Basic Algebraic Geometry

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Lecture 40

How Local Rings Detect Smoothness or Non-singularity in Algebraic Geometry

So this is again back to non-singularity, okay. So if you recall that if t is a point in X which is an affine variety in say A n, okay, then P is called non-singular point of X, okay. Or we say X is non-singular at P, okay. We say X is non-singular at P if for some given set of generators f 1, etc. f L of the ideal of X, okay, the rank of the Jacobian matrix of f 1, etc. f L with respect to the variables X 1, etc. X n at the point P is equal to co-dimension of X which means dimension of the affine space, the ambient space minus the dimension of X which means it is n minus dimension of X, okay.

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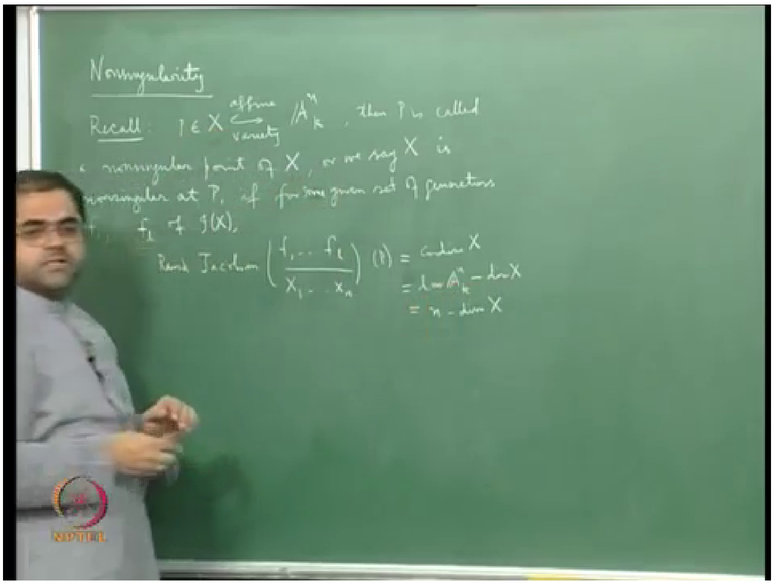

So this was the definition of non-singular point. A point of an affine variety is called a non-singular point or smooth point or we say that the variety is non-singular at that point, okay, if for some given set of generators for the ideal of the variety if we copy this Jacobian which is given by the partial derivatives of all these with respect to these variables. That turns out to be equal to the co-dimension of X, okay.

And co-dimension is the dimension of X taken away from the bigger space in which X sits which is affine space, okay. So you know this is just doh f j by doh x i at P. It is just this matrix, okay. And where of course the x i are all affine coordinates of A n.

So X 1, etc. X n if you take their coordinates on A n, okay, so this is the definition of non-singularity and then you know this can be used to generalize to see when a point on any variety is a non-singular point and the definition will be that given any point on any variety it always lies in open set which is an affine variety because any variety is called by a finite (()) (05:57) of open subspace which are isomorphic affine varieties.

So you take the point on the variety to also to be a point on an affine open set containing that variety. And then you define the point to be non-singular if it is non-singular as a point of that affine open set, okay. And so in a way you know this non-singularity should not depend on the open set immature considering it. So you know the non-singularity or the smoothness at a point is a local notion, okay. You know when you say a curve is smooth at a point it means actually there is a sufficiently small neighbourhood of the point where it is smooth, okay.

And similarly you know when you say surface is smooth at a point it means there is a sufficiently small neighbourhood of the surface where it has to be smooth. So you see this smoothness at a point is something. It is in fact in classical geometry you know if something is smooth at a point then it is also smooth in a neighbourhood of that point, okay. And this is true also in algebraic geometry, okay. But it requires more competitive algebra. It requires more algebra to establish, okay.

The fact is that something being smooth at a point is something that spreads out to that neighbourhood of the point, alright. Therefore so the moral of the story is that when you say X is non-singular at a point, it will happen not only at that point but also in an open set containing that point, alright, in the neighbourhood of that point. And but you know any open set in the Zariski Topology is huge, okay.

Therefore what it will tell you is that the moment you have one smooth point, okay, then there is a huge open set, there is a dense open set plus smooth points, okay. And this is what happens. In truth what happens is that the set of points which are singular points, points which are not non-singular, that is only a small closed set, okay. And that means all points on a huge open set they are all non-singular, they are all smooth points, okay.

But anyway these are all geometric facts and one needs to have sufficient amount of competitive algebra to establish them, alright. So anyway the level of this definition itself there are problems since that you know this variety X could sit in some other affine space. And if I change the affine space then the ideal of X in affine space will change, okay. And that is one ambiguity and the other ambiguity is for this ideal I am taking some set of generators.

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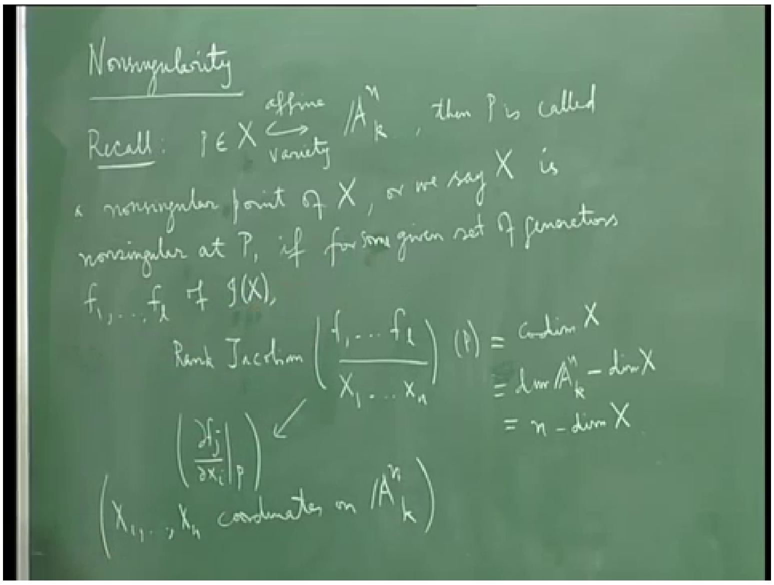

If I take another set of generators then it may be different number of generators and this matrix changes. But I am just saying that this is the condition on the rank of this matrix, okay. So there are lot of ambiguities in this definition. Why should this definition be a good definition? Because, there are so many choices to be made. First of all I have to make a choice of embedding X into some affine space. Namely have to think of X as (())(09:23) closed subset of some affine space, okay.

And variety can be embedded in so many ways, okay. I can think of variety is a line of course I can think of it as a line in the plane or a line in space, in three space or even higher dimensional space. So there is this ambiguity in embedding it in some A n and then once you embed it if you look at the ideal of X for that embedding that ideal of course already depends on embedding but then it could have different sets of generators, okay.

The same ideal could have different sets of generators and the number of generators in these sets could also be different, okay. Therefore this Jacobian matrix you know which is a Jacobian of all these generators, this matrix itself could change, okay. But then the definition is that you know whatever it is you calculate its rank. If its rank is equal to co-dimension then the point is a smooth point, a non-singular point.

So it looks like this definition involves too many ambiguous things and you know it might not be very consistent if we change the various choices. But that is not true. The point is because this is got to do geometrically with trying to you know look at the tangent space to the variety X at that point, okay. And because of that geometric reason that this definition actually works.

