Basic Algebraic Geometry

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Lecture – 39

Why Local Rings Provide Calculus Without Limits for Algebraic Geometry – Pun Intended!

Okay so what I am going to do now is you know in continuation with our discussion of the importance of local rings I am going to explain the notion of non-singularity okay which is in you know analytical language, classical analytical language you are trying to say when something is a manifold okay so or when something is smooth okay so the, so in other words you know the idea is that an object is smooth at a given point if the dimension of the object is the same as the dimension of the tangent space at that point.

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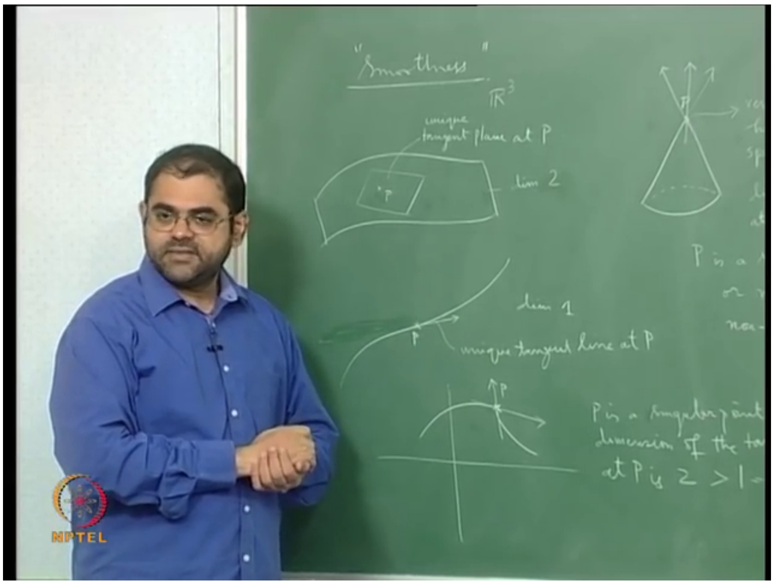

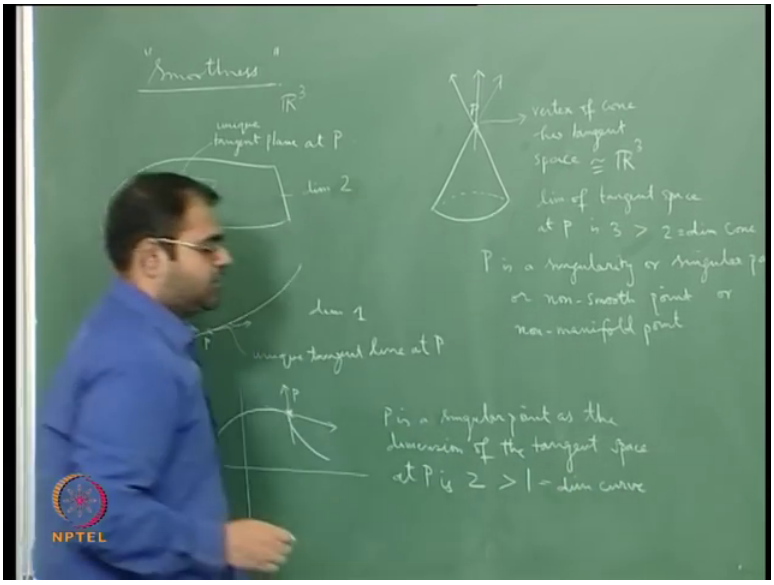

Usually what happens is that you know the dimension of the tangent space will be more and if the dimension of the tangent space is more at a given point then the dimension of the object that you are studying then that point is a singular point, it is not a smooth point so you know for example its smoothness in terms of in the language of classical you know the language of analysis.

So you know if you take a surface and you know you take a point on surface then the surface is, so suppose you are looking at a you know the usual topology and you are in Euclidean space and you look at a surface for example surface of sphere or surface of cylinder or whatever it is okay then you know if you look at the, so there is a dimension of the surface which is say m okay.

So if you are I mean if you are looking at the surface in if you want in so if you are in R3 okay (())(3:37) Euclidean space and you are looking at a surface it will have dimension 2 okay your curve will have dimension 1 alright so a surface will be dimension 2, your curve in 3 space will be dimension 1 and you if you take a point on the surface alright and draw, you try to draw all possible tangents to the surface at that point you get the tangents space at that point.

And if the surface is smooth at the given point P okay then what happens is that the tangent space will also be 2 dimensional okay so what will happen is that you will get a unique tangent plane at the point P okay and if the surface is not smooth at a given point what will happen is that if you look at the tangents, the tangent space could have higher dimension, so for example you know if you take something like a cone and if you consider the vortex to be the point okay.

