Basic Algebraic Geometry

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Lec 36

The Importance of Local Rings - A Rational Function in Every Local Ring is Globally Regular

Ok so now what will do now is, and you know the importance continuing importance of local rings so, so again you know just recall you start with variety capital x and the point small x then you have local ring at capital x small x which contains the regular functions global regular functions then the regular functions defined all of capital x and which in turn contained in the function field of x ok, which (())(02:07) this quotient field of the local ring as well and now the local ring so the point is that local rings lot of information that's the ideal.

So what we saw in the last lecture how local rings captures isomorphism off course isomorphism induces isomorphism at the level of local rings and conversely if a morphisms induces isomorphism at the level of local rings and it is a homeomorphism then it is an isomorphism ok that's what we saw, now I am going to show is I am going to discuss local rings and regular functions and I am going to say that if you take an element here namely an element which is a rational function that means it's a function is a regular and open set and suppose this element is here for every local ring then it is actually define on all of x so it's a regular function on all of x.

So it's very clear if you take a regular function on all of x then regular function is off course rational function because after all for by definition of rational function is just a regular function define on an open set and since if I take a global regular function is define on all of x which is also an open set so any regular function is always rational function and every any regular function is going to live in every local ring namely that regular function is going to give you its germ in its local ring.

So something here is going to be in each local ring and off course is going to be here as well then something here then it is here the condition is that this if the sum is that is lives in every local ring then it's here ok so, let me state that that's all the important thing so to check that function is defined everywhere you check and somehow check that it belongs to every local ring then its define everywhere ok and reason why I able to do that is because all the local rings in sit inside the quotient field all the local rings sit inside the quotient field.

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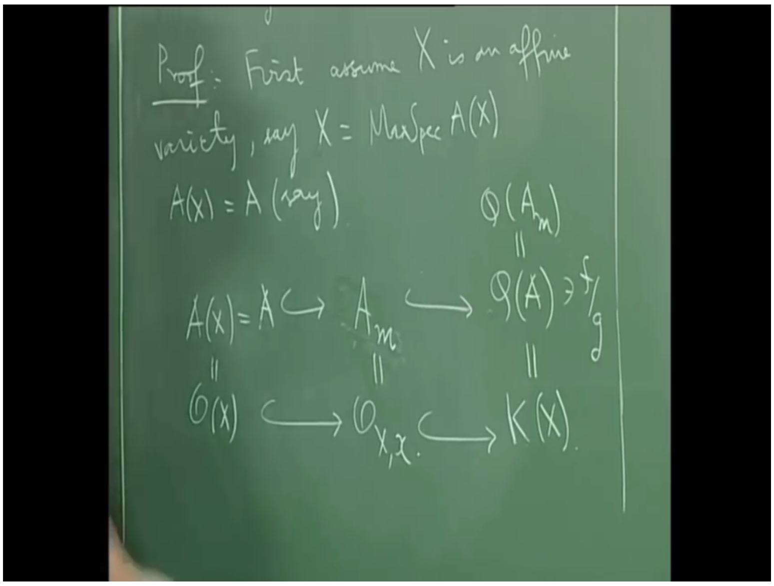

So let me write that here local rings and regular functions, so here the theorem, theorem is let x, p, a variety and element of k, x, that is in a sub ring o, x, x, for all small x in capital x is actually in o, x, so this is a theorem it says that you take an element here, then the rational function if it is in every local ring then it has to be a global regular function ok, so let me the proof, so what I want to tell you is that see if you think it at philosophically it quit you can even think as this oblivious or topological ok.

Philosophically what you are saying is that you have a rational function namely function is define an open set but you are seeing that it is equal to the germ of a regular function at every point which means you are saying that lives in a neighborhood of every point and that is same as saying that its domain of definition that is a maximal open set where it is define its all of x which means it’s actually the open set where it is define it is not just a proper open subset of x but it is all of x.