But then if you want really verify that, one uses of course you know you translate from geometry into competitive algebra and then necessary competitive algebra is given by the following theorem. So here is the theorem. The theorem is that a point P of variety X is a non-singular if and only if the dimension of m p by m p square over k, it is a k vector space, is equal to dimension of O X x is equal to dimensions of X, okay. So here is the theorem, alright, where of course m p inside O X x is the unique maximal ideal, okay.

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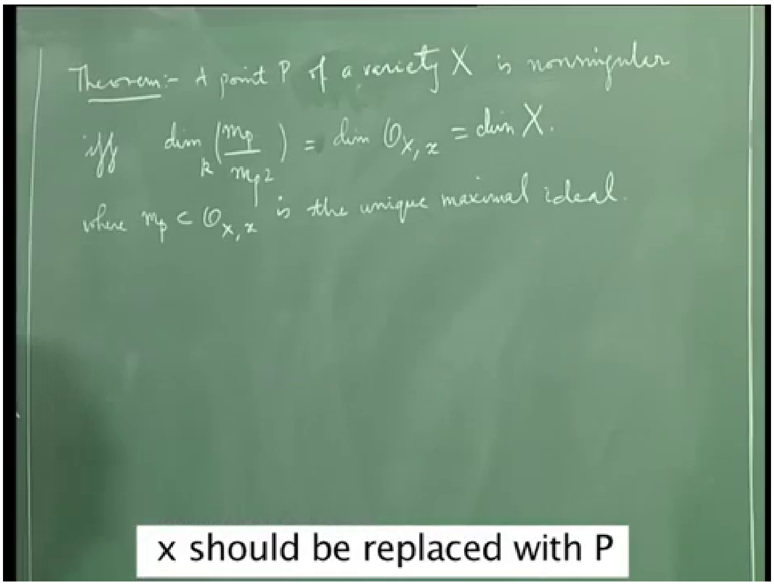

So basically this is due to Oscar Zariski and it is a fundamental result. And here you know that the definition of non-singularity we are using is this definition which is full of ambiguities, okay. This definition is full of ambiguities, alright. And whereas this theorem gives a condition, the condition is only on the local ring of the variety at that point.

And you know you can see immediately one thing because it is a condition of local ring I can assume that this point is on an affine variety because I can go to an affine open subset of this variety and look which contains the given point, okay. So you know it depends only on the local ring, alright. And you know the beautiful thing is that this is intrinsic to the variety.

See here nobody is bothered about whether X is affine or cos-affine or if it is affine or cos-affine, which affine space it is embedded inside. I mean all these things one is not bother about. The condition is completely in terms of things which are intrinsic to the variety, okay.

And which do not depend on you know the variety being seen as a subset either (())(13:42) close or open subset of any reducible close in some affine space or some (())(13:47) space.

Nobody is bothered about these things, okay. So that is the reason why this theorem is very important. And so interestingly the proof is many algebra and but actually it has got to do with geometry because what you are trying to do is you are actually you know if you look at it geometrically you are trying to calculate the tangent space for the variety at that point, okay.

And you are also trying to look at the normal space, okay. The space of vectors in the ambient space, the tangent space at the point for the ambient object and you are looking at the normal direction also, okay. So you know the idea is that, yes?

Student: Sir it should be O X P.

Professor: Oh, okay yes. That is very important. Yes, thank you. It should be O X P because all that time I have used small x and then that is a bad notation. Thank you. So this should be O X P.

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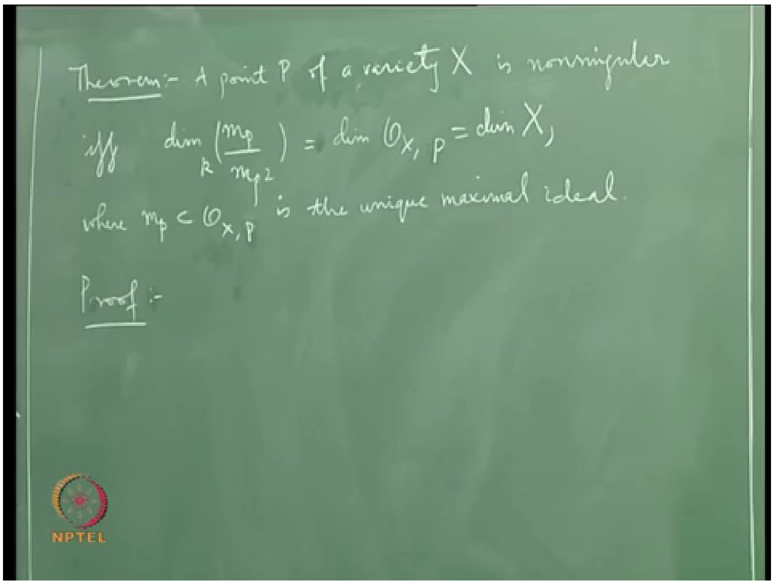

So you know roughly geometrically this is what is happening. What happens is that when you think of an object embedded inside a space, okay, then when you take a point on that object, okay, then it will have a tangent space, okay. And then the normal space will be the normal vectors in the bigger space and the sum will add up to the dimension of the bigger space provided the point is smooth, okay.

So you know if I take a smooth surface in three space, take a point on it and draw the tangent space, I will get a tangent plane. I will get a neat two dimensional plane and I will get a unique normal to the surface which is given by the gradient, okay, if the surface is defined by a single equation. It will be given by the gradient.

And then therefore the normal space will be one dimensional, the tangent space will be two dimensional, some of these spaces will be three dimensions which should be the ambient space in which the object is embedded. But however if you take a singular point this will not happen. The singular point the tangent space itself may be everything. You may not have any normal vectors.

For example, if you take the cone in three space and you take the vertex of the cone which is a non smooth point, if you draw the tangents space then you will get the whole three space. So you will not be able to find at the vertex there are no normal vectors. All vectors are tangent vectors. There are no normal vectors. So if you see that this happens because that point is a singular point, okay.

So if you have a smooth point what will happen is that you will get a tangent space at that point which is equal to the dimension of the object and then the normal space which consists of vectors perpendicular to the tangent vectors, okay, that will give you a subspace of the tangent space of the point in the bigger space in which it is sitting, okay, such that the sum of these two spaces will be the whole tangent space, okay. And this will happen for smooth points. It will not happen for non-smooth points, okay.

And actually it is that calculation which is being reflected here but it is being reflected using algebra alright. So let us do the following thing. So without loss of generality we assume X is affine, okay. And why is this correct? This is correct because after all my definition of non-singularity is the point P of a variety is called non-singular as a point of an open (sub) subset which is affine because I have defined non-singularity only for a point of an affine variety.

And any variety can be covered by open subset which are affine varieties. So without loss of generality I can assume that X is affine, alright. And what I mean by this is you go to an open subset which is affine, okay. By going to an open affine subset containing the point P, okay, and you know by going to an affine open set which contains the point P the local ring is not going to be changed because local ring does not depend on whether you go to a smaller open set which contains the point or not, okay.

