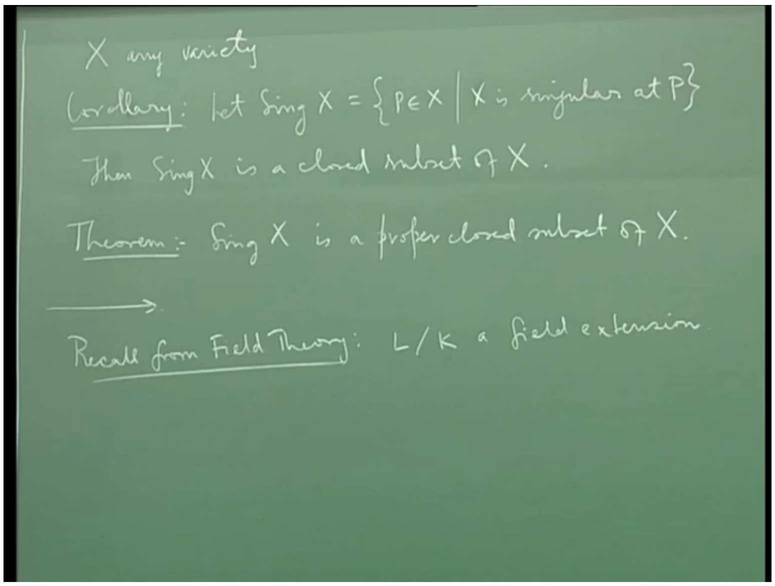

I am going to remind you about few things from Field Theory. All right. So recall from Field Theory. So if you have L over K, the field extension, you take a field extension L over K, then we say when an element of L over K is algebraic, when an element of L is an algebraic element over K, it is an element of L is said to be algebraic over K if it satisfies polynomial, non-trivial polynomial with K coefficients in one variable.

So and if every element of L satisfies such a polynomial over K, then we say L over K is an (algeb), we say L over K is an algebraic extension. That is every element of L is algebraic over K. And if you have elements of L which do not satisfy any polynomial, any such polynomial with coefficients in K, we call such elements as transcendental elements. And well, if L over K is finite extension in the sense that if you treat L as vector space over K, if dimension of L as vector space over K is finite, then a finite extension is always algebraic.

And in fact, but of course an algebraic extension need not be finite. Now you see if L over K is not algebraic, then it has transcendental elements. There are elements which are non-algebraic and then given a set of transcendental elements, you can define when that set is algebraically independent over K. And this is analogous to the definition of linear independence over K. So a set of elements of vector space is said to be linearly independent if they, if no finite subset of that set satisfies a non-trivial linear relation with coefficients in K.

So in the same way a set of elements of L which is said to be algebraic over, is said to be algebraically independent over K, if any finite subset of that set does not satisfy any polynomial relation with coefficients in K. So if you give me finitely many elements of L, then those finitely many elements are said to be algebraically independent over K if those finitely many elements are not zeroes of some polynomial over K in those finitely many variables, in as many finitely many variables. Okay, a non-trivial, it should not be, they should, that element, that tuple of elements should not be zero of a polynomial in as many variables with K coefficients.

That is when you say those elements are algebraically independent and this is when you take a finite set of elements. And an infinite set of elements is said to be algebraically independent if every finite subset of that infinite set is algebraically independent. And then there is this just like in the case of vector space you define the dimension as the max, it is the cardinality of your maximal set of linearly independent elements.

In the same way if you have a field extension which is not algebraic, then you can define it is so called transcendence degree. It is degree of transcendence to be the cardinality of your maximal algebraically independent set. So if there are elements L which are not algebraic over K, you can try to, your transcendental elements and then you try to take, try to find a maximal subset of these such elements which are algebraically independent.

And that cardinality will always be the same. Any two, just like if you take any two bases of a vector space are bijective, the same way any two sets of maximally, any two maximal sets of algebraically independent elements will be bijective and that and the cardinality of that set is called the transcendence degree of L over K. And we say that, L over K has, we say that L over K has a separating transcendence base. If you can find that, you can find that set of elements which forms a transcendence basis, a transcendence basis is just given by maximal set of algebraically independent elements. If you can find your transcendental basis, transcendence basis, such that L over the field adjoined to K are given by those, given by the transcendence basis is finite algebraic extension.

