Basic Algebraic Geometry
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Module 2
Lecture 4
Irreducibility in the Zariski Topology

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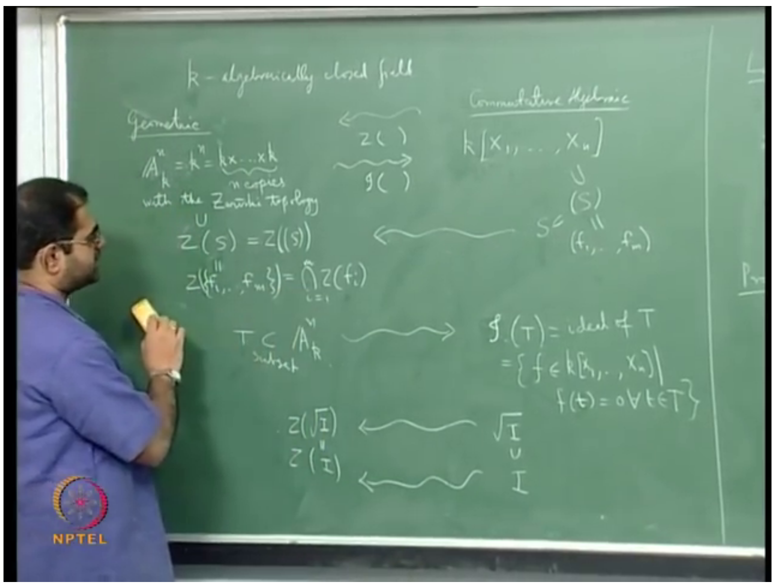

Okay, so let us continue with our discussion so like I will go back to this diagram on this side, okay and what I am going to do is I am going to just abort several things, so let me rub this off and also rub this side and let me just write down the implication of this of this lemma here, okay.

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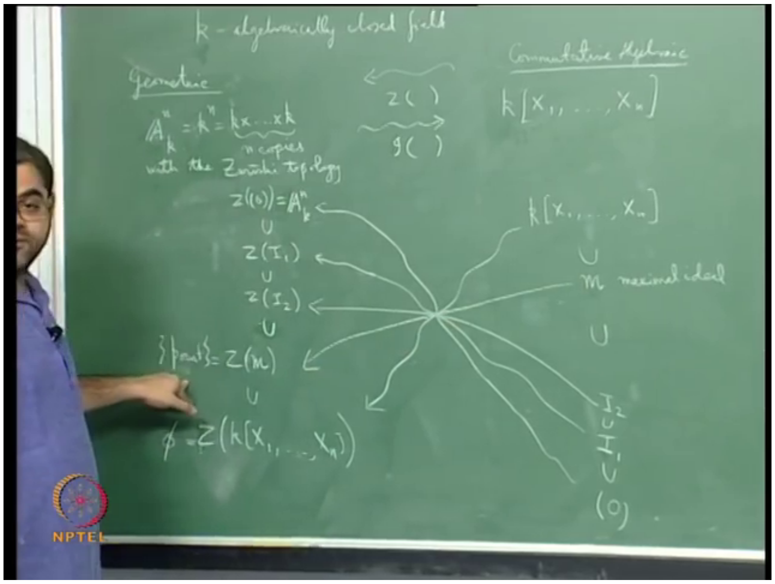

So basically on this side the biggest side you can think of is k k is the whole ring first and well the instead of that is of course a null set which is the kind of the smallest set here, okay and maybe I should because there is an inclusion reversal I should write it here the null set is here which is the 0 set of k X1 Xn, okay.

And you know well then of course if I take a maximal ideal, okay m is a maximal ideal so more generally I could have started with an ideal I and rather let me write it, okay so let me write it here does not matter so I have an ideal I here I would get 0 set of the ideal, okay and of course you know if you start with this 0 ideal, okay then the that would give me the whole space this will be just be 0 set of 0 is the whole space, okay. And if I start with an ideal I1 I end up with well the 0 set of I1 which is the sub of the whole space, right? And you know if I take a bigger ideal I end up with a smaller set of zeros, okay so it goes like this, right?