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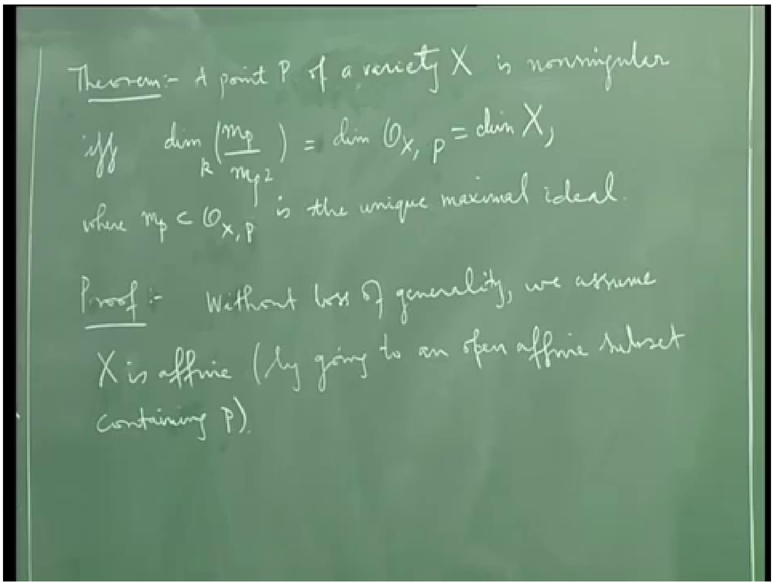

So the conditions of the theorem are both the hypothesis and conclusions for both ways of the theorem they are not affected if you go to an affine open subset, okay. So it is enough to go to consider X affine, okay. So say we are in this situation. P is a point of X which is inside affine n space and now it is affine variety means it is (())(19:53) closed subset of A n, alright. And then you know now we will do some calculations, okay.

So now first of all let P be the point lambda 1, etc. lambda n, okay. P is a point of n space anyway. So take its coordinates, okay. So the maximum ideal of P will be by the (())(20:23) you know I mean the point P corresponds to any maximal ideal in the affine co-ordinate ring of the affine space which is k X 1 to X n. This is the affine co-ordinate ring of A n, alright.

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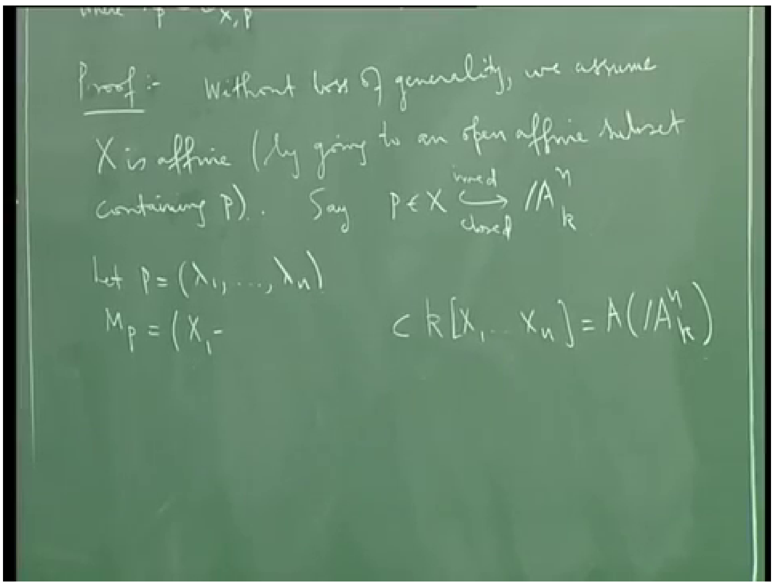

And here I will get a maximal ideal which corresponds to this point by (())(20:48) that will be this maximal ideal generated by the X i minus lambda i, okay. So this is the maximal ideal that corresponds to this point, alright. And you see what we will do is you define this map k X 1 through X n. You define this map psi, okay, into k n. So here is my map. The map is very simple. Take any polynomial g in n variables, t just take the gradient at P, okay. Which means take doh g by doh X 1 at P, doh g by doh X n, just do this.

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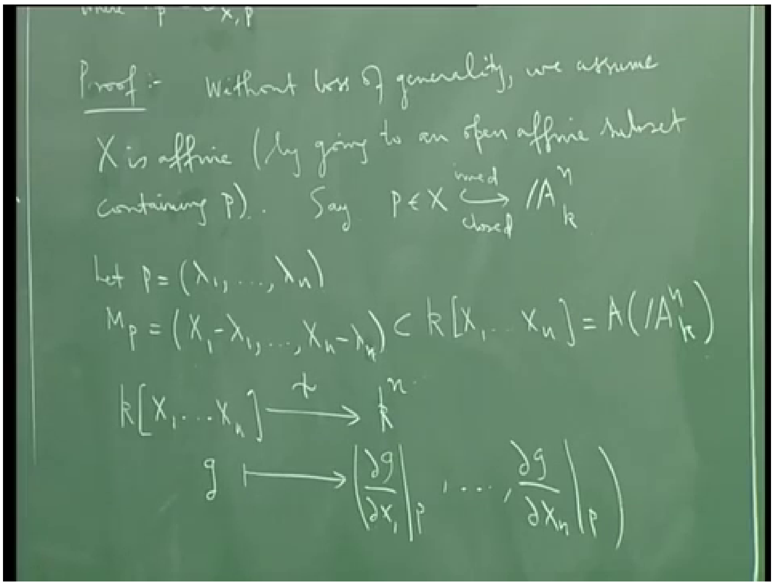

So just take the partial derivatives with respect to each of the variables in this order and then evaluate them at P, okay. So of course here when I say partial derivative it is a derivative that is in the formal sense you know how to write the derivative of polynomial without having to get the derivative from calculus methods which involve epsilons and deltas and limiting processes, okay. So we just use a standard rule for differentiation, formal rules for differentiation of variable and you write out these.

So each of these doh g by doh X i are all polynomials again in the X i. And then you just substitute the point P. Namely you substitute instead of each X i you substitute lambda i. You will get some n triple and that is this n triple of k n, okay. Now the nice thing about this map psi is that it is k linear and surjective, okay. See, the map psi is k linear because you know if instead of g if I replace g by g 1 plus g 2, okay, then all the partial differential operators they are all k linear.

I mean they are all linear and therefore if I replace g by g 1 plus g 2 I will get a sum of n triple here and if I multiply this g by a constant that constant will come out there, okay.

Therefore it is k linear and it is surjective because you know if you take these generators as a maximal ideal, okay, if I take X 1 minus lambda 1, alright, for g then I will get 1, 0, 0, 0, 0 which is the first basis vector, okay.

Because if I differentiate this with respect to doh with X 1 I will get 1 and substituting P has no effect. It will still remain 1. If I differentiate with respect to other variables I get 0. So if I take g equal to X 1 minus lambda 1 I will get 1 0, 0, 0. If I take X 2 minus lambda 2 instead of g I will get 0, 1, 0, etc. So in this way I will get a basis for k n, okay. And since the image contains a basis it is surjective, okay.

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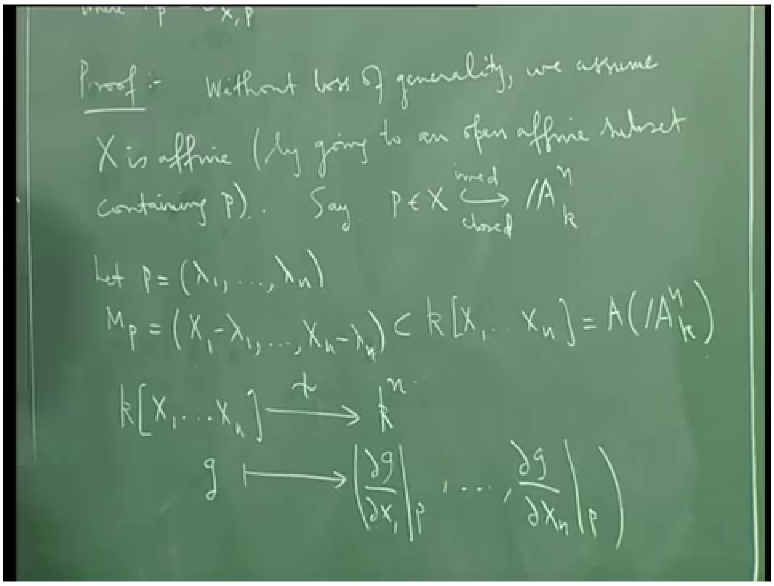

So the moral of the story is that psi is k linear surjective. Psi of X j minus lambda j is equal to standard jth basis vector, okay. So it is linear surjective map, alright. And now the point is that the kernel is exactly the square of this maximal ideal, okay. So in other words what I am saying is that psi induces an isomorphism of the n dimensional vector space k with M p mod M p squared, okay. So, M p mod M p squared is just the n dimensional vector space k.