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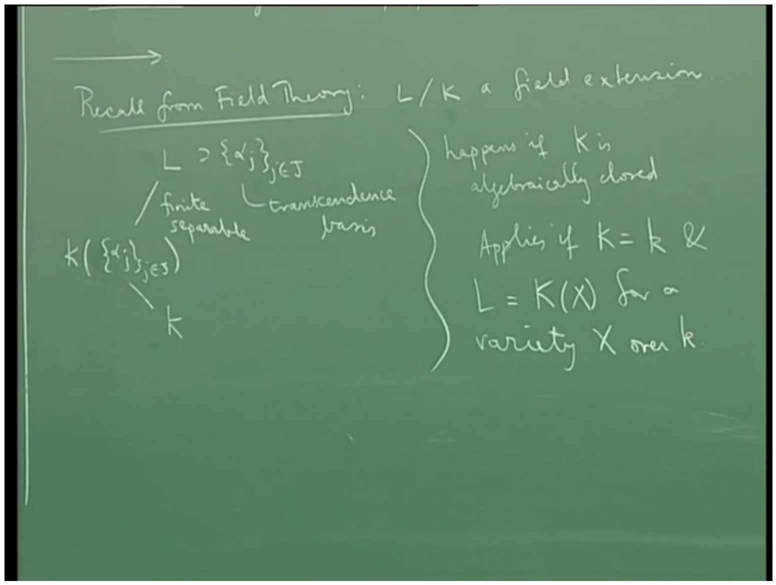

So there is this, you want picture like this, so L, so you have K and in L you have this family of elements alpha j where j is in some indexing set J. Then you have the field generated by these alpha js. So this is the, so this is a transcendence basis. And transcendence basis is, it just a, it is just the analog of basis. So it consists of maximal subset of algebraically independent elements. A usual basis will consists of maximal subset of linearly independent elements.

A transcendence basis will consist of maximal set of algebraically independent elements. So you want a situation where you can find a transcendence basis and you want this extension to be finite, separable. So you want, I mean this is the best thing that can happen with the field extension. The best thing that can happen with the field extension is that the field extension splits up into two extensions like this.

The first one is given, we say that this extension is purely transcendental extension because it is the extension which contains only, it is gotten by simply adjoining all the transcendental elements from that all the elements of transcendence basis. And then this over this will only be a finite extension. This will be algebraic because there will not be, already this contains a maximal set of algebraically independent elements. This so, this cannot be transcendental. Because if there is an element of this which is transcendental over this, then I can take that element and add it to this to get a bigger transcendence basis and that is not possible.

So this over this has to be algebraic and the fact is that you can make, it would be nice if this is a finite separable extension. And that is, that always happens if the field K is algebraically closed. So this, so all this happens if K is algebraically closed. But in fact it happens even under more weaker condition. It happens if K is what is called a perfect field. All right. But let us not worry about it. Basically what the definition of a perfect field is that, if a field is of characteristic zero, it is called perfect and if it is of characteristic p, it is called perfect if you can always find pth roots.

If you take K power p, you should get K. And you should be able to find pth roots for all your elements. So an algebraically closed field is always a perfect field because finding pth roots is just amounting to solve equations and over an algebraically closed field you can always solve equations because that is the definition of algebraically closed. Therefore an algebraically closed field is always perfect. And for a perfect field you have this very beautiful situation. Of course it is important that I need a finite, separable extension here.

And this part will be purely transcendental extension. And this is some field theory, all right but important thing is that we will not apply it when K is small k or algebraically closed field and when L is a function field of a variety. So applies if K is k for k is the algebraically closed field over which we are doing algebraic geometry. That is the field over which we are studying varieties. And L is the function field of X for a variety X over k. So this is our application.

So in all these things our viewpoint is that this k, this K is our k and this L is a function field of variety and then that theorem is that the function of variety is a finite, separable extension of purely transcendental extension of k. You know that if L is KX, then the transcendence degree of KX will over k, will give the dimension of the variety. Therefore but the transcendental degree will be just the cardinality of this J because the transcendence degree is just the size or cardinality of a transcendence basis. So you know if the variety X has dimension R, then this J will have R elements. So this will just be, this will be k and this will be k adjoined with R in determinants or R transcendental elements.