Which means it is a global regular function ok so that’s the philosophically but then it inverse little bit of commutative Algebra and that’s what I want to tell you see the important thing is that you know trying to say that rational function is define at as a regular function in a neighborhood of a point is this statement that you know that this actually belongs to the sub ring that the fact an element here belongs to sub ring like this is another way is actually the commutative Algebraic way of saying that the rational function is actually define the neighborhood of point small x ok.

That is or you can be extend it to a neighborhood of point small x ok, so, so you know trying to say that the rational function is define at every point is if you want to say it commutative Algebraic how do you say it, you say it like this, you say that the element of the rational function to fines an element k, x, in here, you are just require in that it is in each of these sub rings I mean if it is in one such sub ring then it is equivalent to saying that it is define in neighborhood of the point small x ok, so the problem with rational function that it could be maramorphic ok namely it could have basically it could be a quotient of regular functions and may not be define where the denominator function is vanish ok, and therefore it need not extend to all of x.

But the fact that it extends to a point is effected by saying that it belongs to sub rings ok and that's a contained of the theorem so let me do the following thing so what I first like to say is so we use this fact namely any variety is caught by affine varieties ok, any variety is covered by open subsets in fact finitely many of them each of which is isomorphic to affine variety so what I will do is, I will go from x to you know I will first take the case when x is an affine variety ok I will first solve theorem for the case when x is an affine variety and then I will say that therefore also solve the case when x is a general variety because general variety is a finite union of a affine varieties ok.

So first assume the x is an affine variety say x equal to max spec k, e affix, e affix is equal to a say, so you know there is how to recover an affine variety from which coordinate ring if x is an affine variety you are a, x, is a fine coordinate ring and if you take maximal spectrum then you get back the affine variety ok so in fact, so you know if you want x sitting inside on a affine space as an reducible close subset and a, x, is the coordinate ring of x ok it's the regular functions on affine space namely polynomial ring in as many way which is a dimension of an affine space module o, the ideal of the close subset reducible close subset which is the prime ideal ok.

So e is an integral domain it is finitely generated K Algebra right and now what happens is that you have so you have situation like this you have k, x, which equal to o, x, that is a, for us, ok, because f, x, is affine a, x, is same as o, x, that is a, and that is sitting inside well o, x, x, local ring ok, and mind you local ring well I will draw the right, I will write it here ok, and this is sitting inside k, x, this is a function field right and how does this translate well o, x, is a, x, which is a and then local ring is a. m. ok where m, is the maximal ideal of a, that corresponds to small x.

So off course small x is an element in capital x, which is, which can be taught as maximal ideal in a, x, ok, which a, this is just a, m, ok this is expression for the local ring at a point of a affine variety where the point corresponds to the this maximal ideal m, ok and what about the quotient field this will be you (())(12:03) q of a, you have the quotient field of x, and that will be same as the quotient field of x, the function field of x ok, so this mind you q, a, is the same as q, a, m, if you have an integral domain then the integral domain every localization of integral domain is also integral domain and the integral domain sits inside as a sub ring in its localizations and all the localizations have and the integral domain itself the all have one, the same quotient field ok.

So this is a, is an integral domain a, m, is localization of the maximal ideal m, which means that you invert everything outside the maximal ideal and both a, and a, m, have the same quotient field ok q, of a, ok and that identifies the function field of x ok and now what's happening is that I have so what is given to me is a rational function which so this is an little bit of simple commutative Algebra but the point I want to make is that whatever commutative Algebra little simple commutative Algebra that we are doing is beautiful enough to give as this result alright.

So you start with rational function here, I mean you start with rational function here by definition it is here so it is a form of f, by g, ok so, I start with an f by g, here ok I start with f by g here, because it is see because it is a rational function is an element here and this is identified with this and easy able to write as f by g, because this is a quotient field of a, where f and g belong a, and what is given to me is that this f by g belongs to a, m, to every maximal ideal m, that's what is given to me ok.