And if I go all the way to a maximal ideal then you see the corresponding zero set of a maximal ideal is just a point is a single point, okay. And well if the ideal is contained in the maximal ideal then this point belongs to this 0 set, okay so it is like this and of course the null set is contained everywhere so you have on this side the subsets increasing from the null set which is smallest possible to the whole space and on the other side you have the ideals decreasing from the largest possible ideal which is the whole ring to the smallest possible ideal which is the 0 ideal, okay.

And there is this order as you can see this is an order reversing correspondence. So the point I want to make is I want to explain first of all that if you take a maximal ideal you will end up with a point, okay. And why is that so that is again basically if you want due to the Nullstellensatz, okay.

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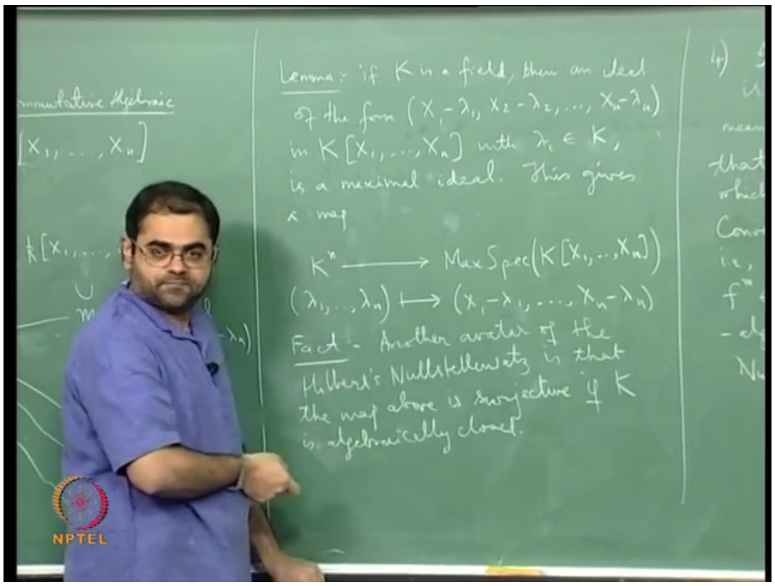

So let me tell you few more lines about that, so here is a lemma this is the lemma from probably from field theory which you could have come across in a first course in commutative algebra by then it is even otherwise it is easy to prove if capital k is a field so capital k is a field and mind you this is any field is not necessarily algebraically closed, then an ideal of the form X1 minus lambda 1, X2 minus lambda 2, Xn minus lambda n in K of X1 etcetera Xn with all the lambda is in K is an maximal ideal, okay.

So the point is that so if you take an n tuple of points of the field then there is an automatically a maximal ideal associated to that. So what this tells you is that there is a so let me write that, this gives a map from Kn to let me write max spec capital K X1 etcetera Xn which sends the point lambda 1 etcetera lambda n to the maximal ideal given by so there is a map like this and you know let me quickly tell you how to prove this, so what you do is you assume that this point is a origin namely you assume all the lambda is a 0 then this becomes ideal generated by all the variables and you can see that this polynomial ring in this polynomial ring the ideal generated by all the variables is a maximal ideal and the reason for that is how do you check is a following how do you check an ideal in a ring is a maximal ideal you just if you go mod the ideal you should get a field you can show that if you take this polynomial ring and go modal of the ideal generated by the variables you end with the field k, okay.

And therefore you will get that the ideal generated by the variables is a maximal ideal and then all coordinates being zeros not a not something special this also holds for other points, okay because you can always find an auto morphism of the ring which maps which translates any given point to the origin and the auto morphism of the ring is a self-isomorphism of the ring so it will carry maximal ideals to maximal ideals.

So the fact that the all the variable generate the maximal ideal will also tell you that ideals like this are also maximal, okay. So that this is the very lemma that you can work out, okay so the moral of the story is well you know that if I take a maximal ideal of this form then you know what is a point you are going to get if this maximal ideal m is going to be X1 minus lambda 1 etcetera Xn minus lambda n then the point you get is going to be just the point lambda 1 with coordinates lambda 1 etcetera lambda n this is the single point you are going to get because what is the point of kn which is the common 0 of all these polynomials for such a point the first equation is 0 such a point is 0 of the equation means that the first coordinate has to be lambda 1 the second is also 0 the second equation means the second coordinate has to be lambda 2 and so on that will tell you that the point has to be just lambda 1, lambda 2, lambda n, okay the ith coordinate has to be lambda.