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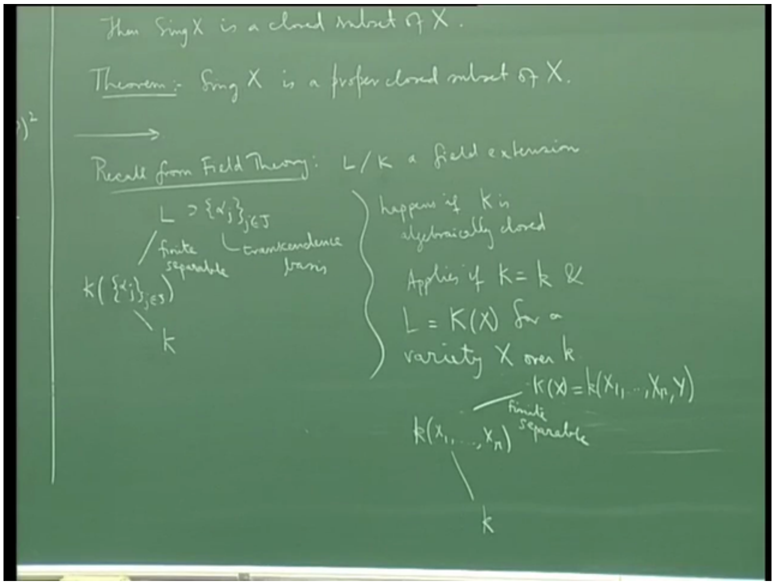

So the picture will look like this, you will get k and then you will have k of X1, etcetera Xr and then you will have L of, L which is actually KX. I do not have to use L, so let me just and this part will be finite, separable. This is it, so this is the picture that we need. If you take the function field of any variety X, then that function field what kind of an extension is it of K, you can break it up into two pieces. The first piece is purely transcendental extension. It is an extension which is gotten by simply adding as many variables as the dimension of X.

So this part will correspond, will contribute to the transcendence degree. This will give you the transcendence degree. And this part will not have any transcendence, this will, because all the (transc) because X1 through Xr are already your maximal set of algebraically independent elements. So this over this will be algebraic and not only algebraic, it will actually be finite. And of course finite extension is always algebraic and not only that it will actually be a separable extension.

And the separability is a technical condition and it is very, very important. And the reason why it is important for example in Field Theory is that whenever you have finite, separable extension that can always be generated by single, by adjoining a single element and that is called the so called theorem of the primitive element. The theorem of the primitive element says that whenever you have field extension which is finite and separable, then the bigger extension can be gotten by adjoining a single element to the smaller field. So this finite, separable extension actually tells you that this K of X is the k X1, etcetera Xr, Y. You can find out Y in KX, such that you adjoin this Y to this field, you get this field and that is all of KX.

So this you get this Y because of the fact that this over this is finite, separable and you are using so called theorem of the primitive element which says that a finite, separable extension can be gotten by adjoining just one element. And in fact you see the theorem of the primitive element says more, it says that if you take set of finite, finitely many generators for this, then that primitive element is even linear combination of those generators with coefficients coming from the smaller fields. So the moral of the story is that you have very nice description in terms of fields if you are working with variety, I mean if you are working with function field of a variety.

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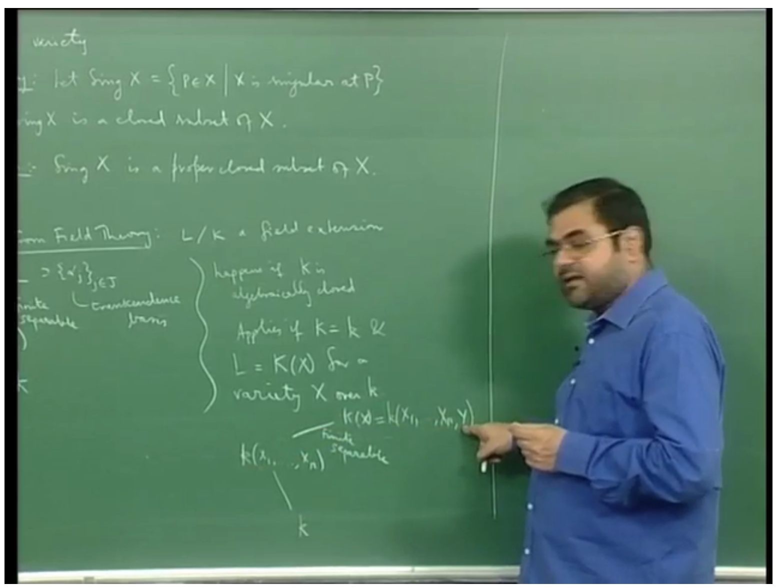

Now you see, now what I want to you to understand is that well, you see you take, now you take X to be a variety over K. Suppose the dimension of X is r, and you take KX as field of rational functions, then you can find of course X1 through Xr which you can think of as are algebraically independent rational functions on X. So X1 through Xr are algebraically independent rational functions on X.