So you see if we take an element of M p squared, okay, then you know it is going to be of the form sigma i sub g i times X i. So I know in fact I can put i comma j, X i minus lambda i. If I put g i j X i minus lambda i into j minus lambda j. Yes it is right, okay. You see what is an element of P square of an ideal? It is a finite sum of products where each term in the product is product of two terms, one each from the ideal, okay.

So if I take an element of M p square it is a finite sum where each term has to be an element of this ideal. But you know any element of this ideal will be generated by the X i minus lambda i with polynomial coefficients. So it will look like this, alright. And now the point is you know to this if I apply the operator doh by doh X L, okay. Suppose I apply the operator doh by doh X L, alright, then you know what will I get?

You know I have to differentiate product of three terms, alright, and therefore you use the product rule, alright, which also is valid in this case, okay, formally, alright. Therefore so if you know if L equal to i, suppose L is i then what I will get is so let me write it here. So what I will get is well sigma i comma j, I will get doh g i j by doh X L at P. And then I apply P, okay.

So you know this is the Lth component of this map psi. The map psi is applying all the partial differential operators and then substitute the point P. So the Lth component of this map that is this map followed by Lth projection that is this map, alright. And what is it? If you apply the chain rule this is you know this will be X i minus lambda i, X j minus lambda j at P, alright. This is one term.

Then the other term will you know it will have g i j at P and then I will have doh by doh X L of X i minus lambda i at P into X j minus lambda j at P. And then I will have one more term. I will have X i minus lambda i at P into doh by doh X L of X j minus lambda j at P, okay. This is what I get, alright.

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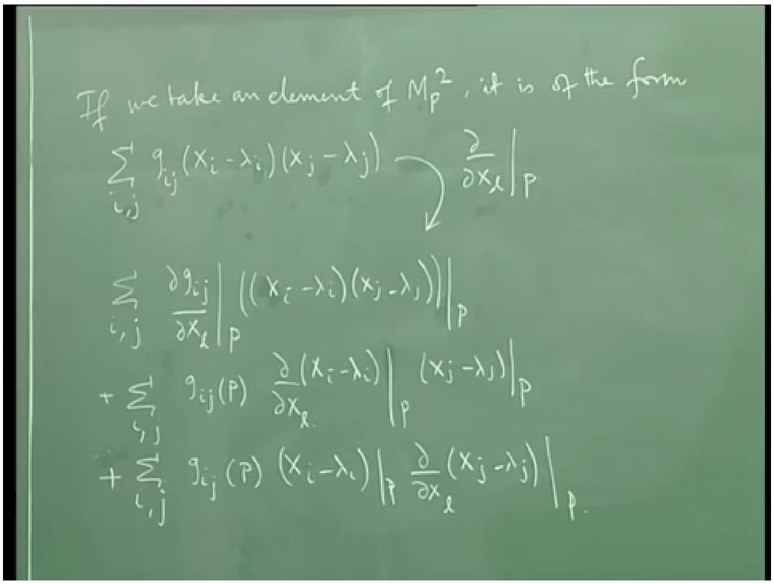

And now if you look at it carefully this is going to vanish because you see the first term will vanish because I have these two fellow surviving. If this is 0 then I am not worried, okay. But even if this is not 0, if I substitute X i equal to lambda i or X j, when I substitute P I have to put X i equal to lambda i and X j equal to lambda j. So this is gone. In this you know this will go and in this, this will go. So these terms will vanish, these underline terms. So this is actually 0, okay.

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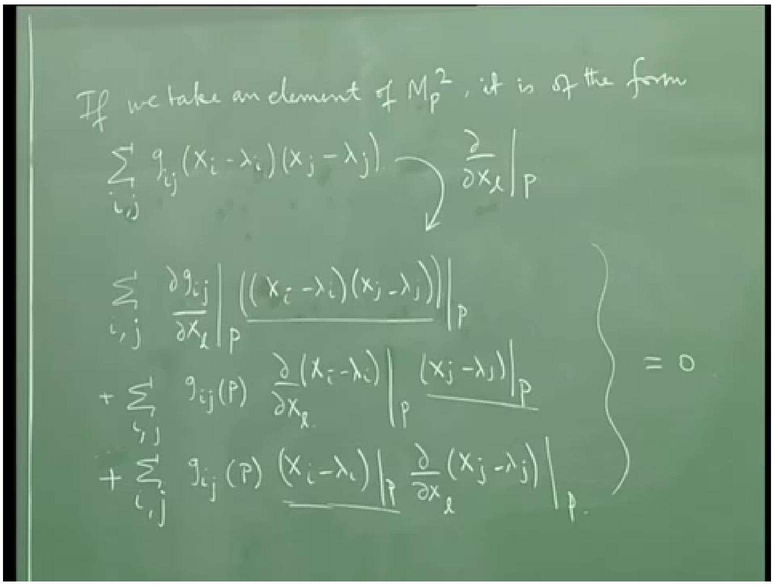

So the moral of the story is that if you take any element of M p squared, this linear map will kill it, okay. So in other words what it will tell you is that M p squared is contained in the kernel of psi, okay. M p squared is (con) contained in the kernel of psi. But the more important thing is M p squared is exactly equal to the kernel of psi, okay. Conversely let h belonging to M p be in kernel of psi, okay. Now h is an element of M p so h is actually given by sigma g i X i minus lambda i. This is how an element of M the maximal ideal will look like.

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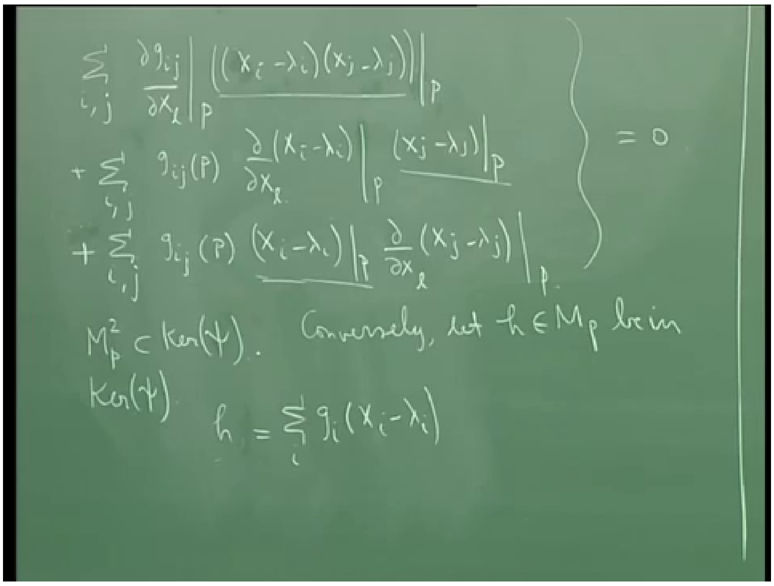

It is just a combination of all the generators X i minus lambda i multiplied by polynomial coefficients. J is our polynomial coefficients, right? This is g i, this is g i j so they are different things, right? So h looks like this, alright, and I am saying that I have taken my h to be in the kernel of psi so psi of h is 0, okay. So what is psi of h? Psi of h is well I have to partially differentiate this with respect to each variable and then substitute the point P then I should get 0.