So it is clear that if you take a maximal ideal like this the corresponding point you get is this, okay and what is more serious is that every maximal ideal is of this point and that is also another avatar of the Hilbert’s Nullstellensatz, okay. So fact another avatar of Hilbert's Nullstellensatz is that the map above is surjective if capital K is algebraically closed, okay. So this is so in other words you take any maximal ideal in the polynomial ring in n variables it is of this form it arises from a point in this way.

And of course I also not remarked that it is easy to see that this map is inject, okay because if you take two different points they will go to different maximal ideals, okay or you can show that if you have lambdas here as coordinates of one point and lambda primes here as coordinates of another point such that the maximal ideals coincide then the lambdas have to be equal to the corresponding lambda primes, okay.

So this map is injective is trivial, okay it is the surjectivity which is more serious and that surjectivity is also another avatar of the Hilbert's Nullstellensatz, okay. So what this really tells you is that in our case since we are working with an algebraically closed field the points of the affine space are precise they correspond the points of the affine space on this side as closed subsets if suppose they are closed subsets and they correspond to maximal ideals on this set, okay.

And the other thing that I want to tell you is that if you take close sets here they will correspond to radical ideals there, okay.

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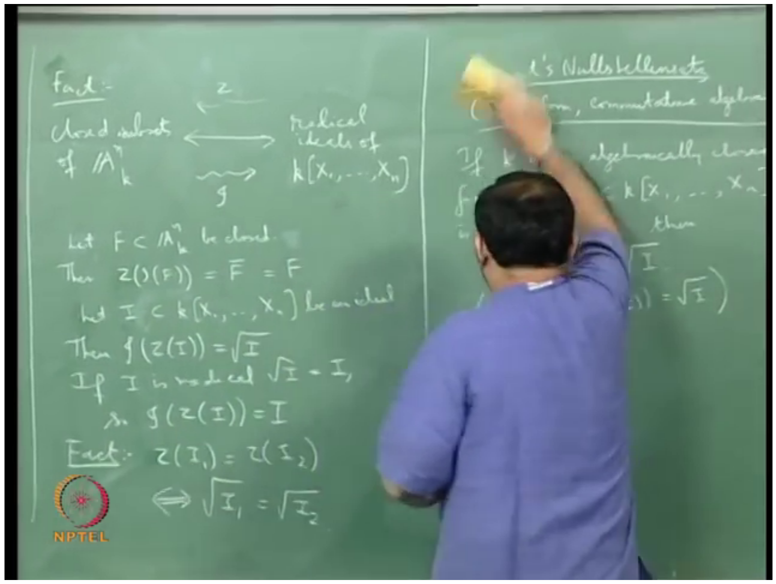

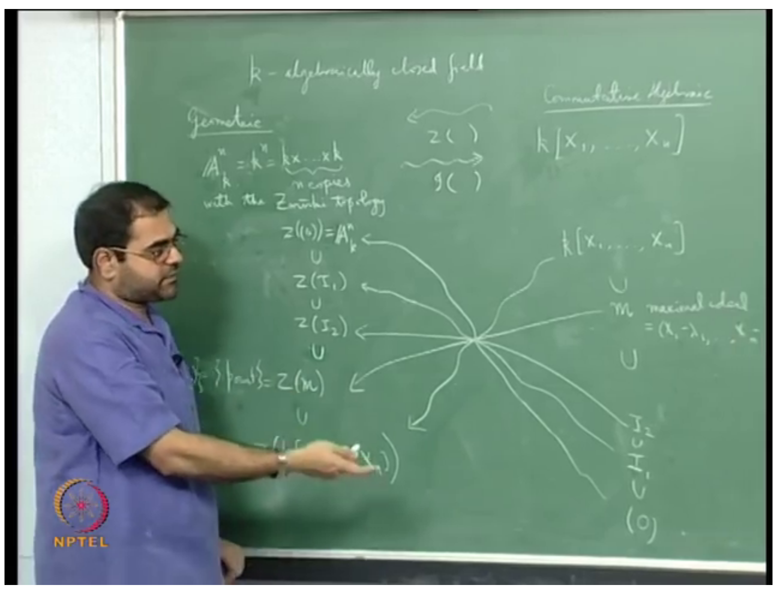

So let me make that statement so again lemma let me just put it as fact close subsets of An they correspond to radical ideals of K X1 etcetera Xn, okay. So you see I am just trying to concentrate on this dictionary the first thing is that its order reversing the second thing is that you if you want an exact equivalence a bijection then on this side you have to take radical ideals and on this side you have to take close sets, okay. And in fact you can have closed sets along with inclusion that as a partial order and you can have radical ideals along with inclusion here as a partial order and then this bijective correspondence will be it will be bijective and it will be inclusion reversing, okay.