Then what happens is that if you, how do you calculate the tangent space, it is a space plank by all tangent vectors and what are tangent vectors, I mean they lie on tangent lines passing through the point okay so you know if I try to draw tangent lines passing through this point I can easily draw three you know I can easily draw three linearly independent vectors I can draw three different lines which are given which lie in the directions of three linearly independent vectors okay.

So you know if the cone is like this right I can draw one like this and then I can draw one like this and then I can draw one like this okay so I can easily draw three of them, I get three linearly independent vectors and therefore you know if you take all such vectors the space that they span will be R3, so you know tangent space here this is the vertex of the cone has tangent space isomorphic to R3 as a vector space.

Because you are easily able to find three linearly independent vectors and you know and these are vectors in 3 space alright and therefore the subspace that they span will be all of three spaces so what happens is that the, so here the dimension of tangent space at the point P is 3 which is greater than or equal to 2 which is the dimension of the surface, the cone is 2 dimensional.

But you take that point which is the vertex of the cone there you look at the tangent space namely the space, the vector space spanned by all tangent vectors, the tangent space is 3 dimensional so the tangent space dimension is more than the dimension of the cone and so this tells you that P is singular point, P is a singularity or singular point or a non-smooth point, sometimes in classical, I mean in analysis the language (())(7:54) we also say it is a non-manifold point, it is also called as a non-manifold point alright.

So the idea is that at any point if you want to check whether it is a smooth point or not okay what you do is that you try to look, measure the dimension of the tangent space at that point, if the dimension of the tangent space is equal to dimension of the object, the dimension of the space on which you are considering the point, then the point is a smooth point otherwise the dimension could very well be more.

If it is more then the point is not a smooth point, it is a singular point, so it is also the case, there is also a case with the line I mean with a curve see if you take a point like this on the curve then you know if it is a smooth point I will get a unique tangent direction to the curve at that point and the tangent space will just be a single line, it will just be the line spanned, it will just be the space spanned by a single vector.

So you see the tangent space you will get a unique tangent line at the point P, the tangent space at the point P has dimension 1 and that is equal to the dimension of the curve on which the point P is lying that tells you that the point P is a smooth point okay but however you know if I take a curve which is not a smooth curve then things can be different for example you know vertex something like you know on the plane I can easily draw a curve which is not smooth.

So you know for example if I purposely draw something like this with a kink here if I draw a curve like this and take this point then at this point if you calculate the tangent space you will easily see that you can draw two tangents you know from if I approach from the left okay the I will get a tangent like this, if I approach from the right I will get a tangent like this at this point okay and these two are two linearly independent directions.

Therefore the directions of the tangent space at this point is 2 whereas the point is lying on a curve which is 1 dimensional so the dimension of the tangent is more than the dimension of the curve the dimension of the object on which the point lies and that tells you that point is not a smooth point okay so here what happens again P is singular point as the dimension of the tangent space of the tangent space at the point P is 2 which is strictly greater than.

So here also I should not put greater than equal to, in fact I should put strictly greater than, 3 is strictly greater than 2 and here it is 2 is strictly greater than 1 which is the dimension of the curve, so this is the curve on which the point is lying the curve is 1 dimensional okay but the tangent space at that point is 2 dimension where the tangent space dimension exceeds the dimension these are the singular points okay.

So this is what happens in so I have of course looked at dimension 2, dimension 1 you can therefore say if you are looking at an N dimensional hypersurface okay and in say some which has to be thought of in some Euclidean space of dimension greater than N okay then at a point how do you say the point is smooth or not what you do is that you check the dimension of the tangent space at that point if it is strictly greater than the dimension of the space on which the point lies then it is not a smooth point if it is equal then it is a smooth point okay.

So this is the idea from at least from calculus and geometry usual analysis okay now the analogue for this in algebraic geometry is of course there and everything is I mean this business of estimating this business of calculating the tangent space and its dimension at the point P is done by looking at things connected with the local ring at the point okay so here is a definition.