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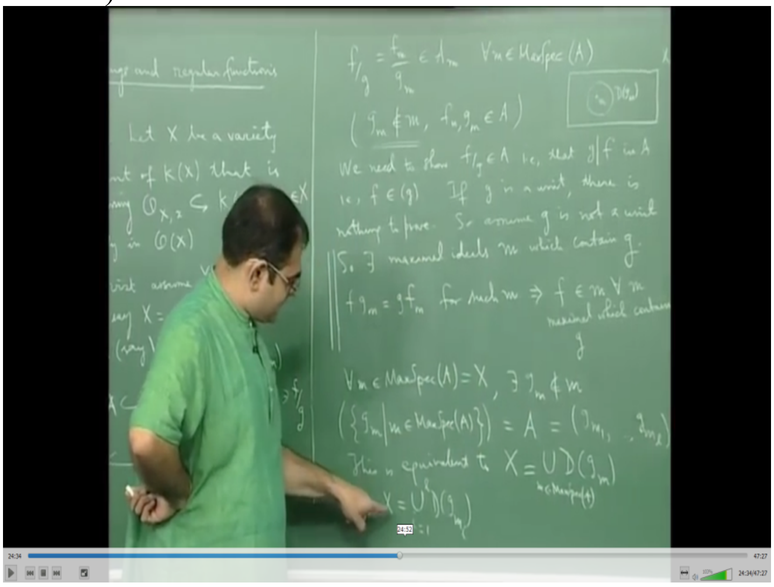

So f by g is equal to well an element here ok and an element here well let me write this as f sub m by g sub m, in a, m, for every m, maximal ideal so max spec is set up to for maximal ideals of a, and for every maximal ideal I have this local ring and this f by g, belongs to the local rings means the f by g is equal to an element of the local ring alright and element of the local ring is written like this where the denominator is not in the maximal ideal ok this is given to me.

What I will have to do, I will have to show that this f by g, is actually in a, ok, I have to show that this rational function is started with is actually here ok so I have to show that f by g belongs to a, which means I will have to show that you know I have to actually g, I have to show that g divides f, ok, I have to show that, if I show that g divides f, ok, then I am just that's same as saying f by g, is an element of a, and that will that f by g belongs to a, so I will have to show that g divides f, alright.

So we need to show f by g belongs to a, that is that well g divides f in a, or in other words you just have to show that f belongs to ideal generated by g, this what you will have to show which f is multiple of g, so f by g is an element of a, ok that's what you will have to show now you see now let's analyze this off course you know if g is a unit there is nothing to prove which is unit there is nothing to prove because then f by g is f by a unit ok, and f by a unit is also an element of a, ok.

So assume that g is not a unit so assume g is not an element since g is not a unit it means that a you know every non-unit is contain in a maximal ideal ok so there aces, there aces maximal ideal and which contain g, ok, there aces maximal ideal which contain g, and you know for such maximal ideal n, you look at this equation so what you will get is you will get is f g n is equal to g f m, for such m, implies what does it implies?

So you see g is in the maximal ideal m, ok and therefore this right hand side is maximal ideal m, therefore left hand side but maximal ideal is prime and therefore one of them has to be in m, but g n is not in m, therefore f has to be in m, ok, so this is tell you that f will have to be in m, for every m, maximal which contains g, you have this ok and any way but what is very very important in this condition even if I take an m which may or may not contain g, ok.

Perhaps this is just an observation that comes from this equation but what is more important is this condition that for every maximal ideal getting an element outside that maximal ideal and the beautiful thing about that, is that you know whenever you give me one element out of each maximal ideal then this correction of this element actually generated the unit ideal ok that crucial topological fact it is same as saying that all the basic open sets define the g, m's, they cover the whole space ok.

See now let us not very about this observation for the movement ok let's look at this see you take g, m, does not for every m, in my spec a, which is actually x, there aces g, m, which is not in m, that's what you have here alright, and now the claim is the set of all, you take the all these g m's. Where m belongs to, m is a maximal ideal of a, the ideal generated by this is actually a itself which is unit ideal ok and on the, the answer is, I mean why this is true and very very simple,

Because you see if you take the ideal generated all these g m's the ideal generated by g, m's if it is not a, if it si not the unit ideal then it is a proper ideal alright some, an ideal which is not the unit ideal it is a proper ideal and you know any proper ideal is always contain in a maximal ideal therefore this ideal will then will be contain in a m not, ok, but then g m not will be in this ideal which is contain in m not but by definition g m not is not support to contain, is not supposed to contain, be contained in m not that's a contradiction ok.