And I think with what we have seen so far we have we can more or less you can more or less why this is true of course I should tell you the map on this direction is Z and map in this direction is script I, okay but you can check why both these maps are inverse of each other it might require a couple of results so let me do that if you start with let F in An be closed then if I go here and come back I will get Z of I of F this is what I get when I go I apply script I when I come back I applies Z so I get Z of I of F but what is this this is supposed to be F bar you have already seen that and but then F is already closed if a set is closed then its closure is equal to its, okay.

So taking the closure of the set is essentially adding the boundary, okay the limits in a very nice sense, okay. So I of Z of I of F is F bar that is equal to F since F is closed, so what it means is that if you go in this direction and come back you get the identity map on the set of closed subsets of An, okay and let us go from this direction so from this direction if I start with let I in K of X1 Xn be an ideal then if I take Z of I and then take script I of that you know that this is because of Nullstellensatz this is radii radical of I this is the enlarged ideal which consist of all those elements some integral power of which is in the given original ideal, okay.

But then if I is already radical it means that radical of the ideal is same as the ideal itself an ideal if the ideal is we say an ideal is a radical ideal if you take the radical of the ideal you do not get anything bigger, okay what it means is that if some power of an element is in that ideal then that element is already in the ideal it means that you do not have to expand the ideal further by taking all possible nth roots for all possible n’s, okay that is what it means.

So if I is radical radii is equal to I, so I of Z of script I of Z of I is just I what it means is that if we start with the radical ideal here I go back go here and come back I end up with this. Now therefore these two statements should tell you that these two are inverse maps of each other from this set to that set, okay that gives you the bijective correspondence here, okay. And there is well there is one more point that needs to be noted you can ask when suppose you are not worried about radical ideals suppose you are just worried about any two ideals you know that already any two ideals can still have the same 0 set here, for example an ideal which is not radical and it is radical they can have the same 0 set so you can ask more generally if what is the condition when two ideals here have the same 0 set and the answer is that they should have the same radical in other words they are radicals of the same, okay.

So here is one more fact so this is another fact that you can try out as a simple exercise Z of I1 is equal to Z of I2 if and only if radical of I1 is same as radical of I2 two ideals will have the same 0 locus same common locus with zeros if and only if there radical coincide, okay. So this is something that you can easily check so that clarifies the bijective correspondence, okay. Now there is one question that one can answer of course coming to the commutative algebraic part the question that you can ask is well maximal ideal have come then of course you know the other important ideals you are worried about are the prime ideals.

So you can ask well if you have a prime ideal here on this side what is so special about the 0 set here, okay. So you can ask so you know now once you start building this dictionary you can take properties here and ask what they correspond to on this side, so for example you can take Z of I where I is radical ideal and suppose Z of I have some geometric property you can ask what does it mean for the ideal I that is trying to come from the geometry side of the commutative algebra side.

On the other hand you could do the other thing you could sort an ideal here which are certain property, okay for example maximality of an ideal is a property, okay and of course primness of an ideal is also a property and you can ask what does that property correspond to on this when you take the 0 set of the ideal, so that is the question I am asking if you start with the prime ideal here what do you get here, what is so special about what you get here, the answer to that is what you get on that side is a strong form of connectedness of the corresponding 0 set and this strong form of connectedness is called irreducibility, okay.

So the answer is that the prime ideals here they correspond to sets on the other side close of course close sets on the other side but these close sets are topologically going to be what are called as irreducible sets and these and irreducibility is a very strong form of connectivity connectedness, okay so will explain that next.