That is what it means to say that psi of h is 0 which is h is in the kernel of psi, okay. I am trying to show that kernel psi is also contained in M p squared. So I am trying to show kernel psi is equal to M p squared. That is what I am trying to show, okay. So psi of h is what? If I take psi of h, psi of h equal to 0 so this will tell you that you know if I take doh h by doh X L at P is 0 for all L. This is what it means, okay.

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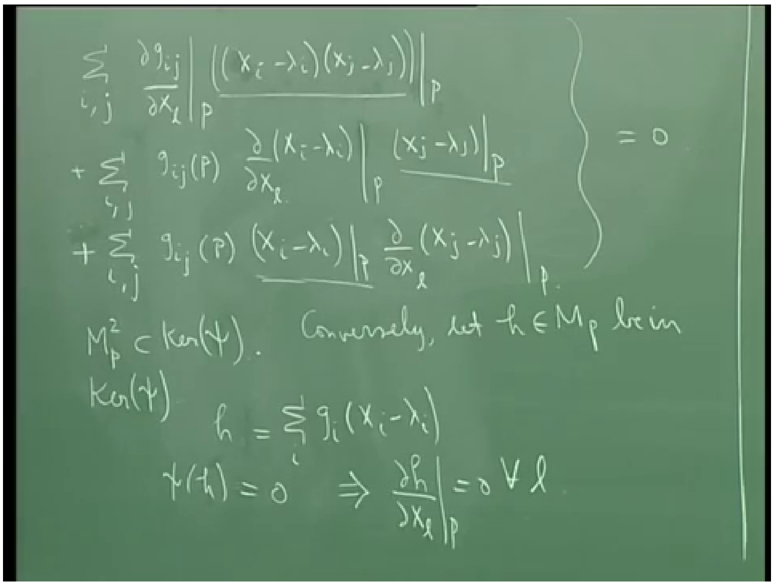

And now write that out. See doh h by doh X L at P 0 is equal to this but see if I get differentiate this I will get sigma over i, okay. I will get doh g i by doh X L at P into X i minus lambda i at P plus I will get g i at P into doh by doh X L of X i minus lambda i at P. This is what I get, okay.

And what you must understand is that you see this is 0. Whenever X i minus lambda i comes if I substitute the point P it is going to vanish, okay. So this term is gone, alright. And this term if you look at it, if L is not equal to i this is 0. So this term will be killed and this term will only survive when L is equal to i. When L is equal to i this is 1, okay.

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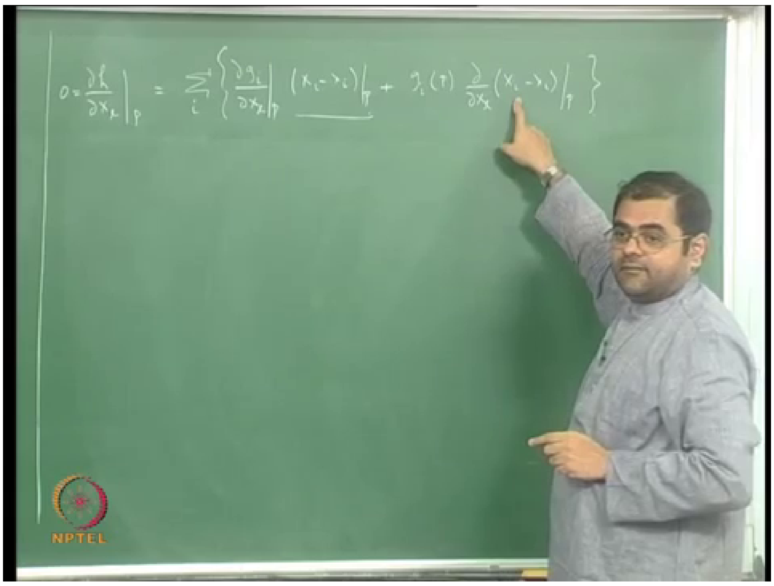

And I will get g i of P. So this crazy thing will tell you that g L of P is 0 for all L. This happens for every L. So g L of P is 0 but that means what? It means that see if a polynomial vanish is a P then it is precisely in a maximal ideal of P because M p is exactly the ideal of polynomial which vanish at P. So what this will tell you is that it will just tell you that you know g L is in M p, okay. But you know that is for all L.

So you know all these g i are in the maximal ideal and these terms are also in the maximal ideal therefore the product is in M squared therefore h is in M squared. So this implies that h is equal to sigma g i into X i minus lambda i is in M p squared. So all these things put together tell you that M p squared is exactly the kernel of psi, okay. So I take h in M p. No, if h is not in M p I cannot write it in this form. It has to be generated by the generators of M p.

See h is in this form if and only if it is in M p because X i minus lambda i are the generators of M p. A general h need not be like that. Any polynomial need not be like that. What I forgot to tell you is that you see this psi is from this polynomial ring into this. Mind you every ideal is also a k subspace, okay. So the psi also gives you a linear map and restricted to a subspace. So psi can also be restricted to M p.

And so in fact what I want to say is that psi restricted to M p itself is also k linear and surjective, okay. So I will come to that. So you know note that M p squared is a k subspace of M p which is also k subspace. Subspace means vector subspace and where is this map psi into k n, okay. And in fact you see psi restricted to M p is also surjective because you know I have already told you that psi takes all the generators of M p into the standard basis.

So psi restricted to M p is also surjective, okay. So psi restricted to M p from M p to k n, okay. So here of course as you said because I am taking h in M p I am looking at the kernel of psi restricted to M p, right? So if you take psi restricted to M p is a surjective and induces.

I mean it is surjective because the generators of M p give their images under psi give you the standard basis, right? And it induces a kill in your isomorphism from M p mod M p squared to k n. And let me call this isomorphism as psi prime, okay.

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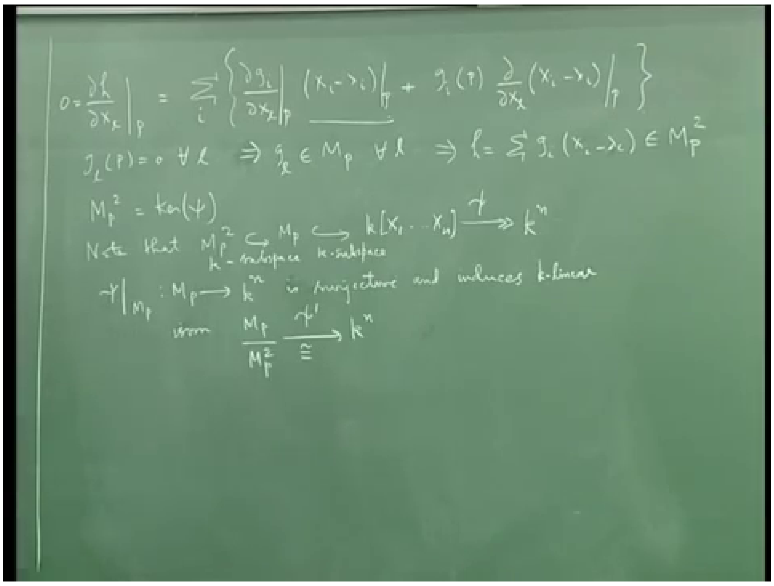

So you know this simple linear algebra in this first step gives the calculation that you know you take a point P in affine space, okay. Take the maximal ideal corresponding to the point. Then M p mod M p squared is just n dimensional affine space, okay. That is the calculation, alright. And you know if you look at it in fact in retrospect you know in view of the theorem that we are going to prove what you are saying is that every point of affine space is non-singular, okay.