And then if you take the subfield generated by these r independent, algebraically independent rational functions, you get this field and then KX over that is a finite, separable extension. Therefore by the theorem of the primitive element, you can get this from this by adjoining a single element Y which is yet another rational function on X. And the point is that this Y is actually belonging to KX and KX is, KX over this intermediate field is finite, so it is algebraic. So in fact Y satisfies your polynomial with coefficients here, Y satisfies polynomial with coefficients here.

And the moral of the story is that Y if you clear denominators, Y will satisfy your polynomial with coefficients in the, polynomial ring in r variables. You can have that polynomial to be an irreducible polynomial. And therefore you know that polynomial in affine space with so many variables, its zero set will give hypersurface. And that hypersurface, its function field will exactly be this. So this argument tells you that the functions field of X is the same as the function field of hypersurface in r plus 1 dimensional affine space where r is a dimension of X.

And we have already seen last lecture that I think last lecture or maybe couple of lectures ago, that if two varieties have the same function fields, then they are birational. So all this argument tells you together that any variety X of dimension r is birational to hypersurface in affine r plus 1 space.

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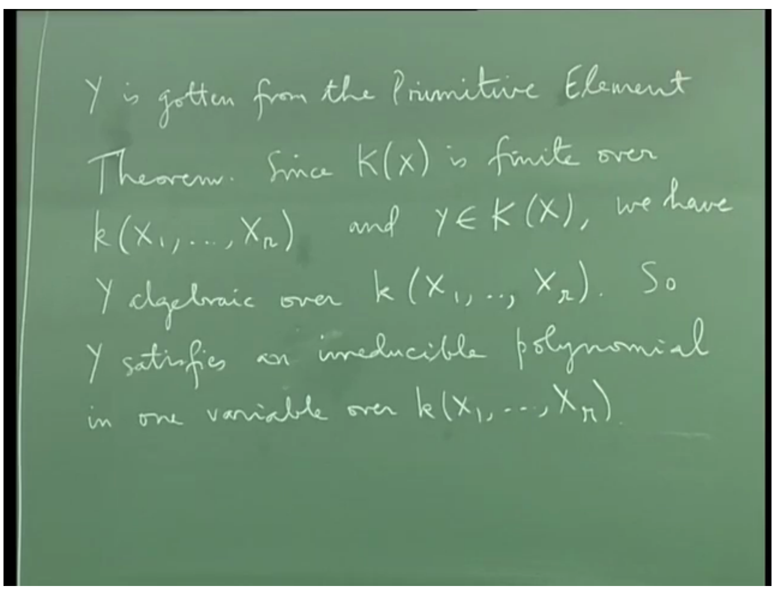

So let me write that down. See Y is gotten from the primitive element theorem. And in fact since K of X is finite over k of X1, etcetera Xr and Y is in KX, we have Y algebraic over k of X1, etcetera Xr. So Y satisfies a polynomial, an irreducible polynomial. It satisfies an irreducible polynomial in one variable over kX1, etcetera Xr. But what are the elements of k round bracket X1 etcetera Xr, they are actually quotients of polynomials in those r variables. When you put square brackets, it is the polynomial ring in r variables but when you put round brackets, you go into its quotient field. So Y satisfies an irreducible polynomial in one variable.

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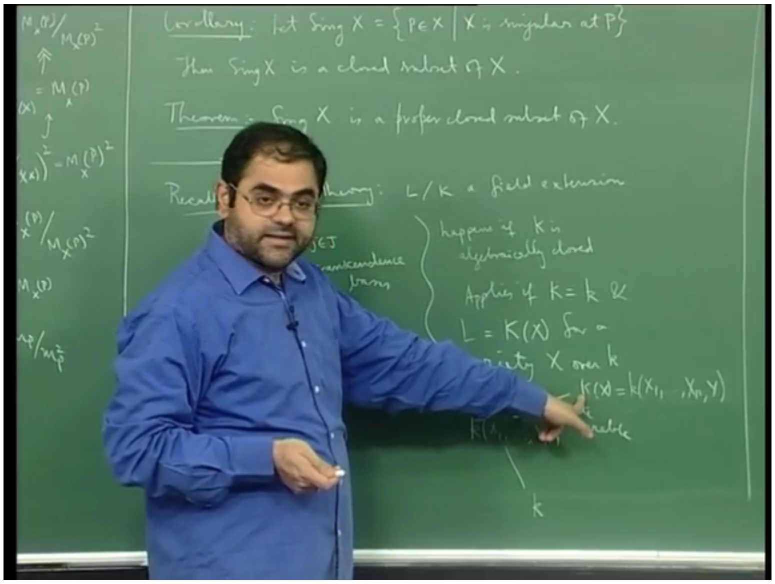

And in fact if you take the degree of that polynomial, that degree will be the same as the degree of this finite extension. For any finite extension generated by a single element, the degree of the finite extension is the same as degree of the minimal polynomial of that element and that the minimal polynomial of element is the unique irreducible polynomial that element satisfies. It is the polynomial of least degree that, that element satisfies.