So let me repeat it if this is not a then this is contained in m not, where m not is a maximal ideal ok, and then the problem is that g m not is not in this collection and that will be contained in this, which is contained in m not but g m not is not supposed to contain, be contained in m not, you get a contradiction that for this ideal generated by all these g m's is a ok this is just equivalent to so let me also write, say that this is equivalent to x is given by the union all the d g m’s where m belongs to max spec a because what you should understand you see.

You see if g m is not if g m is not in m, ok you look at d g m, what is d g m, d g m is the basic of affine open set where the function g m does not vanish ok so it same as saying that the point corresponding to m is in the basic open set define by g m ok so but I will cover all the points I have taken all the maximal ideals so I have covered all the points and for each point I have this d g m so you know if you draw picture this condition is actually like this and it is a bad picture which is very only sytheortically descriptive but it is not a correct picture but it is correct enough to think a little so you have basic idea for things are so you have x even any point m ok you are thinking of maximal ideal in a as a point of x ok and for this every point you are able to find is d g m.

Which is an open subset which contains at that point alright that is what is means to say that g m is not in m that what is geometrically need and since you have covered every such point the union of all the d g m’s will be x ok but then you know Zariski Topology is quasi compact therefore only final key many g m’s have to cover x but that fact finitely many g m’s cover x as same as saying that those finitely many g m’s will generate the unit ideal ok so which if and only if x is union of I is equal to one to l, the g m i, so here I will have those g, the ideal generated by d g m one etcetera up to d I mean up to g, g m sub l.

So if something is union, something is unique ideal then one can be generated by finitely g m’s call those g m as g m one through g m l, then the ideal generated by g m one to g m l contains one so it is full ideally but then writing a is a equal to the ideal generated by g m one through g m l, is topological equivalent to writing max spec of a which is x is union of affine space by g m I’s that is what is topologically is that ok.

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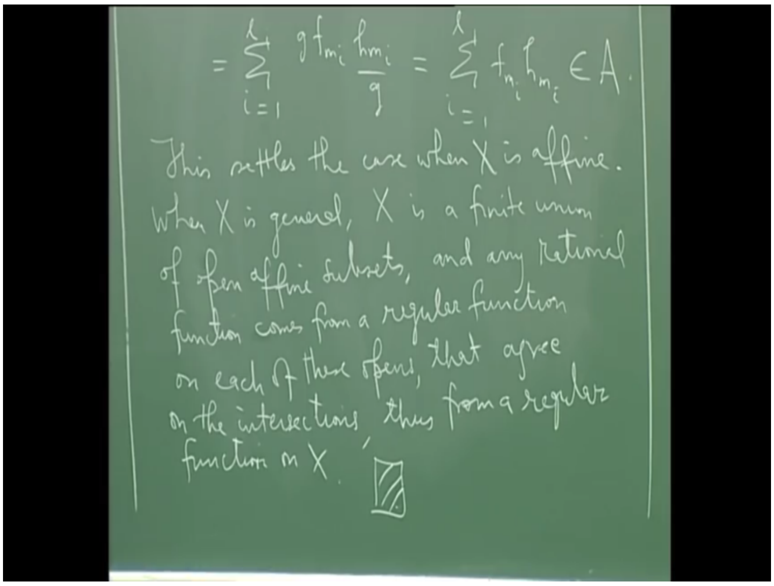

So you can see side by side Algebra and topological going hand in hand for Zariski Topology now if you look at carefully what you get is well therefore what you get is therefore there asec f m i, so let me use something h m i, in a such that sigma h m i, g m i, is equal to one or equal to one tera this is what is mean to say that g m i’s generates one ok, that means a ring linear combination of g m I’s is equal to one for a suitable ring linear combination ok.