Because you know if you take the local ring of affine space at that point is just the polynomial ring localised at M p, okay. And its maximal ideal is generated by the image of M p in that localisation. And what you are saying is that well if you take this quotient and localise, okay, you will get M p mod M p squared at the local ring, okay, for the point in affine space. And you are seeing that the dimension of that is equal to the dimension of affine space which is n. So you know if you take X equal to A n you have got this condition.

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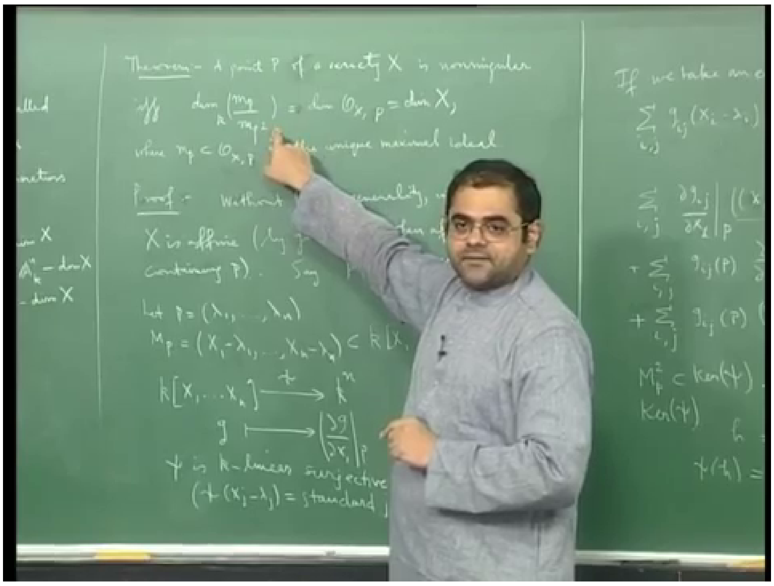

And what you are saying according to the theorem is that affine space is smooth, okay. So the significance of this calculation is that affine space is smooth, is non-singular. Every point of affine space is non-singular, okay. The affine spaces are smooth, right? So well anyway I need to go down to this subset, okay. So we will see for that I do the following thing. You see you have see P is in X and that is the same as saying that I X is contained in M p, okay.

This is again you know this is just applying the script i and the Z (())(40:01) to P and X, right? And using the (())(40:05). So you know P is in X. So you are saying the subset P is the subset of X, okay. You apply I to this so you will get I of the point P contains I of X, okay. But what is ideal of the point? The ideal of the point is exactly M p. So M p contains I X, okay. So (())(40:35) it has gotten by getting the ideal of the subset, okay.

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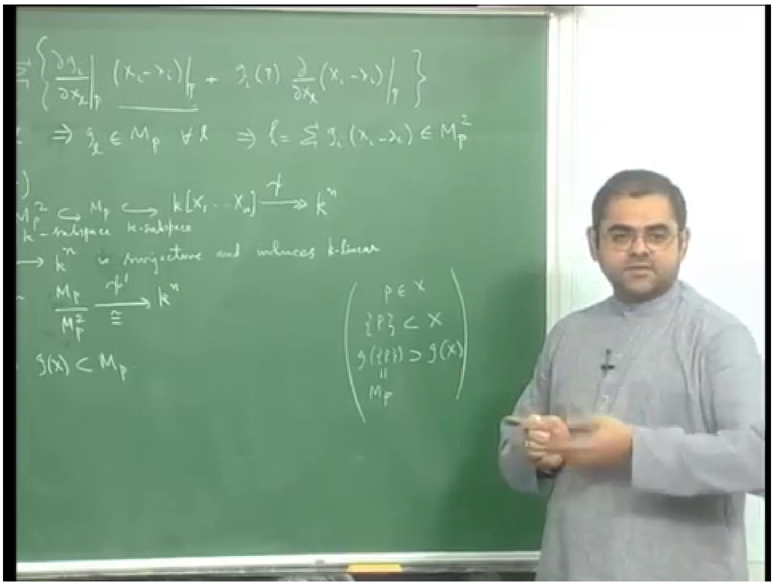

And of course the ideal of point is this maximal ideal, okay, M p. And so you know what will happen is that if you take I X so I X is inside M p, right? And in fact if you take I X plus M p squared, okay, that will also be a sub of M p, okay. Because M p squared is subspace of M p. I X is also sub of M p. The sum of two subspaces is again a subspace, alright. And in fact I can even divide now by M p squared. So I divide by M p squared and I get this. This is a subspace. It is a subspace of this k vector space which is isomorphic by psi prime to k n, okay alright.

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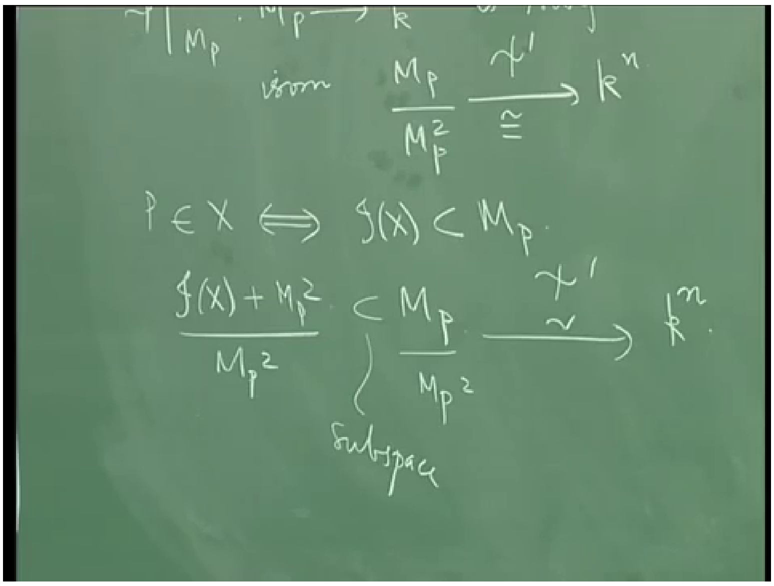

And the beautiful thing is if I take the image of psi prime and image of this under psi prime you know what I will get. Of course if the psi prime is an isomorphism and this is subspace so the image of the subspace under psi prime will be subspace of k n, okay. And what is the dimension of the subspace? What is the dimension of the image of a linear map? It is you simply take set of generators for the subspace, okay, and apply the linear map.

The image of a subspace is just given by taking the span of the images of the generator of the subspace, okay. So if I want the image of this I have to just take a set of generators for this, apply psi prime to that and take the span of that, okay. In other words of course instead of taking span of a set of vectors you can write those vectors in column form, take the matrix and take the rank of that matrix because you know the rank of the matrix will give you the dimension of the image, okay.

This is the standard rank in nullity theorem in you know linear algebra. So it is part of that theorem. So but you see now you know for I X if I take a set of generators f 1 through f L of I X, okay, then those generators will also give me generators here, okay. And applying psi prime is actually calculating this Jacobian matrix because actually what is happening is when you apply to each generator, okay, what you are doing is you are going to get one row or one column of that depending on the way you write it of the Jacobian matrix.

