Basic Algebraic Geometry

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The Importance of Local Rings - A Morphisms is an Isomorphism if it is a Homeomorphism and induces Isomorphism at the level of local Rings.

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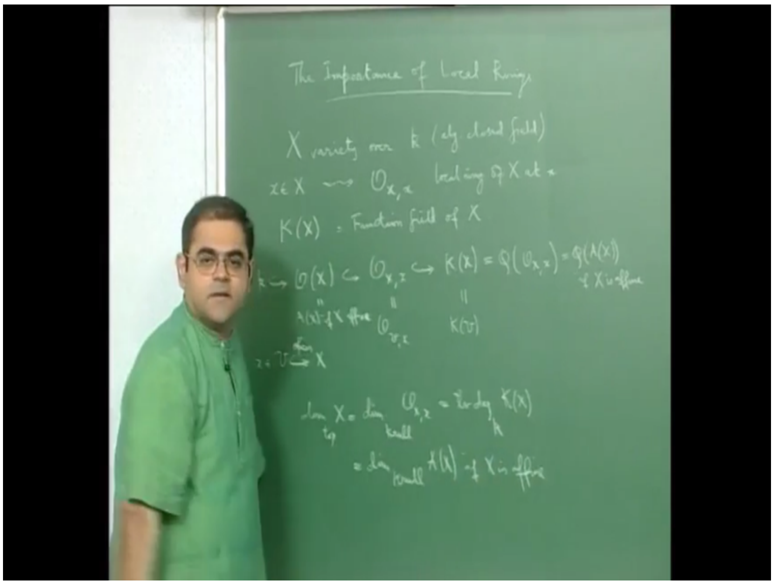

The importance of local rings, so we have already seen this that if x is a variety over on algebraically close field k and you take a point small x in capital x then we have defined the local ring of the variety capital x at the point small x ok, and so this is the local ring of capital x and small x then you also define k x which is the so called function field of x, function field of x, ok, and see we have we have seen that k x consist of equivalence classes of rational functions namely functions that define regular functions which are define not on all of x but on an open subset of x ok, and off course the equivalences that too.

Two functions are equivalent if they co-inside on the intersection of their domains of definition on the other hand the local ring is suppose to be gotten by doing the same thing but concentrating only attention to regular functions in a neighborhood of a given points small x ok and we have seen that you know o x which is the set up all global regular functions on x this is is a sub ring of off course of the local ring of every local ring and that is content in a quotient field and in fact off course k is off course setting inside regular functions as constant functions ok and and in fact this k x is this k x is actually the quotient field of o x x, the quotient field of this exactly this.

In fact if you go to any open set u, then which is non-empty then k x is same as k u and o u x is same as o x x ok so let me also write that, this is equal to o u x and this is also equal to k u where u is an open subset of x and yes small x and capital u and so local ring does not change we got a small open set and the function field also does not change if you go to smaller open set because so called smaller open set is actually not so small ok.

Any open any non-empty open set of a variety is reducible in fact it is dense so what you are thronging out the boundary it just a very set ok so, so the, and we have seen that the local ring already there is not a information stored in these things in fact so I also mention that this o x, so I also say that ox is, this is equal to a x if x affine so that is also, you also have that o x is same as a x if x is affine and then and in that case this is also equal to the quotient field of a x off course when I say, when I write a xi this is define only when x is affine that is x is isomorphic to an reducible close subset of some affine space and off course there are so many other things, off course if x is projective we have seen that o x is, this is equal to this ok.

If x is projective then o x is just k alright then because there are no non-constant global regular functions on a projective variety right, then we also seen that this is local ring stores lot of information that, so in fact the dimension of x a topological dimension of x is a same as the krull dimension of the local ring at each of its points and this also equal to the transcendence degree over a small k of the extension capital k x, capital k x is a function field of x, is an extension of k, it contains small k after all this k is a field these things, this is a finite, I means this is a this is a K Algebra and this is local ring.