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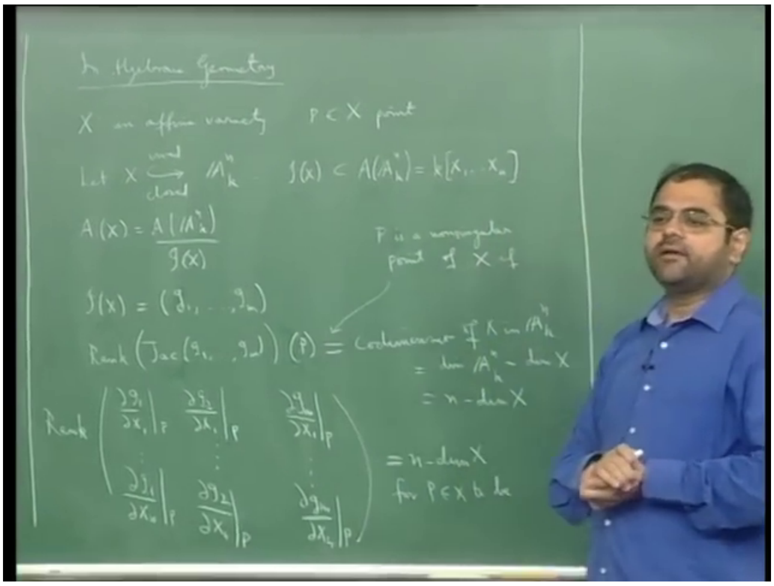

So now I am switching you know some classical or analysis based situation I am going from there to algebraic geometry so you know x, I am going to take the following thing I am going to define when your point of an affine variety is smooth, I mean when it is non-singular okay and when it is singular right so here is, so in algebraic geometry so how do you do it so what you do is that you take x to be an affine variety okay.

Let X be an affine variety, P a point of X okay, how do you define that P is a smooth point or a non-singular point okay so for that what you do is you do the following thing so let X sit inside some An, affine space over k, k is of course an algebraically closed field, X is an affine variety so it is an irreducible subset isomorphic to some irreducible closed subset of some affine space.

So this is by definition X is isomorphic to an irreducible closed subset of affine space okay and then you know you have the you have the ideal of X which is the ideal inside the affine co-ordinate ring of An okay which is actually you know well it can be identified with all the polynomial ring in N variables if you want okay if you take capital X1 through capital Xn to be the co-ordinates, co-ordinate functions then you take the polynomial ring in those co-ordinate functions that is the affine co-ordinate ring of affine space okay.

And X is an irreducible closed subset so it corresponds to a prime ideal so I of X is its prime ideal and of course you know the affine co-ordinate ring of X is given by the affine co-ordinate ring of affine space namely the polynomial ring mod the ideal of X, there is a finitely generated K algebra which is an integral domain okay but the point is more importantly the point is about this ideal see the ideal of X, this is an ideal this polynomial ring which is noetherian ring so it is finitely generated okay.

So let us look at set of generators so f1 say g1 etc upto gm is ideal generated by finitely many polynomials okay this is true because in a noetherian ring any ideal is finitely generated and the polynomial ring is noetherian because that is as you know Hilbert's Basis Theorem or Emmy Noether's Theorem so you choose a set of generators alright now what you do is you do the following thing.

You compute, calculate the Jacobian of this (())(16:39) of functions with respect to these N variables okay you get M by N matrix of polynomials, a matrix with polynomial entries and that you evaluate at the point P okay and then you get a numerical matrix, a matrix with the entries in the field and calculate its rank okay so this is the thing that you will have to do, so you calculate rank of Jacobian of g1, gm with respect to these variables.

So let me just write it like this, calculate rank of the Jacobian of all this at the point P, calculate this number okay so what you are doing is well basically what you are doing is you are taking g1 partially differentiating it with respect to x1 and then you know evaluating it at P okay and then you do it so on with g1 with respect to well all the variables Xn and then now you repeat it with g2 with respect to X1 at P and this is dou g2 at Xn at P okay and then you do it like this, you calculate this, you have this matrix okay.

Now mind you when I say partial derivative, you do not have to think of derivative in the sense of calculus because derivative in the sense of calculus will require a limiting process but do not think of it as derivative in the sense of calculus but think of it as formal derivative because you know you can always take any polynomial in so many variables and you know how to define the derivative okay using the usual rules of differentiation.

So in calculating this derivatives there is no need for I mean you are not going you are not actually computing derivative in the calculus sense okay but you are directly using the formula for the derivative which so you know formula is for differentiation of polynomials, I mean these are the same formulas that you get in calculus but then they make sense even without calculus you take those differentiation formulas as the definition rather than getting them by using a limiting process okay.

So you have this matrix okay this is a matrix you calculate these rank okay and what you do is you, now you do the following thing we say P is a non-singular point of X if, so this is the definition the rank of the Jacobian of the generators okay of the ideal of X at the point P should be equal to the co-dimension of X in An and what is co-dimension it is co-dimension is just dimension of An minus dimension of X so it is just n minus dimension X.

So co-dimension of a subspace is just the difference of the X, you take away the dimension of the subspace from the dimension of the ambient space okay the ambient space here is affine space An the subspace is X which is embedded sitting inside An and you take the dimension of the ambient space minus the dimension of X that is called the dimension of the bigger space minus the dimension of the smaller space is called the co-dimension of the smaller space and the bigger space okay.