So if you take the images of all these generators I am just going to get to the Jacobian matrix. And each rank is precisely going to be the image of this, okay. That is the connection. So moral of the story is dimension over k of I X plus M p squared by M p squared is equal to dimension over k of psi prime applied to that, I X plus M p squared over M p squared, okay. And this is equal to rank of the Jacobian of doh F i by doh X j at P, okay. Where F 1, F 2, etc. up to F L are a set of generators for the ideal of X as here, right?

So you see now if you look at it like this it is very clear that no matter what set of generators I use. Instead of using F 1 through F L suppose I use some h 1 through h L prime, another set of generators. No matter how many generators they are I do not care, okay. But in any case I am only going to get dimension of this subspace and that is the reason why this rank of this Jacobian does not depend on what your generators are, okay.

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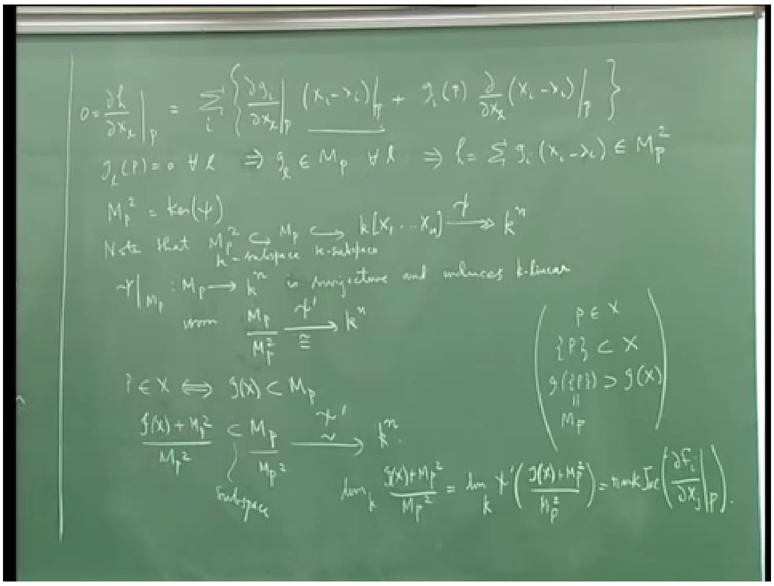

You are always going to get only dimension of this subspace, okay. That is the reason why even for a fixed embedding for the ideal or affine variety in the embedding into affine space, even if you change the set of generators, when you calculate the rank of the Jacobian of the generators with respect to the variables you are going to get only the same dimension. You are going to get only the same rank because it is a dimension of one and only one subspace under this isomorphism, okay.

Now it is a matter of just a little bit more of linear algebra, alright. So let me keep this statement here as it is. I am cutting you here, okay. It is only probably a couple of lines more. So what will happen is see what you should remember is you see M p mod M p squared divided by I X plus M p squared mod M p squared, okay.

I have this is space and this is subspace. If I take the quotient my claim is that this is the same as M p mod M p squared where this M p is a unique maximal ideal in the local ring of X at P, alright.

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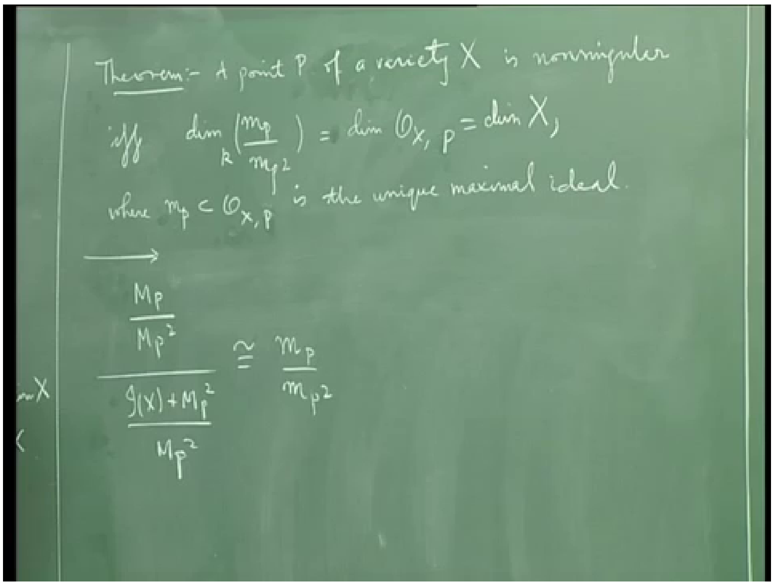

So you see we have an isomorphism of k vector spaces in this way, okay, because how do you get the local ring? You get the local ring because the local ring of X at P is got to know. What you do is you take the affine coordinate ring of X and you localise it at the maximal ideal that corresponds to the point P inside X. You know this is the definition of the local. This is how you get the local ring of an affine variety at a point.

You simply take the affine coordinate ring of the variety and you localise at the maximal ideal that corresponds to that point. The unique maximal ideal of the affine coordinate ring which corresponds to that point by (())(48:48), okay. But what is A X? It is just polynomial ring. You go ideal I X and then what is this maximal ideal? This is just you localise at M p mod I X, alright.

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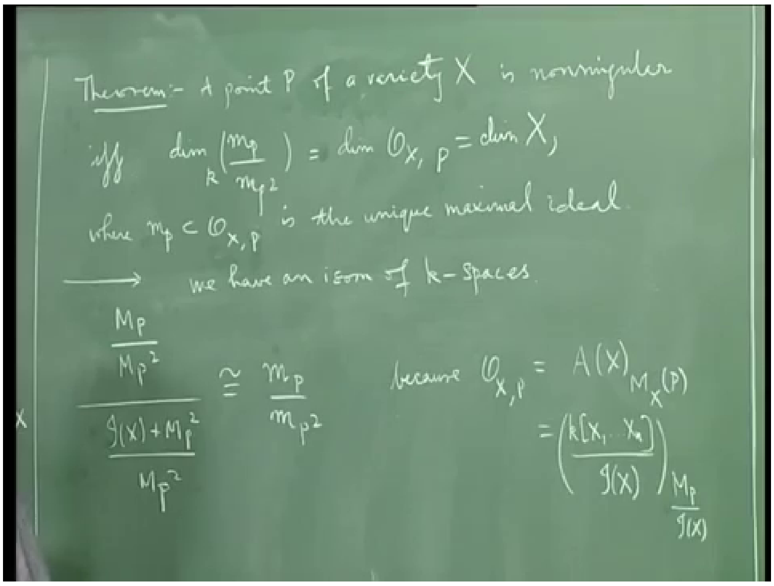

And because of this if you take this quotient M p mod M p squared, I X plus M p squared by mod M p squared, this will be the same as small m p where small m p is the unique maximal ideal in this because you see small M p is actually this, you take the ideal M p X dot take the ideal generated by this by its image in O X P, okay. So this quotient is the same as this, alright, as k vector spaces.

And therefore if you count dimensions, dimension of this minus dimension of this is equal to dimension of this so you will get dimension of M p mod M p squared over k is equal to dimension of capital M p mod M p squared over k minus dimension of I X plus M p squared by M p squared, okay alright. And that would be equal to, what is this?

This is n dimension because we have already proved capital M p by capital M p squared is isomorphic to k n as vector spaces therefore because of the isomorphism psi prime its dimension is n. So you see this is n, okay, minus this part is just the rank of the Jacobian matrix at P.