This is a local K Algebra, local ring which a K Algebra where this is a, n of filed this is large, and everything is contain in this big filed ok, and small k to capital k is a field extension ok because it is a bigger field containing a smaller field and then the transcend degree of this over this namely the by that for means the cardinality of a transcend basis of this over this ok, which measures a number of maximum number of Algebraically independent you know transcend elements over small k ok.

That is a measure of the dimension topological dimension ok, and the dimension also show up as a krull dimension of the ring ok. And you also know that this is also equal to the krull dimension of a x, if off course if x is affine ok, so already there are so many information but then my the object of this lecture is tell you that this local ring is really very powerful thing it contains it stores lot of information.

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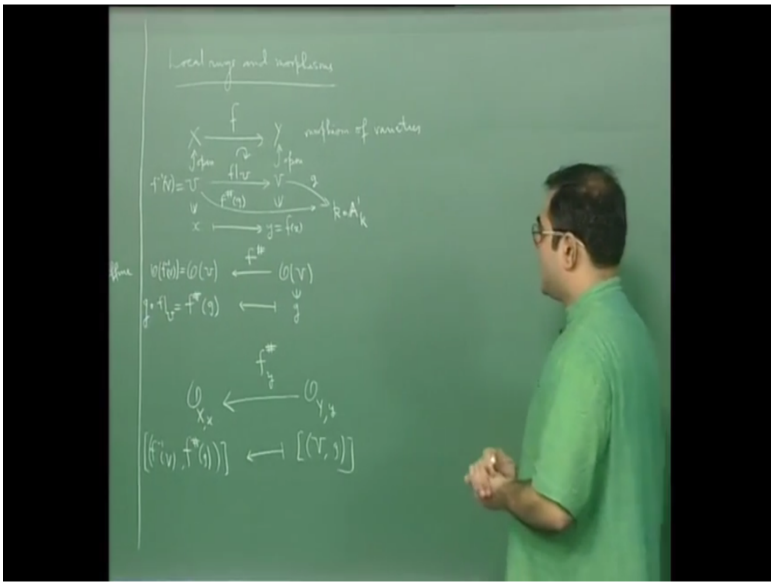

So let me tell you the first thing, so it’s about the so alright so, so let me talk first about local rings and morphisms ok, so you know suppose x and y are varieties and f is morphisms of varieties you know that, that means at x and y could be both x and y could, each of x and y could be affine, quasi affine or projective or quasi projective and the fact isomorphism morphisms if it is continues and f pulls back regular functions to regular functions.

So the point is that you know if I take a point small x here and it goes to point small y which is the image of x under f, and you take an, you take an open set u, an open set here which contains a point x and you take well rather I will do it the other wrong, let me take an open set v which is a neighborhood of point y and let me set u equal to f inverse v, then off course f being continues inverse image of, Fi is an open set I called that as u, off course f factors like this is this is, f restricted to u this diagram (())(10:20) ok, and now you have a map from o, you have map from o v, to, o u which is actually o of f inverse of v and the map is just pull back of regular functions.

So the notation is f hash and what does it do if you give me a regular function g on o v what I do is that I mean what this does is that it, pulls back g to f hash g and what is f hash g, f hash g is just first apply first apply f restricted to u and then apply it. So f restricted to u followed by g, so this what, so this is pull back up functions ok.

So basically what’s happing is that giving a g on o v, means you are giving function from v to k which thought of us a one and you know Zariski Topology on k, you have a one and g is morphisms so regular function is a just morphisms into a one so g is a morphisms at a one then you compose into this and then you get a morphisms like this and this is precisely hash of g this is a pullback of g by f ok, and the condition for morphisms is that not only f should be continues but whenever you start with regular function on target on open subset of the target.

When you pull it back you should get a regular function on the pullback of that open set on the source ok that's part of the definition of a morphisms and one of the important thing is that what this leads to is that, from f this induces a map hash y which will go from a local ring of y at small y to the local ring of capital x at small x ok.