So the condition for P to be non-singular point X is that the rank is equal to the co-dimension alright so this is the condition and by the way rank of the Jacobian at the point P is actually rank of this matrix okay so this is equal to n minus dimension of X, so this is the condition for P in X to be non-singular for it to be smooth point okay now the beautiful thing about varieties is that you know they are not always smooth okay they will involve singularities.

But the point is that the set of points which are singular will form a very small subset where you should take the set of points which are non-singular that will be a huge open set they be a dense open set okay so you know if you want to compare a variety with classical smooth object in analysis okay the comparison should be the if you want to think like that what you should think of is that a variety is something like a smooth object on an open set plus a boundary which is the compliment of the open set which will have singular points.

So you know something like a cone okay, if you throw away the point which is the vertex of the cone the rest of it is all smooth okay and that is a dense open set and the boundary is this point which is a singular point so a variety also looks like that there is a big open set which is full of smooth points okay where it is like smooth where it is analogue of a smooth object in analysis okay.

These are all the points where the dimension of the variety is same as the dimension of the tangent space okay and that is what is actually being said in the definition but then we will have to we will unravel this definitions and try to literally see that this is the same as that okay but there is some translation that one has to do okay which we will do okay so when you think of a variety what one needs to remember is that there is an open set, dense open set where it is smooth okay, where it is like a manifold, a smooth object in analysis.

And then the compliment of the open set is a closed set, it is a boundary and that closed set will consists of singular points okay of course there could be varieties which are totally smooth that also can happen and such varieties which are totally smooth are called non-singular varieties right so now let me, this definition looks a little involved okay but the advantage of this definition is that you can do some calculations okay.

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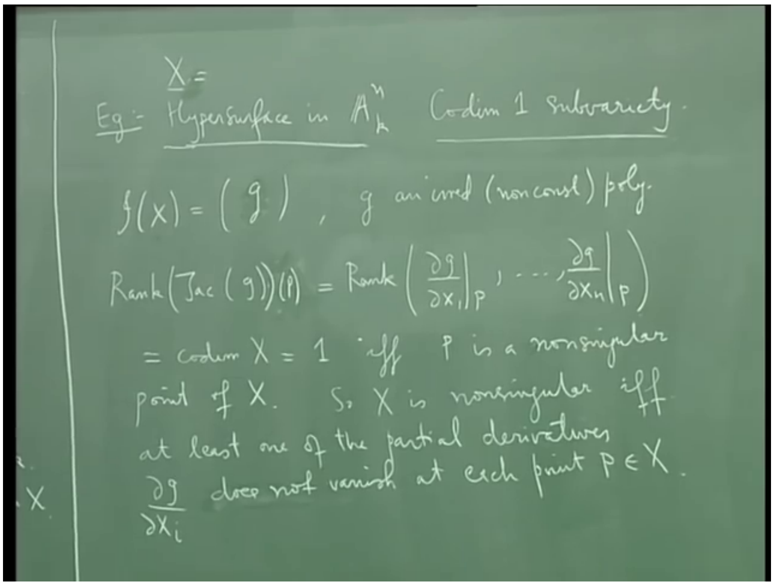

So for example you know so if you want to apply it so let me take example of hypersurface in An okay what is the hypersurface it is a co-dimension 1 subvariety of An, it is a co-dimension 1 subvariety and we have seen this a co-dimension 1 subvariety means that it is an invisible closed subvariety of dimension 1 less so it is dimension n minus 1 and we have seen that this will happen if and only if the ideal of the variety is generated by a single non-constant irreducible polynomial okay.

So you know so if I call that as X is equal to hypersurface in An so ideal effects is generated by f where or let me put g itself, g an irreducible non-constant polynomial so we have seen this and now what is if you take a point P on this hypersurface when is the point P smooth so if I apply this condition so I will get I will have to look at rank of the Jacobian of g which will be just at the point P and that is just going to be, that is equal to rank of dou g by dou x1, dou g by dou xn okay.

So are just looking at all the first partial derivatives of that polynomial and then you are evaluating them at a point P and this is equal to the co-dimension of X and what is the co-dimension of X is 1 okay because X is a hypersurface so the co-dimension is 1just if and only if P is a non-singular point of X, so you know if you want the hypersurface to be non-singular that means you want all the points on the hyper surface to be smooth non-singular points then the condition is that all the first partial derivatives of g should not simultaneously vanish at any given point okay.

And such polynomials is called a non-singular polynomial okay, it is called a smooth polynomial so X is non-singular if, so X is non-singular means X is every point at every point of X is non-singular that is what it means okay so X is non-singular if and only if the rank of this is always 1 okay that means that given any point at least one of the partial derivatives should not vanish okay, at least one of the partial derivatives dou g by dou xi does not vanish at each point of X okay.