So this as we have seen here is dimension of I X plus capital M p squared by capital M p squared is actually the image under psi prime, dimension of the image under psi prime of that subspace, okay. And that is the rank of Jacobian. So here I will get n minus rank of the Jacobian of doh F i by doh X j at P. I will get this, okay.

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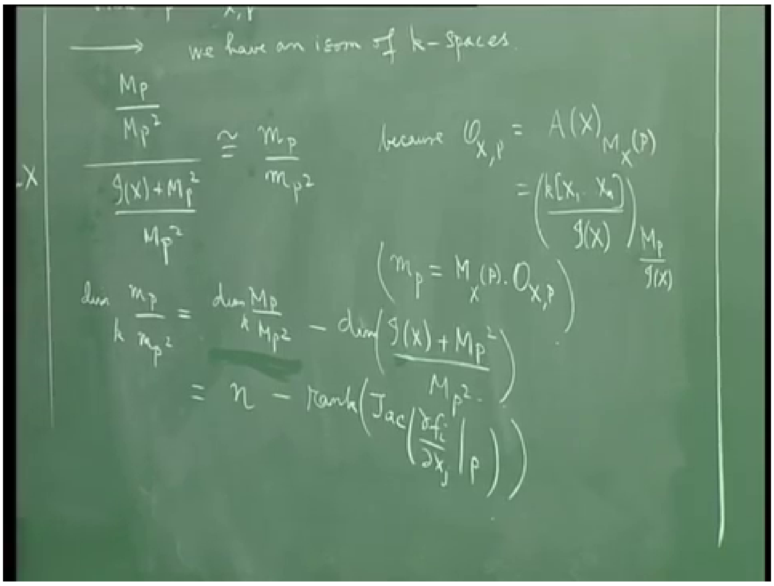

And this is equal to n minus n minus dimension of X if and only if P as a point of X is non-singular. That is our definition of non-singularity. The definition of non-singularity is that the rank of the Jacobian should be equal to the co-dimension for a point which is a smooth point, point which is a non-singular point. So this is equal to n minus n minus of dimension of X which is equal to dimension of x if and only if the point P is non-singular. So that proves the theorem.

So what you have proved is dimension of M p mod M p squared where M p is the unique maximal ideal of the local ring corresponding to the point P on the variety X. That is equal to the dimension of X if and only if the point P is a smooth point, okay. And the importance of that is that this gives me the dimension of tangent space at that M mod squared, okay, dimension of M mod M squared that actually gives the tangent space at the point, okay.

And therefore you are saying a point is a smooth point if and only if the dimension of the tangent space is equal to the dimension of the variety on which it lies, okay. In general what will happen is that the dimension which is not a smooth point, this dimension can go bigger. You can get more tangent vectors. Your tangent space can become bigger.

If your tangent space becomes bigger then if you calculate the rank of the Jacobian that will become smaller, okay. And the point will be a non-singular point. I mean the point will become a singular point, okay. So the moral of the story is that this gives you that tangent space, okay, and this actually calculates the normal space.

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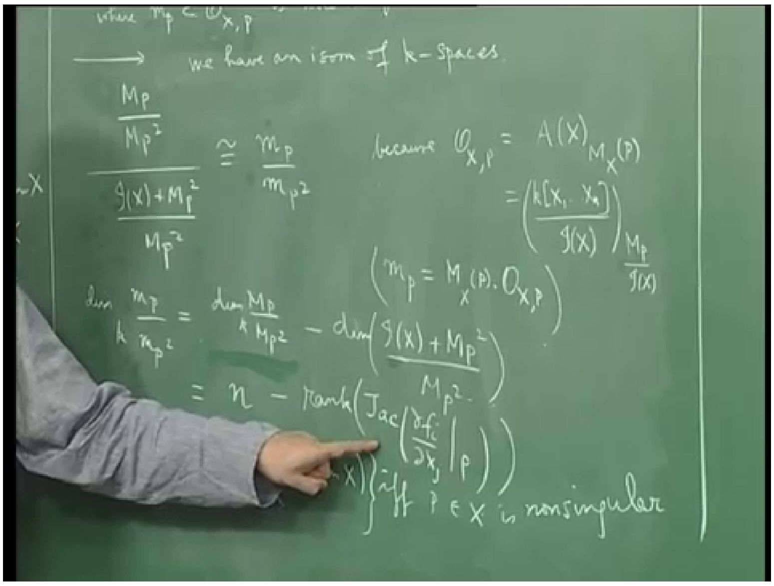

This is a set of normal directions at that point in the ambient space in which this object is sitting inside, okay. So if the point is a smooth point and X is an r dimensional variety in n dimensional space then the dimension of the tangent space will be exactly r. The dimension of the normal space, all the vectors which are perpendicular to the tangent space, tangent vectors that will be n minus r. And that should be equal to the rank of the Jacobian for any set of generators.

And that is what geometrically it is happening and that is what it says, okay. So that finishes the proof of this theorem. So in this connection I needed to tell you that the set of points where the set of singular points that is a closed subset, okay. So that is also a fact that can be seen immediately from this argument. If you look at all those points which are singular points, see those are points on the variety where this rank of this matrix falls, okay.

And you know where the rank of matrix falls is just given by the locals of the vanishing of all the maximal minors, alright. So if you take this Jacobian matrix, okay, it is a matrix having polynomial entries, alright. And you know if you vary the point P, what are the points P which are singular those are the points for which if you evaluate the Jacobian matrix you are going to get rank less than, you will not get rank n minus r, okay.

But you will get lesser rank, okay. And what does that correspond to? That corresponds you take all the maximal square minors of the matrix, there determines should vanish. If all the maximal square minus vanish, at that point the rank has to fall. And only at such points your rank will fall. And therefore that is the closed subset, okay. So this argument tells you that if you look at all the points P which are singular points that is a closed subset of X.

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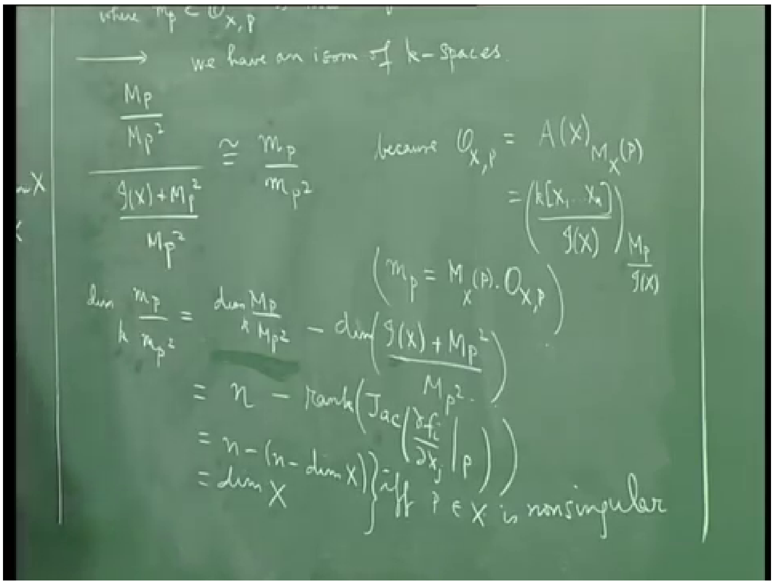

That will be a closed subset of X, okay. So the set of singular points is a closed subset but the more important fact is that this closed subset is by no means the whole space. It can only be a proper closed subset which means it will be of smaller dimension and that needs proof and I will prove it in the next lecture.