You have the map at the level of local rings alright and this map is off course again induces very same thing namely what you do is, what is, what is an element here and an element of the local ring is just an equivalence class of a function which is regular on a open neighborhood of point y so you have something like v comma g, this is what an element here looks like some equivalence class where g is a regular function v, g belongs to o v, and v is neighborhood of y, will contains the points small y and what you do its very simple.

You simply send it to the class that corresponds to the pullback up of the functions which is also regular function, so I will take f, f hash g and I take, I take it on u which f inverse of v, and I take this, then I take the class, so this is how this map is define so all I am trying to say is here morphisms is a varieties, that morphisms induces for every point small x and you take its image in capital y, you get in the reverse direction you get the morphisms of local and you get one morphisms like this q zero full bunch of morphisms of local rings go in the other directions ok.

One for each point here and its image there alright, so, so you have this situation and now you can see that it's all right its rather easy to see that you know if f is an isomorphism then you have map going other directions ok, and that map will induce that map in this direction now and you will see that f is an isomorphism then all these maps on local ring are all isomorphism ok, so in fact I am still use all this ring maps could be talking about here are, not just ring maps they are in fact K Algebra Homomorphism's ok they are, their identity on K and they are K linear ok.

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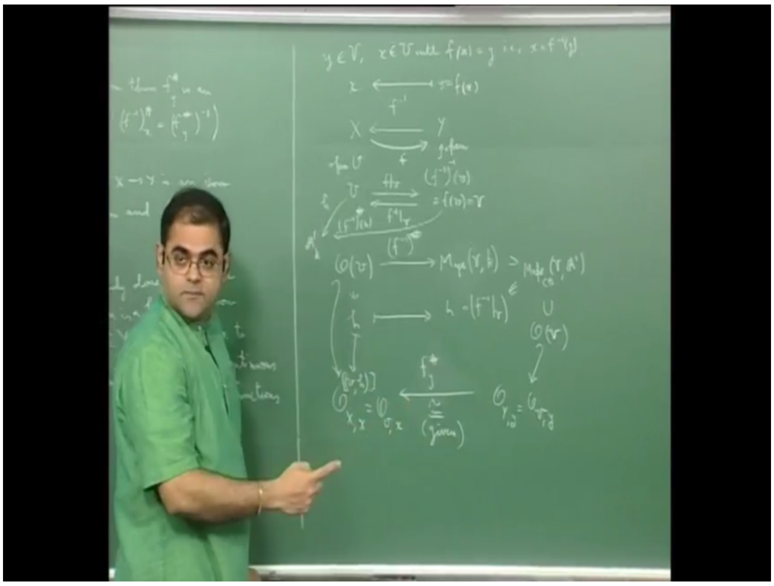

So it is clearly, it is clear, clearly, if f is an isomorphism, isomorphism then f hash y is an isomorphism, for all small y in capital y for all small y and capital y, so you have this, this is very clear ok, because yo just have to f induces f hash y in this direction then f will induces a map in this direction which will be f inverse hash x, and you will have f inverse hash x will be the inverse of, f hash y ok.

So let me write that since f inverse hash x will be just f hash y inverse yo will get this ok, so because you know from this diagram if f is up if some isomorphism g pulls back to f hah g under f then f hash g pulls back to g under f inverse ok, it is quite obvious so but the beautiful thing about local ring is that you can decide whether something is a morphisms you see the fact that it is a Homeomorphism, it is a morphisms which is the Homeomorphism ok.

That is a topological isomorphism plus it induces at the local ring isomorphism so that is the big fact so here is the theorem the theorem is a morphisms f from x to, y of varieties is an isomorphism, if and only if f is a Homeomorphism and f hash y is an isomorphism, for all y one, so here is a power a very power theorem that tells you about the power of local rings, so about tells you is that if you want check a morphisms and isomorphism’s then you first check it is topological isomorphism ok then check at the local rings the morphisms induces the level of local rings they are all isomorphism’s then morphisms and isomorphism's.