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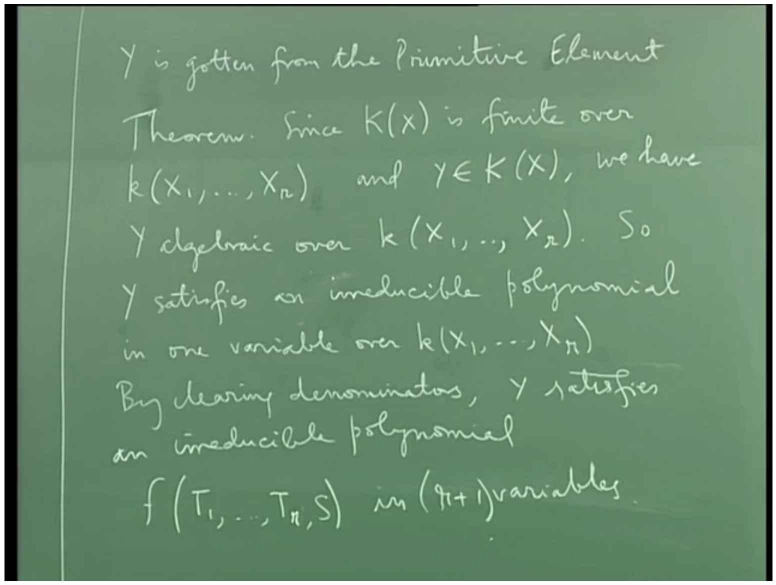

So this, so here I am looking at the minimal polynomial of Y over this extension. And so now the coefficients of this, in fact you can even make that polynomial monic if you want. But the point is that is because the leading term can always be made 1 by dividing by its coefficient. But the fact is that if, but even if you think of these as coefficients of polynomials in r variables and you clear denominators, what you will get is that you will get that Y satisfies your polynomial in r plus 1 variables.

If you clear, by clearing denominators, Y satisfies a polynomial, an irreducible polynomial in r plus 1 variables. So what you are doing is Y satisfies an irreducible polynomial in one variable, call that one variable as S, and the coefficients are all here. And but the coefficients are therefore coefficients of polynomials in the X size. If you clear denominators, finally and you rewrite instead of the X size, you put the Ts. You will get a reducible polynomial in r plus 1 variables which the whole tuple, X1 through Xr upto Y satisfies.

And now see the, this is the polynomial which is an irreducible polynomial in n, so I should not put n here, it should be r, sorry it should r. So this is an irreducible polynomial in r plus 1 variables. What does zero, what is that zero set? It is a hypersurface in A r plus 1.

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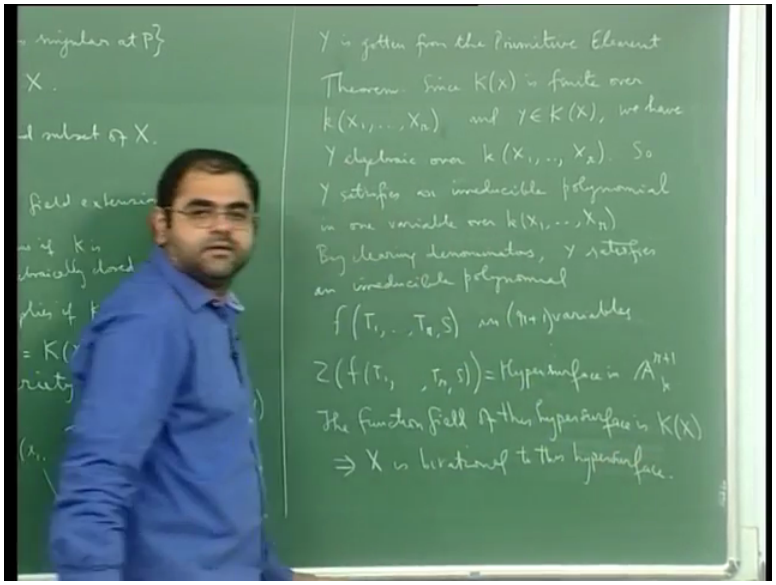

So the zero set of, f of T1 etcetera Tr, S, this is a hypersurface in A r plus 1. We have already seen this, hypersurface in affine space is simply given by the zero set of a single irreducible polynomial. Hypersurface is by definition, codimension one sub-variety. An irreducible closed subset of dimension 1 less, then the dimensional affine space. So this is a hypersurface in A r and for this hypersurface what is the function field?