Now you know but then you see now I can use this I can now use now I can take f by g is equal to f by g times one where you know I am doing all this computation here in the quotient field ok whatever equation I am writing now are valued in the quotient field f by g is f by g times one and the one, the one in a, is same as one in the a m is same as a quotient field ok, so this make sense of quotient field so it is f by g sigma I is equal to l, h m I, g m I, because after all this is one now I push inside I will get sigma, I is equal to one to l, f by g times h m I, g sub m I, but you see I have f g m I, is g f m I, alright so so this is going to be sigma.

So here f I, I can write a sigma I equal to one to l, f g m I, h m I g, but that is equal to sigma I equal one to l, f g m I, is actually g f m I, by g, and when g cancels out what you get I will get sigma I equal one to l, f m I, h m I, which is actually need this is an element of a, ok so what I have done is I started with f by g, then k x, and I put actually in a x, so it is actually a regular function ok.

So this set is the case when x is a affine ok now when x is not affine, you know that x can be covered by finitely open sets each of which is affine and what this argument will tell you is that on each of those affine’s the rational functional will be regular function ok and what will tell you is that the regular function lives on all open sets of a cover ok, and it co insides and it will off course co inside on the intersections therefore it will given, at give a global regular function ok.

So that finishes the proof of general case ok so let me write that down, this settles the case when x is affine ok when x is general x is a finite union of open subset open affine subsets ok, if x is a general variety then you have seen that any general variety is a finite union of an open affine subsets and what we, what we just prove the rational function ok, the rational functions will not change whether you consider rational functions of x or rational functions of any open subsets ok therefore what you will get is that the rational functions started with actually is a regular function on a each affine open which form, all of which form a cover for x, and off course on intersections also it will be given the same function.

Because everything is finally happing in quotient field which is a same ok, so basically what will happen is that you will get a regular function on each affine open subset by the previous argument and all the regular function will co inside on the intersection therefore they will define a global regular function, after all global regular function is a global function which is locally regular ok so to define a regular function you have to just define it on an open cover ok, so the case when x is general also covered ok, and so that is a general case ok you know if you want maybe I can even write something simple so that it will be more convinced comfortably.

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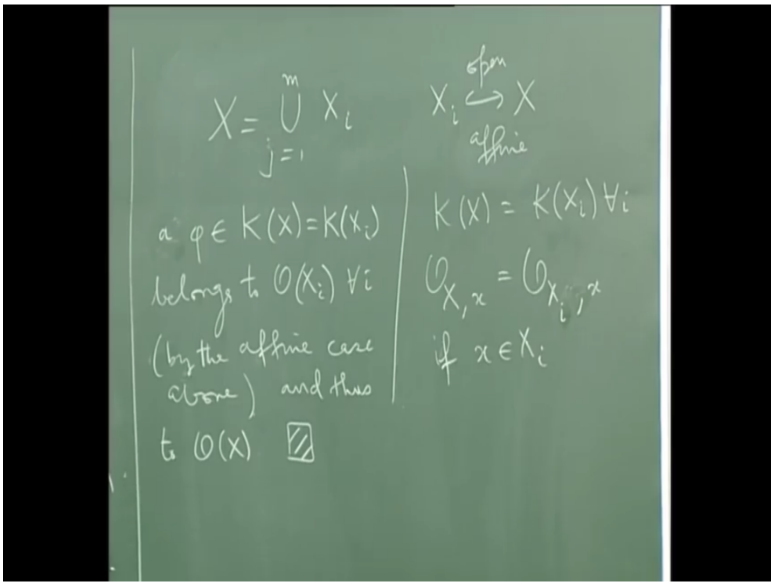

So you know x is union I see j equal to one to m xi where each xi inside x is open affine which means that each xi is an affine variety, each xi is an open set of x which is isomorphic to affine variety, and then you must understand that you know kx is the same as k xi for every I, ok, and o capital x small x is same as o capital x small xi, o capital, small x, capital xi comma small x, if small x is an xi, you know the local rings and function field do not change if you go to open sets ok.