So for the proof of this you see one way we have already discussed that if f is an isomorphism, then off course f is a Homeomorphism, f y hash is an isomorphism, for everywhere that something we have already seen ok, you have to proof the way round ok so one way is already done, so assume that f is a Homeomorphism, f is a morphisms which is a Homeomorphism and f hash y is an isomorphism, for all small y and capital y ok.

So what do I have to proof that I just have to proof that if is an isomorphism which mean I have to show f inverse is a morphisms that's all, ok, how do I show f is an isomorphism just by showing that f has not inverse but f already has a sytheortic inverse because it is a homeomorphism f inverse is a continuous map f has a the continuous map f inverse ok, which is also a homeomorphism I just have to show that the f inverse is a morphisms once done I, once I prove that I am done ok.

So we only have to show f inverse is a morphisms but you see f is a homeomorphism, so f in verse is always continuous ok, so the only thing that you have check is that the pulls back regular functions to regular functions ok, how do you check something is a morphisms? You have to check two condition only, one condition is that, it is continuous, the second condition is that you have to show that this pulls back to regular function to regular functions.

So but f inverse is already f is already homeomorphism so f inverse is also homeomorphism so it is continuous, the only thing that I have check is that f inverse pulls back regular functions to regular functions, so since f inverse is already continuous you only have to show f inverse is, f inverse pulls back regular functions to regular functions, I just have to show this ok, so so what do I do to do that, so you know I have this situation like this so here is x, here is f inverse and here is y, I this should, f is already off course is a homeomorphism, I have to show it pulls back regular functions to regular functions.

So what I will do take I will take an open subset u, u here open subset ok, off course whenever we talk about open subset I not mentioning it but I am only interested in non-empty open subset ok so then I take f inverse, inverse of u which actually f of e which v and this is an open subset here because after all f is an open map f is a homeomorphism, so it is an open map so f of u is an open subset so.

So f inverse, inverse of u is just f u which just v, and well f takes u to v so I have two arrows one arrow in this direction which is an isomorphism topological isomorphism which is f restricted to u and I have map like this which is f inverse restricted to v (())(21:54) what I have to show I have to show that start with the regular function here I have to show that f in, f inverse pulls back regular functions to regular functions so I will have to look at o, u, which is a regular, these are regular functions on u, and from here I will get o, v, these are regular functions on v, and this is f inverse hash it is.

I have to check that this f inverse hash which is pulls, pulling back regular function by f inverse, I actually that this has to, have to show that this lands into this ok so, so let me do something if I write like this then I am already assume that its land inside o, v, I have do something I will have to write o, so I will just write maps from v, to k, ok, and mind you I start with h, which is regular functions on u, ok, so I have h like this its morphisms into a one ok, and what I will have to do is the pull back of h under f inverse hash is just composition of h with f inverse ok.

So what I will get, I just have to apply, I will have apply f inverse to v, I have take f inverse restricted to v, I have to first apply that will take me from here to here and then I have to apply h this is what goes to ok and now so this is the composition is this so it is map like this this is f inverse hash of h ok, it is just this f inverse forward by this h, I pull back regular function h to function on v and I will have to show that this function on v, these functions from v with values in k, is not just function I have to show its regular.

So first thing I want you to notice is that is not just a map from v, to k, it is actually goes to the subset of maps continuous to the Zariski topology v to a one certainly it is a continuous map because it is a composition of f inverse which is a continuous map and the map h and the map h is a continuous because mind you an y regular function is continuous, after all a regular function of morphisms into a one and a morphisms is always continuous ok.

So this is a composition of continuous functions so it continuous right but what is have to show so this is this is, dot just here but it actually belongs here but what I have to show is that it's actually belongs e one in o, v, this is what I will have to actually show, I have to show that this is here, I have to show it is regular alright that is a smaller subset not every continuous map is regular function ok.