The function field will precisely be KX. This hypersurface, the function field of this hypersurface, how do you get it? What you have to do is you have to first get its affine coordinate ring which is gotten by taking this polynomial ring in r plus 1 variables and you have to divide by f, because that is the ideal of the hypersurface. The ideal of the hypersurface is generated by f, so you have to divide by the ideal generated by f and then you will get affine coordinate ring of the hypersurface and then you have to take its quotient field.

But then if you divide by f, you the moment you divide by f and then invert everything, you will get back KX because that is how f was gotten. So what you will get is that the function field of this hypersurface is KX by definition, the function field of this hypersurface is exactly KX. Because it will be the polynomial ring in this r plus 1 variables. You divide by the ideal generated by f, okay and then you will get finitely, you will get integral domain.

You take its quotient field, so essentially what you are doing is you are inverting everything except that you are inverting, it is like taking the quotient field of the polynomial ring in this r plus 1 variables but putting additional condition that this, f of this is equal to 0. But putting the condition that f of that is equal to 0, will give you the smaller field which is KX by construction. So the moral of the story is that now the function field of the hypersurface is same as function field of X. Now we have already seen that if two varieties of the same function field, then they are birational. So this implies that X is birational to this hypersurface.

So X is birational to this hypersurface, so what this tells you is that I mean the whole purpose of my recalling all this though I did it very quickly and on your part this will demand some more reading, is that, is just to tell you that if you have variety of dimension r, then it is birational to a hypersurface in A r plus 1. So this is the fact that I need to use. I am going to use the fact that given a variety of dimension r, it is having an open set which is isomorphic to an open subset of a hypersurface in r plus 1 dimensional affine space.

So roughly what I am saying is following thing, I am saying take a variety of dimension r, there is an open set where the open set looks like the zero set given by single equation in an affine space of dimension one more. The variety X has dimension r, you take affine space of dimension one more which is A r plus 1. And there is a hypersurface there. And an open subset of the hypersurface which looks like the open subset of your variety. So I am just saying that given a variety of dimension r, there is an open set which looks like the zero set of a single equation in affine space of dimension one more. So that is this fact.

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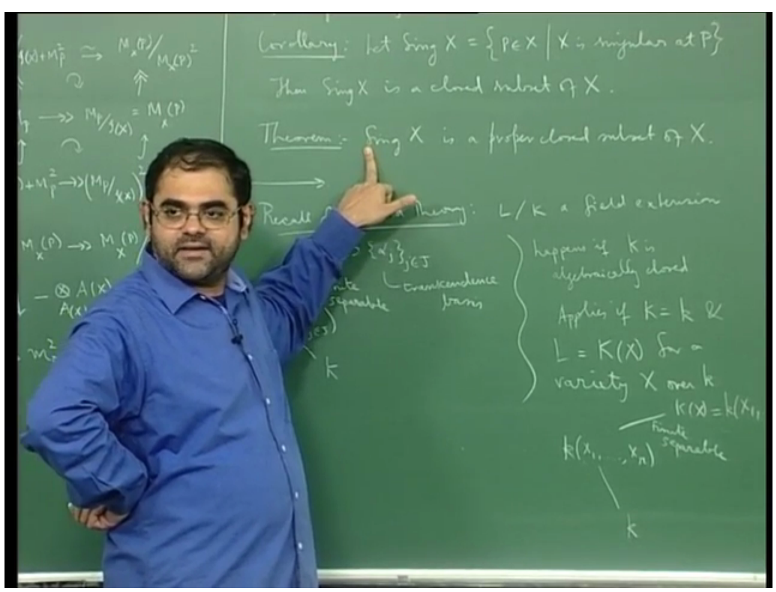

And now I use that to prove this theorem and I do that in the following way. What I say is well, if I take, so this is standard taken algebraic geometry, you want to prove something about an arbitrary variety. You want to prove some open condition, something that is happening on an open set. So for example, here I am trying to, what I am trying to show is that I am trying to say that the condition of non-singularity is open because its compliment is, it is equivalent to saying that the condition of singularity is closed. And I want to say it is nonempty, open. So what I have to show is that the set of points where X is non-singular is actually nonempty, open set.