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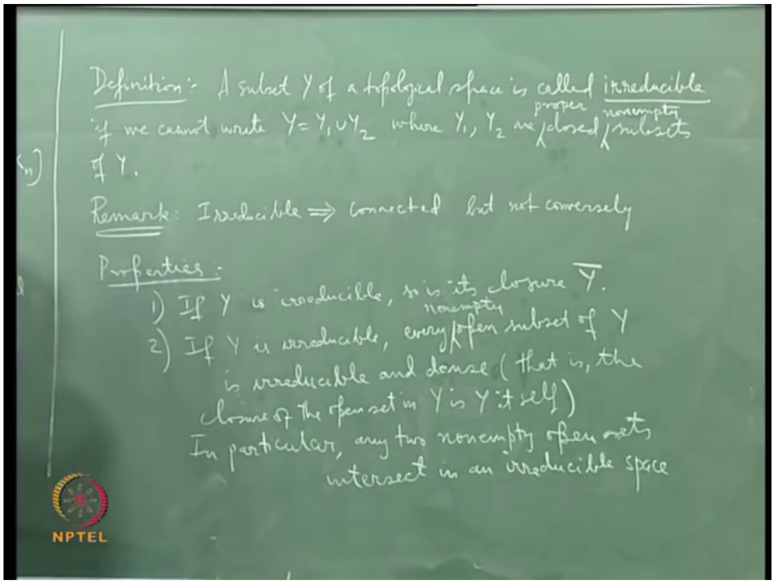

So let us do that so if this again goes so let me so definition a subset y of topological space is called irreducible if we cannot write y is equal to y1 union y2 where y1, y2 are closed subsets of y, okay.

And of course I should say are proper closed subsets here, okay and maybe I should also just to make sure that some silly contradiction do not come I should say proper close non empty sets, okay. So you see I want you to reflect about this with respect to the notion of correctness, see when do you say a topological space is connected, okay you say topological space is connected if cannot be disconnected, and what is a disconnection? A disconnection is breaking a topological space into two disjoint pieces in two pieces which do not intersect so say each piece is open, okay but then since each piece is complement of the other that is because they are disjoint and there union is a whole space it is also same as saying that each piece is closed, okay.

So same the topological space can be disconnected means that you can write it as in two pieces which are closed, okay and mind you it can happen that you may not be able to write the set as two pieces two disjoint pieces which are closed but you might still be able to write it as a union of two pieces which intersect and which are closed, okay. So you know for example you know if you take an interval on the real line an interval on the real line is usual topology is connected and which means you cannot disconnected it if you try to write it in two pieces, okay then you will see that the simplest case one of them will be a half open interval the other will be half closed and therefore you can never break it into pieces with both pieces being disjoint and closed, okay but if you remove the disjoint (())(26:13) then you can do it, okay.

So you know if I have an interval from 0 to 1 I can write it as the union of let us say 0 to 0.8 union say 0.2 to 1 these are two closed subsets take the close interval 0 to 0.8 take the close interval 0.2 to 1 these are two proper closed sets closed subsets because they are closed intervals and they are non-empty the union is again 0, 1, okay. So you see that you can always do but what irreducibility says that even that is not allowed irreducibility is very very strong.

So irreducibility says you cannot write it even as a union forget disjoint union, so when I say that you cannot write it as union of two proper close non-empty sets it follows that you cannot write it as a union disjoint union of two proper non-empty close sets. So what you must understand is that by the very definition irreducibility is very strong condition it is a very very strong form of connectedness and for example I our interval on the real line any closed interval on the real line or for that matter any (())(27:36) on the real line it is not irreducible but it is connected, okay.

So I want you to understand I want you think of irreducibility as a very strong form of connectedness, okay and the definition of irreducible reduces to connectedness when you make this union disjoint, okay. So here is a remark the remark is irreducible implies connected but not conversely so the converse is not (())(28:19), okay so irreducibility is a very strong form of connectedness.

So one more to understand what are the properties of irreducibility? So the there is one thing you can expect some properties that are true of connected sets will also hold for irreducibility. So one of the properties for connected sets is well you know if the set is connected then its closure is also connected, okay this is a very simple exercise in topology the set is connected then its closure is also automatically connected. And the analog of that result also holds for irreducibility if a subset is irreducible then its closure is also irreducible so let me write that properties.

If y is irreducible so is its closure y bar, this is something I mean this is something that you should expect, okay. You know the point is when you take the closure of a set you are only adding the boundary so when you take the closure you are not removing anything and you should always think that trying to remove things might disconnect. So as far you are adding the boundary so it should not disconnect, so that is the reason you take a topological connected subset you take its closure it will be connected but of course you know just adding something will no help if you add something that is away then you already made it into two pieces so you must add something only in the boundary, okay.