So but then the point is how do I check something is regular? How do I check a map is a regular? See the notion of regular is something very local ok, so to check that something is regular, I just have to check it at every point alright so you know so I will have to check that this function is regular, at each point of v, that's all I have to check right and so for that we do the following thing you know, so now comes the, this business about, so now comes, so now

I will have to use hypothesis that f inverse hash I mean these f, y, hash are all isomorphism for every small y, and capital y, well I have to use this.

So what I will do? I start up with y, and v, so I have so I have, you start up with small y, and v, and you take small x in u, with f of x, is equal to y, which is same as seen x is f inverse y, ok, so small x, small y, goes to small x and this y, is f, x, f of small x, ok, and off course I choosing y in v, so you know I have this v, is regular functions on v, then I have this o to, y. this is same as o, v, y, because u told you the local ring is not change if you got to our smaller open set alright, and you know the regular functions are setting inside global ring off course ok, and on the other hand I have also o, o x x, o capital x small x which is a local ring of capital x at small x this is same as o u x, because after all you use smaller open set which contains point small x and the local ring does not change if you go to smaller open set and this is also contains inside this ok and the point is that this f inverse hash induces f inverse hash x, so f inverse so I will have to I will still have to say that f inverse is a so I need to do something here I will not use this I will draw use clear the map that is in the other direction which is given to me.

What is given to me? What is given me, given to me is f y hash is a isomorphism so you know there is a map like this which is f y hash ok, and this is an isomorphism this is given, it's given that this is an isomorphism alright it is given that this an isomorphism and mind you this f y hash induced by f which actually growing in that direction ok, so it's given to me that this you know this f y hash is an isomorphism for every small y ok I have to use that so well so what do I do? I do the following thing see you know, see this h is a regular function on u since h is a regular function on u, it gives me germ at the point small x.

Namely what is the germ, germ is just you take the equivalence class u comma h, are you take par u comma h and take its equivalence class that's an element here ok, this si how you have a map from the regular function on an open set to the local ring at a point on that sit namely you take this par consisting of the regular function and that open set and take the equivalence class and such equivalence class is up to equality on intersections is what constitute local rings at the point ok. So I have this but then you know f hash y is an isomorphism so this counts from something here and that is here that's in a local ring therefore what it means is that.

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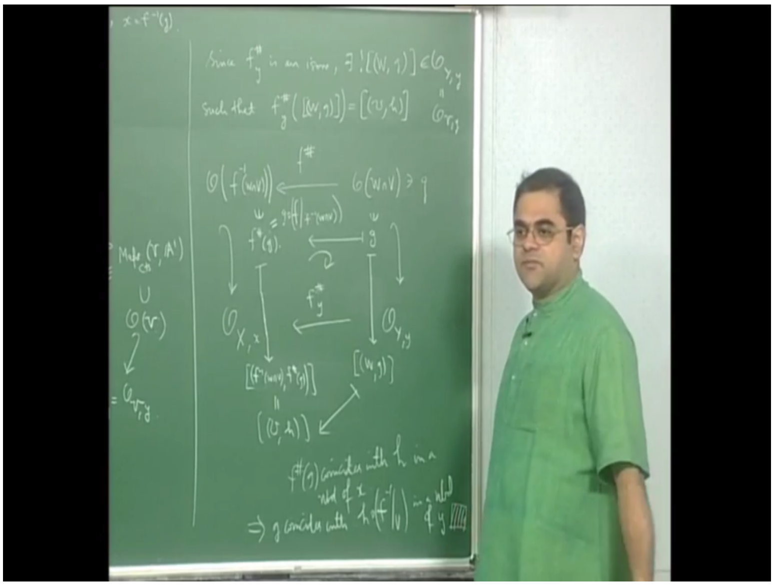

So let me write this down since f hash y is an isomorphism par axis germ here which goes to that and how do, how is a germ here represented. A germ here is represented by regular function on an open subset on y ok, so what I have is well I have there axis w comma let me use some other symbol (())(31:15) I want use g little use to it, the axis w comma g which is a germ an element of the local ring of capital y, small y, which is off course same as o, if you want to v, y, so you know that in w can I can even take w setting inside v, such that in fact this is g because this is an isomorphism such that f hash y, of this w comma g is is this class u comma h.