So I am just saying that if I prove that you take hypersurface, for hypersurface if you show that the singular points is proper closed sets, then I am done. Because if I show for any hypersurface, the set of singular points as proper closed set, then I am saying that there is open set which consist of good points. But opens, but any variety has an open set which is isomorphic to an open subset of the hypersurface. Therefore it will have good points. Because under isomorphism of varieties, local rings are preserved.

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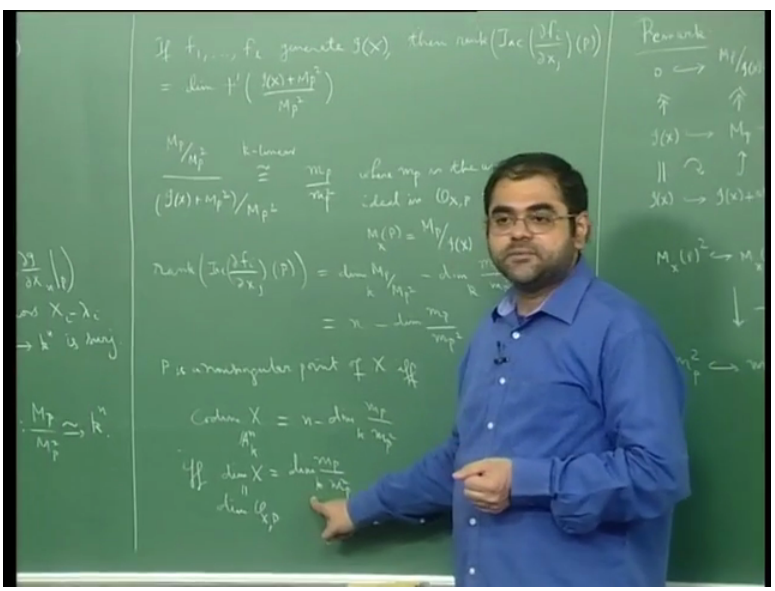

Since local rings are preserved, the smoothness condition will be preserved. So under isomorphism of varieties, a smooth point will go only to a smooth point. So if you give me any variety, then I know that if I can, I know that there is an open subset of that variety which is isomorphic to an open subset of a hypersurface. And I know open subset of the hypersurface will contain smooth points.

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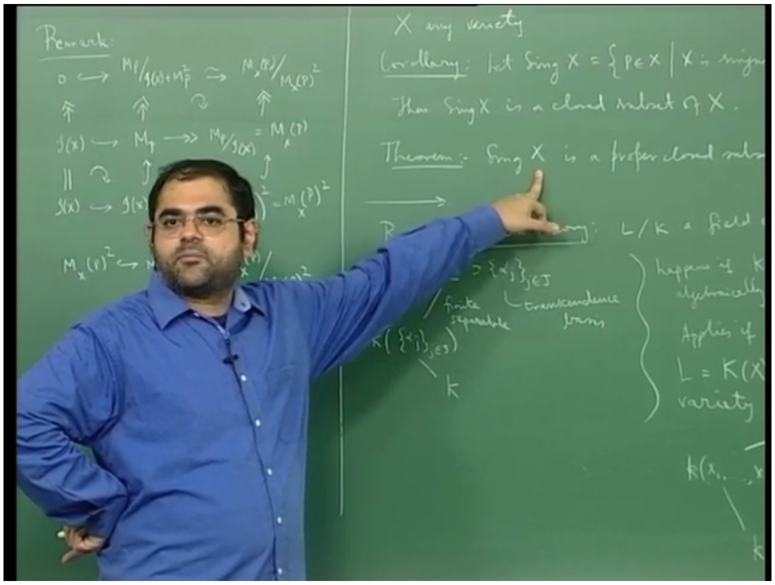

If I know that, then I will know that X will contain smooth points. The moment X contains smooth points, this becomes a proper subset, so it become a proper closed subset. So this theorem is just to show that every variety contains at least one smooth point. And I prove this theorem by reducing to the case of hypersurface because I know every variety of dimension r is birational to hypersurface in r plus 1 dimensional affine space.