So just like adding something in the boundary does not affect connectedness the same way adding something at the boundary is not going to affect irreducibility, okay that is exactly what the statement is that is one thing. The second thing is that something even more serious it is about open subsets of an irreducible set, okay so but before I go to that let me again remind you what you mean by closure, the closure of a sunset is the smallest closure set which contains that subset so it is intersection of all the closed set which contain that subset that is how it is defined, okay fine.

So let me continue with my earlier statement, irreducibility means a lot for open subsets, and what does it mean? It means the following thing you take an you take a subset which is reducible, okay then every open subset is not only irreducible again but it is also dense, okay. So that is a very I mean that tells you how strong irreducibility is, okay. So number 2 if y is irreducible every open subset of y is irreducible and dense, so let me explain so I have to there is something that I wanted to say that I forget to say let me say it now see when I say y is a subset of a topological space and it is called irreducible if you cannot write it as a union of two proper close non-empty subsets what do I mean by close subsets, okay.

So what do I mean by closed is with respect to the induce topology. So if you have a topological space and you have a subset then the subset itself becomes a topological space in what is called the induced topology and what is this induce topology is very easy you simply call you know to define a topology on a subset on a space you have to just give me a class of subsets which you might call as open or closed if you call them as open they should satisfy the schemes for open sets and if you call them as close they should satisfy the schemes for closed sets.

So for example, when I say y1 is closed in y what it means is that y1 is gotten by intersecting y with a closed set in the bigger topological space for which I have not given a name here if you think of y as sitting inside topological space capital X then when do I say a subset y1 of y is closed it is said to be close if it is gotten by intersecting y with a close subset of X the ambient the bigger space, okay.

So whenever you say closed or open with respect to a subset it means it is gotten by taking a closed or open with respect to the big topological space after intersecting with the subset, okay that is what closed or open in a subset means this is called the this is the language of induce topology, okay. So that is what I mean when I say an open subset of y, an open subset of y is nothing but an open subset of the big space in which y sits intersected with y, okay and of course the more important thing is so the point is that any open subset is irreducible, okay which means that irreducibility is a property that passes on to open subsets, okay.

And you see this is not true for connectedness an open subset of a connected set need not be connected, for example if you take the real line, okay if I take union of if you take the whole real line is connected and take the open subset to be a union of two disjoint intervals that is open set because any open set on the real line looks like a union of intervals. So you take two disjoint intervals open intervals that is a subset, but is that connected? It is not, okay. So you see but irreducibility is something that passes onto an open subset, of course I should again say that whenever I say subset I should always keep worrying about the non-emptiness so I should add that every non-empty, okay and you might ask what about the empty set the answer is the empty set is not considered to be an irreducible subset, okay.

So that the reason is it is a matter of logic so the rule in logic is you team a statement to be true if you can test it and prove its truth or if there is nothing to test then also the statement is true. So you know if my y is already empty, okay then I have nothing to test so it will you know probably in that sense it is fair to think of the empty set as not irreducible I think that is the standard but any why let me check once more, yeah the empty set is not considered to be irreducible so I am the book that I am following which is also given as a reference for this course is well the standard book by Robin Hartshorne titled algebraic geometry it is a graduate text in mathematics series gtm52 by Springer Verlag and is more or less the first chapter that I am trying to cover in this course, okay fine.

So as I told you the empty set is not considered to be irreducible, right? So whenever I say open subset of course I am taking a non-empty open subset so you see irreducibility passes onto a non-empty open subset which is not true of connectedness, okay and more importantly this is the more important condition it is dense in other words you take an irreducible space, take an open subset not only that open subset is irreducible but it is dense what does it mean to say it is dense it means that the closure of that will be again the whole (())(37:34) set.

So let me write that down, that is the closure of the open set in y is y itself, okay. So you know why this is so important for algebraic geometry is because you see it is like you are saying every point of y is in the boundary of every open non empty open subset of y that is what you are saying. So what this means is that you know if you want to test things you want to test properties which are going to be preserved under limits, okay for example but when I say limits take it in the naive way, okay because I cannot really talk of limits unless I have a metric and I have notion of conversions and so on and so forth I do not have all that here but I am just thinking of limits as trying to add the boundary, okay.