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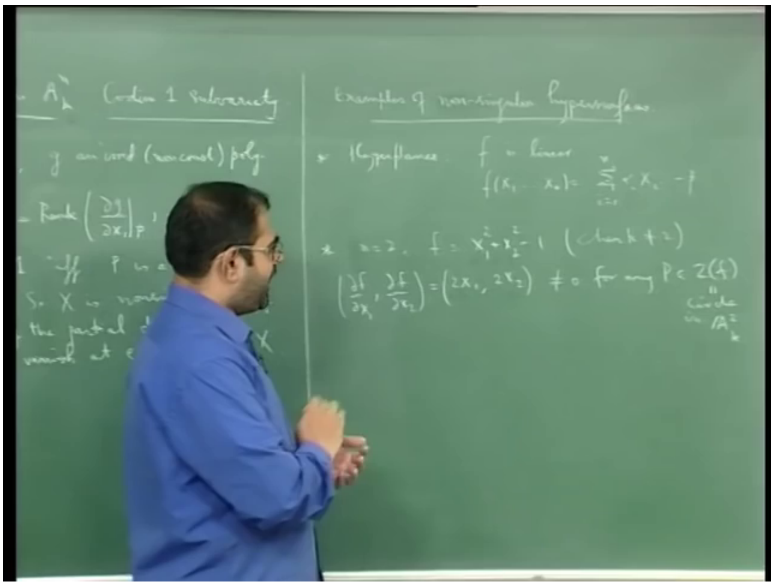

So you know it is very easy to check that a hypersurface is you know is non-singular that is smooth right, so examples of these things are well examples of smooth, I keep using the word smooth but in algebraic geometry the word smooth is a reserved for something more general than this so the word that we use is actually non-singular, so examples of non-singular hypersurfaces or well hyperplanes which are given by you know f is linear homogeneous, f is just a linear polynomial, you take a linear polynomial alright.

So that is f of x1 through xn is just sigma alpha i xi minus some beta 0, i equal to 1 to n, something like this and of course you know I am not looking at the case when all the alpha i's and this beta are all 0 and so I really want a linear polynomial which is not zero polynomial right or a constant polynomial so at least one of the alpha is survives is non-zero, so you if certain alpha j survives then dou f by dou j if I take the partial derivative f with respect to xj, I will get the alpha j which is not 0.

So that will never vanish at any point on this hyperplane so these hyperplanes are non-singular then you can take things like n equal to 2 and you can take f to be x squared plus y squared minus 1 okay then well this is the circle in A2, this is a circle in A2 and you know if you calculate of course I am taking the variables as x and y okay then if you want use standard notation I should take x1 and x2 right.

So let me do that, let me write it as x1 and x2 then you see that if I calculate dou f by dou x1, dou f by dou x2, I get this I will get 2 x1, I will get 2 x2 okay and now you know now you want that at any point of the circle okay one of these should not vanish, so when will both vanish, both will vanish at the origin okay both will vanish at x1 equal to 0, x2 equal to 0 which is the origin in A2 alright.

But then the origin is not the point on the circle so it does not give me, so what it tells me is that this is not going to vanish at any point of the circle but there is a catch, the catch is that your, there is one issue this k could be characteristic 2, k could be an algebraically closed field of characteristic 2 in which case this will be identically zero because in characteristic 2, 2 is zero, so if k is an algebraically closed field of characteristic 2 then this x1 squared plus you know this will become you know this will vanish.

So you have to be careful about the characteristic of the field where you are doing this computations okay and so let me put characteristics is not equal to 2 for safety okay so whenever you get these some integer co-efficients you have to really worry about whenever you are talking about something vanishing okay and you are in algebraic geometry, you are working when algebraically closed field, you should remember that it could be any characteristic.

So if the characteristics divides one of these co-efficients you are in bad shape because it will just vanish okay so if you take characteristics not equal to you know their circle is smooth and well in fact if characteristics is 2 something more serious is happening f is first of all not irreducible if you are in characteristics 2 this is the same as x1 plus x2 plus 1 the whole square because in characteristics P, A plus B whole power P is A power P plus B power P okay.

So you know x1 plus x2 and minus 1 is same as plus 1 in characteristic 2, so this is actually f becomes square of a linear polynomial in characteristics 2 it becomes x1 plus x2 plus 1 the whole square okay and so it is not even irreducible alright, so you have to worry about characteristics alright, of course if you are working over complex numbers one does not worry about these issues but then whatever algebraic geometry we are discussing about is over an algebraically closed field and you know you can have algebraically closed field of any characteristics right.