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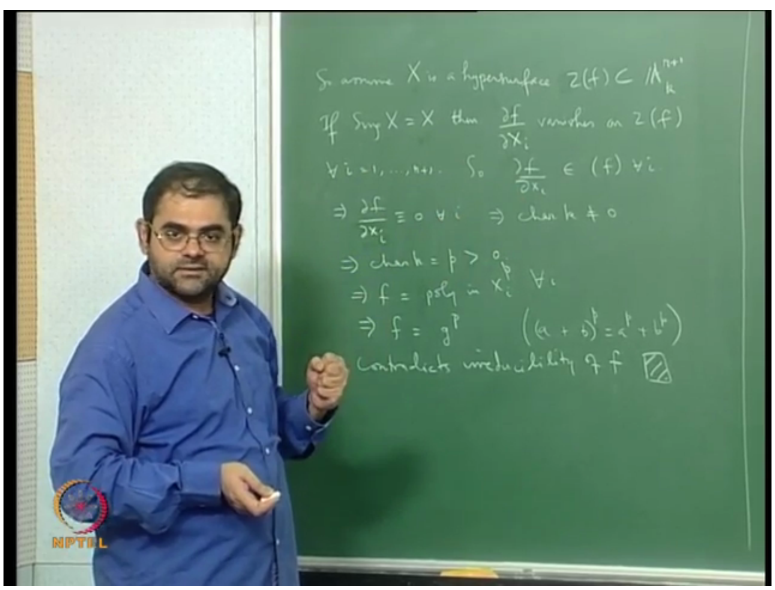

So because of this fact assume that X is, assume that the variety X is hypersurface. So now the proof is just probably few lines. So assume X is hypersurface in, z of f in A r plus 1. What do I have to show? I have to show the singular points of X is a proper subset. So if let us look at this, let us go by contradiction, to the singular points of X is all of X, then what it means is that doh f by doh Xi vanishes on z of f for all i from 1 to etcetera up to r plus 1.

So here now you know the Xis are the affine coordinates in r plus 1 dimensional space. And the singular points is the whole of X, that means these, I mean these components of the gradient of f, they all need to vanish. Your point will be a smooth point if at least one of them does not vanish at that point. But the only way that every point is not smooth is that all of these guys vanish. But then you see, what does this mean if, see after all these are polynomials and if they vanish on the variety, then they have, then you know that by the Hilbert Nullstellensatz they have to be some power of, I mean they have to be in the ideal generated by f.

So you see, so by the Nullstellensatz, you see doh f by doh Xi, they all belong to the ideal generated by f for every i. That this is because Nullstellensatz who says that if a polynomial vanishes on a variety, then some power of that polynomial should be contained in the ideal of the variety. So if this vanishes on z of f, then some power of this is contained in the ideal of f, ideal of z of f but ideal of z of f is just the ideal generated by f. And if some power of that is contained in f, then that itself is, has to be contained in f because f is irreducible.

And therefore the ideal generated by that is prime. All right. So you will get this. But then what is the degree of f? This is more than the degree of, this will have degree at least one less, right? Because we have taken a partial derivative. And therefore this condition will tell you that this is, these are all identically 0. So this will imply that doh f by doh Xi are identically 0 for all i. It will tell you that all these partial derivatives are identically 0. Now you see if you are in characteristic 0, this cannot happen.

Because if, when will the partial derivatives of all the variables be 0, will be 0 for a polynomial, if those variables do not appear in the polynomial at all. Okay, if the variable appears in a polynomial and if you are in characteristic 0, then that variable come with a coefficient. All right. So when you take a partial derivative, the coefficient will not kill it because you are in characteristic 0 and the partial derivatives cannot vanish.

So the fact that all the partial derivatives vanish, so this implies that characteristic of k is not 0 because in characteristic 0 this cannot happen. And in characteristic, so if characteristic is not 0, then the characteristic of k is positive, is a prime positive. And in positive characteristic if a polynomial is in a certain variable, partial derivative is 0, it means that the polynomial should be a polynomial in the pth power of that variable.

So this implies that f is equal to f is polynomial in Xi power p for every i. This is a result from characteristic p. And this will tell you that f is g power p, because since you are in, since your field is algebraically closed, you can take pth roots of all the coefficients and you can use the fact that a plus b plus, I mean if you want a plus b whole power p, is a power p plus b power p. So if a polynomial is a polynomial in all this Xi power p, then you take the pth root of all the coefficients.

Then you can write the polynomial itself as g power p, but this will contradict the fact that f irreducible, contradicts irreducibility of f. So this contradiction tells you that if X is hypersurface, then the singular points cannot be the whole of X. That means there are smooth points. But already we have seen as a corollary, the singular point is a closed set. So that means the compliment of this closed set can be the set of smooth points, is nonempty.

So the set of smooth points of hypersurface is irreducible, nonempty and dense. And since any arbitrary variety is birational to hypersurface, each singular points will also be only a proper closed set. So that proves that on any variety, you have open, dense, irreducible subset consisting of smooth points, non-singular points. Okay and the bad points may be the singular points, they only correspond to a proper closed set. Okay, so I will stop here.