So when I say any non-empty open subset is dens, what I mean is that is closure its boundary I mean you add the boundary to it you get the whole set it means that every point of the set is either boundary point that occurs in a closure or it is in that open set and so any properties that I have preserved when you go to the boundary they can be tested on an open set, okay because testing if there is a property that is true if there is a property which is such that is true on a subset then is also true on its closure then such a property can just be tested on any non-empty open set then it will automatically be true on its closure which will be in the whole space.

So that is the importance of that is one of the important outcomes of being dense, the other thing is that you know if you take any two open sets they would always intersect that is again because of this dense, okay. So what this so that is another thing you cannot you take any two non-empty open subsets they will intersect, okay. So what this tells you is that in an irreducible space the open sets are huge, see in the if you are for example thinking of an open set on the real line, okay I can make the open sets smaller and smaller, okay or if I take the open interval 0, 1, okay I can find two small open sets two small open subintervals of that which are disjoint from each other which do not intersect and then I can make them as small as I want but I cannot do that here in the case of irreducible topological space because if take irreducible topological space any two open subsets will intersect you cannot make them very small.

So this is one seeming disadvantage with irreducible subsets namely that you cannot get very small open sets but then the fact is that the amazing fact about algebraic geometry is even with this much of information you are able to do all the geometry that you want, okay. So let me write that down here in particular any two open sets any two non-empty so I have to keep one has to keep remembering that one is working with non-empty open sets intersect in an irreducible, okay I mean this is the fact that open sets are huge, okay you cannot find a very given a irreducible space and given a part in the irreducible space you should not think of being able to find very small open neighborhoods of that point that would not happen any non-empty open any neighborhood of the point will be non-empty because it contains that point the moment it is non-empty open sets will be dense you cannot expect to have very small neighborhoods.

You see this gives you the felling again let me repeat this gives you the feeling we cannot take you cannot make an infinite decimal study around a point like you do in the usual analysis you cannot take smaller and smaller epsilon neighborhoods and to analysis it gives you that feeling but that is not true the fact is that analysis is not done on the geometry side the analysis is done on the commutative algebra side, okay by studying the so called local rings of the commutative ring which are obtained by localizing the commutative ring with respect various to prime ideas or maximal ideals.

So the limit process and its and what you get from it in calculus is kind of already buried there in studying the local rings, okay so you should not get the negative feeling that well I am not able to get very small open sets as surrounding a point so I cannot do any analysis that is not true, okay. So well so this is the other thing and, okay so let me come back to this question here if I start with a prime ideal here then what you get on that side if you look at the 0 set, so the answer is if you sort the prime ideal here the 0 set of the prime ideal will be an irreducible closed subset and conversely if Z of I is an irreducible closed subset then the radical of I has to be prime and if already chosen I to be radical it means I has to be prime.

So what it tells you is that the prime ideals on this side they correspond to the irreducible closed sets on this side, okay. So that gives you an answer as to what the commutative algebraic property of an ideal being prime means in terms of geometry, the commutative algebra property of an ideal being prime translation to geometry into the geometry that the 0 set of that ideal is an irreducible closed subset, okay.

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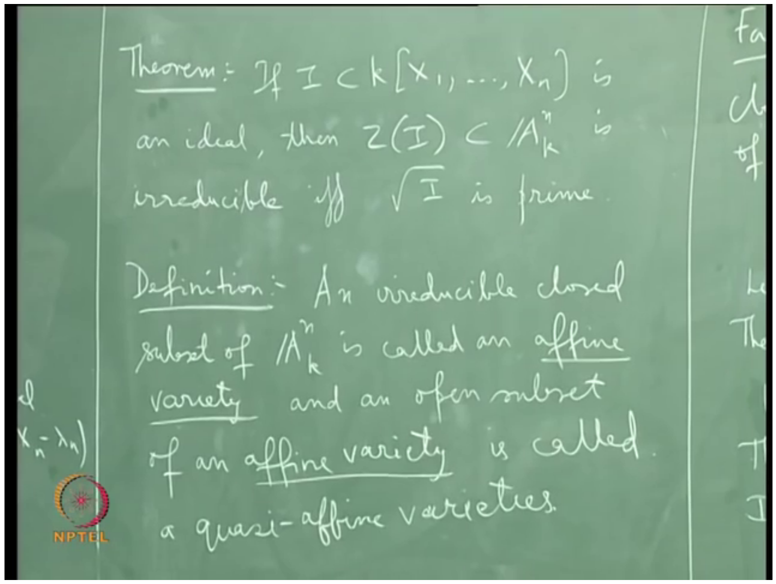