So this is not equal to 0 for any P in the zero set of f which is the circle in A2 and well now you know you can start this is with one equation you can start looking at objects given by several equations okay and start checking which are the points that are smooth points and whether the smooth points that is a non-singular points are all the points or you get some points singular, some points which are non-singular.

So that way this definition is useful for computation alright but you know the problem with this definition is that there are two problems with this definition, so the first problem is I only defined it, defined non-singularity for a point of affine variety okay I have not defined non-singularity at a point for any variety because any variety in general would be non-affine okay it could be quasi affine, it could be projective, it could be quasi projective.

So well I but anyway I can get over this problem by saying that well any of I know that any variety is covered by finitely many open sets which are isomorphic to affine varieties therefore you give me a point on any variety, I can find an open sets surrounding that point which is isomorphic to an affine variety so that point it is now lying on this open set which is an affine variety and then I can say it is non-singular or not based on this definition alright.

So I can get over this problem of extending this definition of non-singularity to any variety just because of the fact that any variety admits a cover, finite cover by open sets which are isomorphic to affine varieties that is an issue that is easily resolved but there is more something more serious, the more serious thing is this numerical business here, you see there is lot of ambiguity here, you see the same affine variety could be embedded in different affine spaces okay.

I could embed the same affine variety in An, I could also embed it in some Am, if I embed it in a different Am then the ideal will change okay so this ideal depends on the embedding, see this ideal of x is the ideal in the affine co-ordinate ring and that affine co-ordinate ring it is in the ideal of the affine co-ordinate ring of the affine space in which x is embedded but if I change this affine space where x is embedded then I am changing this ring therefore this ideal also changes, this ideal is not an invariant.

See what is an invariant, this is the only thing that is an invariant, for an affine variety the affine co-ordinate ring is an invariant okay, whether I embed X as an irreducible closed sub variety of An or Am any affine space if I calculate this, if I calculate the affine co-ordinate ring of X then you know that is an invariant because that is also equal to OX, you know the regular functions on X.

But the ideal can change okay so that is the ambiguity of the embedding if we change the embedding the ideal will change alright that is the first ambiguity, what is the second ambiguity, second ambiguity is here when I write the ideal I write the set of generators for the ideal, the same ideal can have different sets of generators the sets of generators are by no means unique okay.

So if I change these generators then this you know this Jacobian matrix itself will change instead of well even the number of generators I have, I do not know, g1 trough gm maybe one set of generators, I may find these are m generators, I may find some different number of generators and they may be all completely different polynomials and again I do this computation, what is the guaranty that my definition is consistent.

Okay for the same ideal if I keep the same embedding and therefore my ideal is fixed if I take a different set of generators what is the guaranty that if I compute this I will still get the same rank, so well thanks to God that is the case okay and the point is why does that happen, the one way to see it is using the language of local rings okay so this is where the power of local rings comes in to tell you that this definition is independent of the embedding it is independent of the generators for the ideal that you choose, this definition is absolutely correct, it is not going to fail you, it is not going to become inconsistent okay.

So it is to that end that I am going to state something now, so here is the, this is the fact which was discovered and proved by Oscar Zariski who can very welled be called the father of even the grandfather of algebraic geometry, the grand man of algebraic geometry, he was a person who initially wrote papers in the Italian style where he was, the papers were the proofs were based more on you know geometric ideas and there was no proper rigour.

But then being a you know commutative algebra is (())(42:35), he developed the necessary commutative algebra in field theory to translate all that into modern language and then he was able to rewrite everything gradually and show that everything is can be you know made rigorous using commutative algebra.

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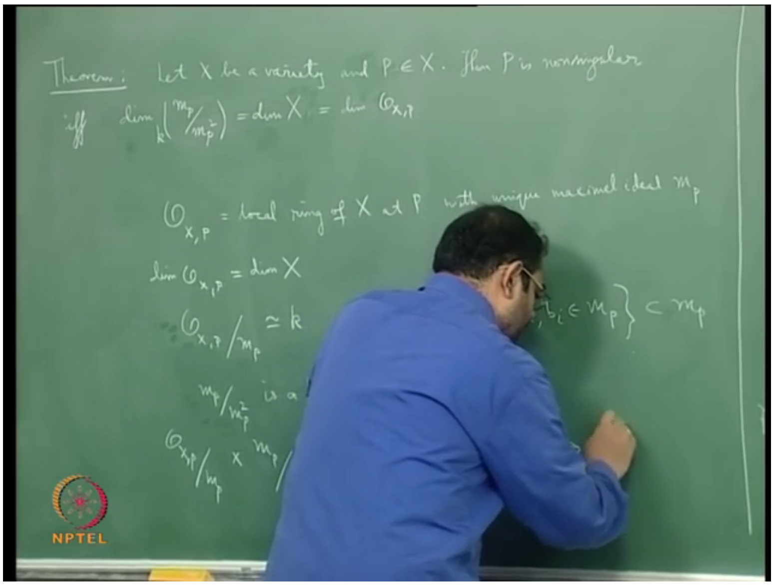