So what will happen is that if you start with the element of k x it will belong to or it will be taught as, k xi for each I but because if xi is affine and it's each local ring by the previous argument it will be o xi for every I, so you have an element k x which is in o xi for every I but then that will be mean it is actually in ox, ok, so and Fi in k x is equal to k xi belongs to o xi, for every I by affine some above and thus to o x that is the end of the proof ok so after all you know to define a function you have to define functions on open sets which agree on intersections ok and similarly you define a regular function you have to give near regular function an open cover which agree on the intersections and that's what happens here ok.

So this is a very nice fact that to conclude the rational function is actually global regular function that it is define everywhere you just have to check that it belongs to every local ring ok so that is another that demonstration another important power of the local ring ok, then the next thing I want to tell you about is something that for more performed it is a following thing see that, you can see that somehow a very word local ring will tempt to you think that it, it somehow in codes all the local information at the point that the problem is local information if you want to think of is information on an open set continuing the point in a simple open sets a huge after all every open set is reducible and dense ok.

therefore if you take an open neighborhood of a point it is not a small neighborhood it is not small in the sense of usual topology it is not like hang a smaller and smaller disc or smaller smaller interval of a point on a real on the real line or having smaller smaller disc of excel on radius of a point on the plane it is not like that every open set is huge therefore the fact that the local ring stores such information should tell you it is actually having much more then local information so the truth is, the answer to that is yes it actually also stores information on a huge subset containing the point so that is a power of the local ring.

So what I am going to next is state is very beautiful result it takes two varieties x and y ok, and take a point one variety take a point of the other variety, take a local ring at this point for this variety and take a local ring at that point for the other variety, if just these local rings isomorphic then there is a whole open subset containing this point in this variety and an open subset containing that point in the other which are isomorphic so if two rings are isomorphic if the local ring of a variety one point just the local ring alone is isomorphic as K Algebra to the local ring of another variety at some other point then a huge neighborhood of this point on this variety has to be completely isomorphic to huge open neighborhood of the other point.

So even isomorphism of local rings is actually in the sense pretty global ok, it tells you that if two varieties have two points which are whose local rings are isomorphic it means that these two varieties are on an open set, on an open sets they are actually isomorphic that means the only place whether they are not isomorphic is a boundary ok, so that is another powerful property of the local ring so let me state that, so local rings they will call as local rings and by rationality.

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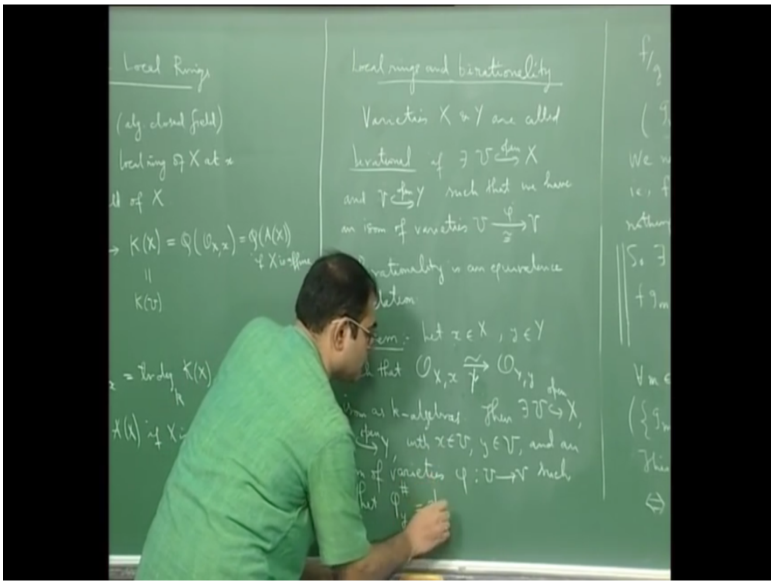

So local rings by rationality ok so let me define this motion by rationality by rationality will you two varieties are call birational if there is a non-empty open subset of each which are isomorphic ok, so x varieties x and y are called birational if where exits u open x off course non-empty and v open y, off course again non-empty such that will have an isomorphism of varieties give it to be ok, so two varieties are birational if there is an open subset of one which is isomorphic to an open subset of the other.