So let me write that so let me keep this diagram here so here is (())(44:58) theorem if I in k expand through Xn is an ideal then Z of I in An is irreducible is irreducible of course it is close by definition but it is irreducible if and only if radii is prime, okay. So that what it means is that if you already have started with a radical ideal then the 0 set of the radical ideal is irreducible if and only if the ideal you started with was already prime, okay and a prime ideal if you take the 0 set of the prime ideal that will also give you an irreducible closed subset, okay and in fact the reason why we are interested in irreducible closed sets of course we got into this notion of irreducibility because it is the translation of primeness, okay but then you might ask in what other ways it is useful, so the answer is it is useful in several ways the first thing is that you know if you start with a prime ideal the advantage is if you go modulo prime ideal you will get a integral domain, okay you take any commutative ring and you go a modulo a prime ideal you will get an integral domain.

And in fact a commutative ring modulo and ideal is integral domain if an only if that ideal is prime. So prime ideals are good because when you take the quotient you get at least domain an integral domain and why are domains good because domains do not have 0 devices and you certainly do not want 0 devices because you see what are 0 devices they are elements on 0 elements whose product becomes 0 but elements in our rings are thought as functions, okay.

So what it means if you have a ring of functions which says 0 devices you are saying that there are two non-zero functions, okay which when you take the product becomes 0, okay which is which never happens in for decent functions if a product of two functions is 0 at a point you expect one of them to at least vanish, okay but so you know it is not good to begin with go modulo any ideal that might result in a quotient which is not an integral domain.

So in that sense if you go modulo only prime ideals then you get integral domains and why integral domain why do you want the rings to be integral domains that is because as I told you you do not want this geometrically non-intuitive situation of having two functions which whose product vanishes at a point but neither function vanishes at a point which you do not expect to happen, okay that is one thing.

Then the other thing is I have already told you that you have for correspond into prime ideals you get irreducible closed subsets and what is so special about these irreducible closed subsets the answer to that is what is called as Noetherian decomposition. So this Noetherian decomposition tells you that you take any close subset you can always break it down into a union of finitely many irreducible closed subsets and this breakup this decomposition as you may call it is unique provided you make sure that you are not repeating any of the subsets as being contained in one another.

So if you so the important result is start with a closed subset here the closed subset can be written as a union of irreducible closed subsets and this union is unique up to permutation of the factors of course if you assume that no component of the union is contained in some other component no piece of the union is contained in some other piece, so this is called a Noetherian decomposition. So what it tells you is that if you want to study if you if you want to study any close subset you can always break it down into a irreducible closed subsets and therefore it is enough to study only irreducible closed subsets, okay. And of course since prime ideals it seems maximal ideals are also prime ideals the subset tells you that a single point is irreducible and that is but that is anyway obvious, okay. So the moral of the story is that we have somehow led to study just irreducible closed subsets, okay and these are what are called as affine varieties, okay.

So the moral of the story is we study the irreducible closed subsets of affine space and we call them as affine varieties, okay and open subsets of such irreducible closed subsets are called cossy affine verities, so the word cossy is used whenever you take an open subset of a whenever you go to an open subset you use the word cossy, okay so let me make that statement so this theorem is quite easy to prove, okay and probably the reason why I am putting this as a theorem is because it also something that probably we will use the Nullstellensatz, okay so this needs to be it is quite easy to prove but let me write down the following thing definition, an irreducible closed subsets of An is called an affine variety and open subset of an affine variety is called a cossy affine verity, okay.

And whole object of the first step in algebraic geometry is to study what is called as affine algebraic geometry and affine algebraic geometry is just study affine verities cossy affine verities and I have already told you why affine verities are important, they are important because any close subset any algebraic set any closed subset can be uniquely decomposed into a finite union of affine verities. So you can analyze any close subset in this way if you analyze affine verities, okay. So probably I will stop here and then I indicate proof of this theorem and proof of other statements that I made in the next lecture.