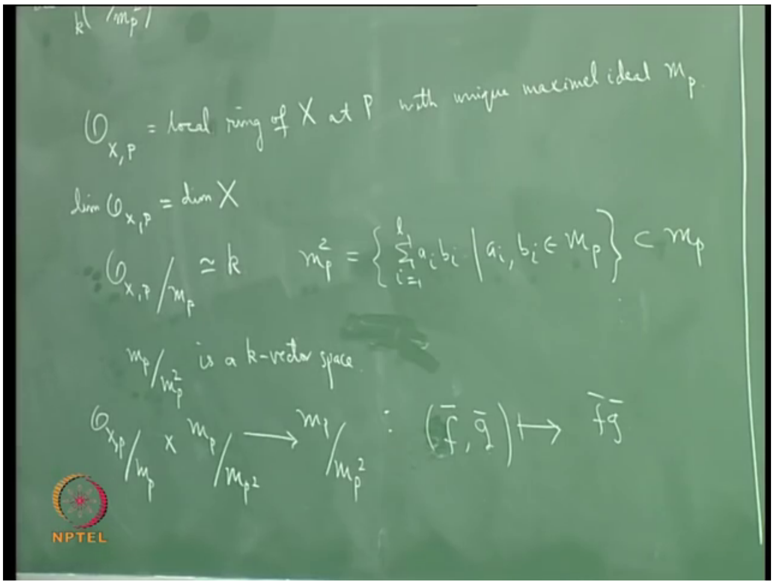

So here is the theorem, let x be a variety and P a point of X then P is non-singular if and only if dimension over k of mp mod mp squared is equal to dimension of X okay so this is the correct statement so what I want to tell you about this is you see you have the local ring O XP okay this is the local ring of X at the point P okay and it is a local ring so it has unique maximal ideal and that maximal ideal is given by this mp okay.

So with, so this is with unique maximal ideal mp right and now what you must understand is you know the dimension of the local ring you know that this is the same as dimensional X this is something that we already know okay, the dimension of the local ring is a same as the dimension of X and so you know you can here I can add if I wanted if I want to reflect the point P, I can also write dimension of O XP okay and so if I remove the, if I do not look at the central term then I have condition which seems to have only to do with the local ring.

I am just saying that the dimension of the local ring is the same as the dimension of m mod m squared so you know what I want you to understand is the local ring modulo the maximal ideal will give you just k okay and what you must understand is that if you look at mp squared okay this is the square of the ideal mp, so you know this is just consisting of elements of the form you know sigma ai bi, i equal to 1, 2 some l where ai and bi are in the maximal ideal okay this is just the squared ideal alright.

Finite sum of products taken two at a time from the ideal okay from this maximal ideal mp and you know this is if you look at it, if you look at and you of course this is contained in mp okay this is certainly contained in mp because if you take two elements of this ideal and multiply them out the product is certainly in the ideal this sums that finite sums are also here, so this is contained inside this but the point is that if you look at mp mod mp squared okay this is a k vector space okay this is a k vector space.

And it is a k vector space just because of this fact because O mod m is k okay so this is a k vector space it is module over k, well in fact you see you take O XP modulo mp you can define scalar multiplication like this by simply you know f, so you know f bar, well g bar going to well f bar g bar, I think this should give you an obvious map which will make mp mod mp square a module over O Xp mod mp.

But O XP mod mp is just k therefore mp mod mp squared is a module over k so it is a vector space okay so you can well you know you can check that this is, this map is well defined where this f bar is, where f is an element of the local ring so it is germ of a regular function at the point P and f bar is its image actually f bar is just that regular function evaluated at the point P okay, it is just evaluation.

And here you are taking g bar is just the image of a g, g is just a regular function in a neighbourhood of P, germ of a regular function in a neighbourhood of P which vanishes at the point P so g is in mp and its image in the quotient mp mod mp squared is g bar, so you are reading g upto you know going mod mp square is just reading only the linear (())(50:10) when you go mod m you are evaluating at the point see these are all actions that are going on with the elements here, the local ring, what are the elements of the local ring?

The elements of the local ring are regular functions, germs of regular functions, how do you get this quotient isomorphic to k, what you do is give me a regular function at that point you evaluate it at that point, you give me a germ of a regular function here you evaluate it at that point that will give you a map from this to k its kernel will be exactly mp, all those germs of those regular functions which vanishes at the point P.