See the varieties them self-need not be isomorphic but the point there is a huge open set of this, of each of these which looked the same ok that is the concept of by rationality now what you must understand is that actually the motion of by rationality actually is an equivalence because you know if x is, if x and y birational that off course the symmetric ok, x is birational to y, is same as y is birational to x, and off course an isomorphism is always if two varieties are isomorphic there are off course birational because you can choose the open sets to be the whole varieties themselves.

So the notion of by rationality actually it's a relation which is equivalence relation if x is birational to y then y is birational to x, any variety x is always birational to itself by the identical map which is isomorphism ok, and if x birational to y, and y is birational to isn't the x is birational to is it ok, because any two open sets, always intersects ok, so if you have an isomorphism if u inside x is isomorphic to v inside y, and vepra inside y is isomorphic to w inside z, then you can then the isomorphism composed in, on v intersection v prime which will be non-empty ok.

Therefore this by rationality of the varieties is an equivalence relation ok, and if go if you take set up of all varieties if you take the category of all varieties and then you go module o, is equivalence you get what is called the set up birationalequivalence classes of varieties and actually it is a the first step in classification of varieties is to fine in each birationalequivalence class varieties say good properties for example variety which is smooth ok, and so you know basically whenever we study objects in mathematics we always ask the question how many non-isomorphic objects are there ok.

For or in other words you try to describe the setup of isomorphism classes of objects ok, this is called the classification problem so you can ask the classification question for varieties, you can ask you take all varieties and you go module isomorphism then you get the setup isomorphism classes of varieties what is that's it? Ok now the fact is that that's not very easy question, ok and in fact a the first answer is to instead of going module o isomorphism which is very strong you try to first go module o weak a required relation like by rationality so you first ask the question what is the set-up birationalequivalence classes of varieties ok.

That brings down the problem to a more tack able level ok so by rationality is very important classification of varieties so let me write this down by rationality it is easy for you to check this equivalence relation and now comes the theorem that I want to state theorem is let x belong to x, small x being a point capital x, small y be in a point in capital y, such that off course x and y are the varieties o x x, the local ring of capital x at small x, is isomorphic to, o y y, isomorphism as K Algebra.

So I just have two varieties capital x and capital y and point small x belonging to capital x small y belonging to capital y, such that the local ring of capital x at small x is isomorphic to the local ring of capital y at small y by isomorphism which is an K Algebra isomorphism ok then the result is that there is a open set u in x in capital x which contains the point small x there is a open set capital v of y, which contain the point small y there is an isomorphism of varieties from u to v, which under pull back regular functions at the local rings induces Fi, ok.

So then there aces u in x open,v in y open with x belonging to u, y belonging to v, and an isomorphism of varieties so let me use so let me use a different single for this so let me call this something, let me call this as Si, Fi, from u to v, such that Fi hash y, is actually sum, (()) (45:30) ok, so it is a very very power theorem, what it says that if one local ring of one variety is isomorphic to another local ring of anther variety then these two varieties are birationalthey are the same on huge open sets on they are the same ok, and that tells you how much information local ring contains just isomorphism we think as local ring as something that's concentrating attention on a point but this tell you much more than that.

Just local ring at one point one variety be an isomorphic to the local ring at another point of another variety is enough to say that these two varieties a same or a huge open set accept for a boundary they are isomorphic ok so that's the fact that given you that the local ring contain lot of information even globe more or less, global information as listens that it contains information on contains information on a big open set ok.

So needs to proof this and again you give the fact is that this also involves the commutative Algebra but pretty easy commutative Algebra involving localization and it is very it's poof is not very difficult but the point is that you, just use nice commutative Algebra to capture these two open sets as nice basic sets ok are isomorphic basic open sets contain toes tow points which are isomorphic ok so let's do that in the next lecture.