So this is this happens because this has a pre image here this is an isomorphism ok, so what I get is a, I have I have a w comma g which goes to this on right hash y, because and that w comma g is off course unique germ because it is because if hash y is an isomorphism K Algebra isomorphism, now what you will have to notice is that see if you now look at g, this g is define on w and this w is an open, see after all w is open subset of y and in fact it contains with the point small y so it also intersect v, even think of w is an open subset of v, but any case if you want to think of it as a open subset of y then always intersects with v.

So what happens you have o, of w intersections v you have this ok and you have fellow here, namely you have this g, g is a regular function there ok and what's happing is that well you know if I take the image of this under f inverse ok, then I will get I will get f inverse of w intersections v, ok, and I will get o of this and this will be f hash I have this f hash ok this f hash, you see f in this direction so f hash will be in this direction, so f will go from f inverse w intersection v, to w intersection v, ok.

So f hash will be pull back regular function in the opposite direction namely a regular function on w intersection v, goes to a regular function on f inverse of w intersection right and so you know if you take this g here what I will get f hash g, that will be here so this f hash g will be a regular function on f inverse w intersection v, and off course f inverse w intersection v, will intersect u any way because x is w intersection v, and x is also in u, ok, and the point is that see it is this f hash which actually induces f hash y, because you should remember that the morphisms the K Algebra homomorphism the local rings are induced from the given morphisms by pull back of regular functions.

So the point is that you have competitive diagram like this so you have if you go through the local ring of x at small x and you go to local ring of y at small y then this is setting inside this, this is setting inside this, and you have this you have this f hash y and this f hash y is precisely going to take see the germ of, f hash g, see to, whatever is going to go to ok, that is a same as taking the germ here and applying f hash y.

But you know if I take the germ here g is going to w it is just going to this germ ok, see it just going to the germ of w comma g and that so maybe I will write little below so that I can have more space when these guys really going to go here to, to germ of w comma g, and this fellow by our definition of f hash y, is going to go to well u comma h, germ of u comma h and that is precisely what this f hash is, f hash g has to go to ok.

F hash of g has to go to u comma h alright so the moral of the story is that you see f hash of g, f hash of g and see but f hash of g will go to what f hash of g will go to w intersection v, intersection u, comma, so let me write that down correctly so what you will get here is so let me have some more space to write so why, what is this going to go to, it just going to go to if you want w, f inverse w intersection, f inverse if w intersection v, comma f hash of g and that is a same as this fellow here u comma h, as germs because this is what goes to that ok and you have something like this, this diagram computes ok.

Now so the moral of the story is that this f hash g, and h co inside when are two germs the same when where are two germs are same, when are the germs of two par's the same by the definition of the local ring two par's define germs if the functions co inside on the on intersections so what this tells you is that f hash of g, co insides with with h in a neighborhood of y, of x, ok f hash of g co insides with h ina neighborhood of x, that is because these two germs are the same alright.

But then but what is f hash of g, f hash of g is just you know you apply you first apply f and then you apply g so if hash of g it just g circle f, g circle off course f restricted to f inverse v intersections w, this is what f hash of g is, f hash of g is just the pull back of function g, so g is a regular function on w intersection v, and f takes f inverse w intersection v, to w intersection v, and you first apply f then you apply g the composition is precisely f hash of g so what you are saying is that g circle f restricted to f inverse w intersection v, co insides with h but that is a same as saying that you know because f is a homeomorphism it is a same as saying that g co insides with h circle f inverse v, in a neighborhood of y.