So the quotient will be k so this isomorphism is just evaluation arises just by evaluation of a germ of a regular function in a neighbourhood of P at the point P okay and what does m mod m square stands for, it is you take a function which vanishes at the point P, germ of a regular function which vanishes at the point P that is what a function that belongs to mp means and reining it mod mp squared means that you take literally you know its derivative.

Because you know you are cutting if you read mod m squared that means you are not, you are only taking the linear tem, you are not taking the degree to term onwards so in a sense this corresponds to taking only reading only the linear term alright and you know the first order term always is the derivative so going m mod m square is reading of the derivative in a certain sense okay.

And therefore this is how m mod m squared becomes a k vector space okay and it is it certainly a finite dimensional vector space because you know after all O XP is a noetherian ring you know this local ring is a noetherian ring and this ideal mp in this noetherian ring is finitely generated you take a set of generators and take their images here they will give you generators for this quotient.

So it is a vector space which has finitely many generators and therefore it is a finite dimension vector space it is a vector space which has a finite spanning set so it is a finite dimensional vector space okay therefore this is a finite dimensional vector space you calculate its dimension okay and this dimension if it is equal to the dimension of x then and only then is the point P a non-singular point.

So you know what this quantity is, you know this quantity is actually the dimension of the tangent space at the point P this quantity dimension of m mod m squared over k actually measures the dimension of the tangent space to the variety x at the point P and what normally will happen is that this will be more than this as we saw in those examples of cone and a line and a curve with a kink at a point, what happened at the singular point was that the dimension of the tangent space shot up.

The dimension of the tangent space became more than the dimension of the object at for the vertex of the cone the dimension of the tangent space is 3 whereas the cone is only 2 dimensional so the vertex is not a smooth point it is a singular point, similarly if you take a line with a kink if you take a curve with a kink at the point where you have the kink you know that the tangent space becomes 2 dimensional.

And the dimension is 2 which is greater than dimension of the curve which is 1 so that point which is the kink is not a smooth point, it is a singular point okay so that is exactly what is happening here so it is a matter of little bit of commutative algebra to check that you know if you have a noetherian local ring with maximal ideal m then the dimension of this vector space will always be greater than or equal to the dimension of the local ring okay.

And if the dimension is greater than the dimension of the local ring then that point P is not a smooth point it is not a non-singular point, it is a singular point it is a singularity if the dimensions are equal then it is a smooth point, so the point I wanted to understand is that what this, see what this theorem is saying is exactly the analogue of what we saw in the analysis calculus situation.

That your point of a variety is smooth if and only if dimension of the tangent space at that point is exactly equal to the dimension of the variety and what will happen if it is a non-singular, if it is a singular point, this will be bigger than this you will have more, tangent space dimension will be more than the dimension of your, of the space on which your point lies okay.

But the nice thing I want you to notice is that whole thing that is done using calculus all that has been captured just using local rings that is what I want you to appreciate okay so when you do usual calculus how do you define a tangent space at a point, you take a point you draw a curve through the point and then you draw the tangent to the curve at that point okay and then like this you try to fill the neighbourhood of the point by curves draw tangents and then now take the all these space of all these tangents that is a tangent space.

So it involves the usual facts from calculus thinking of curve passing through the point and drawing tangents and all that okay and of course even to find the tangent to a curve at a point it is a limiting process right, because you take a point and you take a sufficiently close point and then you draw a cord and then you take the limit as the sufficient closed point tends to the given point so the cord becomes the tangent at that point.

So in usual calculus even the process of getting hold off a tangent is a limiting process and then you in this way you build the tangent space you check out what the tangent space is you calculate these dimension and then you check whether the tangent space dimension is more or whether it is equal to the dimension of the object and then that is how get smoothness or not and the (())(57:31) sort of calculus but you see in algebraic geometry all that limit process is not there.

But still you are able to capture the smoothness, the non-singularity the key is local rings okay that is one fact then the other fact is what this tells you is that you know this based on this definition tells you that this definition is independent of the embedding of X in affine space, if X is an affine variety then a point of X is smooth okay that is a condition which is intrinsic to the point because that condition only depends on the local ring at that point and the local ring at a given point is invariant, it will not change if you no matter how you embed you variety.

The local ring is in invariant and therefore this theorem tell you that the non-singularity that you have defined here actually really does not depend on this embedding or this ideal or I mean this embedding which dictates this ideal and then the ambiguity of what generators you have chosen for these ideal okay so it is a very intrinsic statement okay and that tells you that you do not have to worry about this definition.

But it is useful to make calculations and to check that a given you know variety defined by a bunch of equations is smooth or not at a point so in that way this definition is useful but that theorem tells you that you are not going to go wrong if you use this okay, so I will give you proof of this in my next lecture okay so I will stop here.