So this implies that g co insides with h, h circle f inverse restricted to v, in a neighborhood of y, because you know f is a homeomorphism ok on whichever neighborhood f hash g co insides with h if you apply, if you apply f inverse on the write you will get that g co insides with h restricted h circle f inverse v, so you see what is f hash g, f hash g is, g circle f restricted to f inverse w intersection, f restricted f inverse of w intersection v, ok that co insides with h ok, then if you apply f inverse on the write you will get g co insides with h circle f inverse restricted to v in a neighborhood of y, because you just apply f inverse ok.

So what have we proved? What we have proved is that this h circle f inverse v, in a neighborhood of point small y is equal to g but what is g? g is a regular function, so what you have proved is that this function this function that is define on the whole of v, is at every point equal to regular function that means it is locally regular and a function that is locally regular it is regular because the definition of regularity is local ok therefore we are done so this implies that h circle f inverse v, is actually o, v, ok, so that hence proof, that hence proof.

So let me repeat what we did is we started with a regular function, on h on u, and then we pulled it back the wanted to show that this is a regular function, on v, ok but what we end of proofing is that for every point small y of capital v in a you give me any point of small y of capital v, there is a neighborhood where these function is equal to a regular function, that means these functions locally a regular function, here function which is a locally regular function, is regular because definition of regularity is local this function is already continuous ok.

So we have proved that f inverse hash pulls back regular functions to regular functions we already know it's continuous therefore f inverse morphisms and we are done therefore f is an isomorphism ok, so what you must understand is that it is you must realized that the to check that something is a morphisms the are two steps one is a topological is continuous second thing you have to check it as it pulls back regular functions to regular functions but the point is that the checking that its pulls back regular functions to regular functions can always be done even at the level of the local rings that essentially the idea.

So to check that morphisms is an isomorphism you all, you all you have to check is that the morphisms is topologically isomorphism namely a homeomorphism plus you must take the induces the isomorphism at level of local rings at all at every local ring ok, so that's a power of local rings so now onwards whenever you see a morphisms which is topologically homeomorphism what is an easy way to check that it is actually an isomorphism you just check that at local ring it induces isomorphism and mind you I told you that there are morphisms which are topologically isomorphism that is homeomorphism but for which the inverse map is not a morphisms there are morphisms like that they are bad morphisms ok.

So just because a morphisms is a topological isomorphism need not mean the degree will have an inverse morphisms it need not be an isomorphism it may be so your morphisms can be even isomorphism the topological sense between two varieties but it may not be an isomorphism in the sense of varieties may not be an isomorphism of varieties so you can have two varieties, topologically same topologically same but as varieties they are very different ok.

Ok so this is one important fact the next thing that I want to talk about the other important fact about local rings is you know so you know here we have seen that the somehow the behavior of morphisms is control by all the ring homomorphism that induces the level of local rings right that's what it, that's the whole idea now for example the fact that this isomorphism is captured from the fact that all the local ring at the local ring level it induces isomorphism ok.

So the behavior of the morphisms at the local ring controls everything ok, now that's the level of morphisms what about level of regular functions so you know let me make a very simple statement which seems topological so if you have a rational functions on a variety ok by definition of ration functions on a variety is a (())(45:53) functions field its basically it's a regular functions depend on open set off there could be locus where it is not defined ok.

Namely the compliment there could be some close set where is not define you know when you conclude that the rational function is regular function so you know rational function an element here if capital x is a variety a rational function an element here and when I do conclude that its here, ok when I do conclude that a rational function is regular which means this means what I am trying to say as I am trying to say that the rational function is actually defined everywhere ok, when you say rational function, a rational function is only a regular function that is define only on an open subset it may not be define on the whole variety ok.

But if you ask the question when is the rational function actually regular the answer to that if following to check that something is here is here it's enough to check that it in here in every local ring ok, so it is a very nice result to check that the rational, so you know the ideal is like this, if you have rational function and if it is in every local ring that fact that it is in every local ring means that you know at every point in a neighborhood of the point it can be extended to a regular function that means you can extend it to a regular function everywhere therefore you should be a global regular function so it should be here ok, but then how does one prove it accurately is in competitive Algebra we do that in the next lecture.