# Basic Algebraic Geometry

**Professor Thiruvslloor Eesanaipaadi Venkata Balaji**
**Department of Mathematics**
**Indian Institute of Technology Madras**

**Mod-13**
**Lec 34**

## The d-Uple Embedding and the Non-Intrinsic Nature of the Homogeneous Coordinate Ring

Ok so, I want to make a few remarks about this proof which is, there are no non-constant global regular functions on a projective variety the there is only one issue with proof mainly that I need the y intersection Ui is non-empty for Ui ok and what I want to say is that given Y closed projective sub variety of projective space it could happen that Y may not intersect as at in Ui ok. And therefore in that case we reduce a case we can reduce the N ok to come to a situation where Y is a embedding the projective space and it intersects every Ui ok, and how can we do that is by realizing that if Y dose not hit a certain Ui then Y is completely counties the compliment of that Uibut the compliment of Ui is the locus where Xi vanishes and the locus where Xi vanishes is a projective space of into one dimension less ok.

So Y is embedded in a projective space of one dimension less ok, and you can therefore go to, you can work in that projective space now that projective space will again have an affine cover and you check whether Y intersects each of those a members in affine cover the movement it does not intersect one of the member in affine cover it means it is again contained in some hyper plane in that smaller projective space which is again much more smaller projective space, so you can continue this process, this process will have to stop at some stage giving rise to a Y embedded in suitably smaller dimension of projective space with the property that Y intersection Ui is never empty for every I ok.

So that is the case we (())(03:26) ok so with that in mind this proof covers all the cases ok that something that you have to notice right ok so this so the point is that you know there are no global regular functions on a projective variety alright and as you can see the proof needs the notions it needs the notion of the function field ok without the notion of the function field you cannot give this proof ok.

So what I am going to next is I am going to, go back to, I am going to go back to this issue about projective varieties that the, that there is no proper analog of the affine coordinate ring for the projective case, so the projective case the analog of the affine coordinate ring is a projective homogenous coordinate ring but the being deal is that while affine coordinate ring is a invariant of affine variety the projective coordinate ring, the homogenous coordinate ring over projective variety is not a invariant ok.

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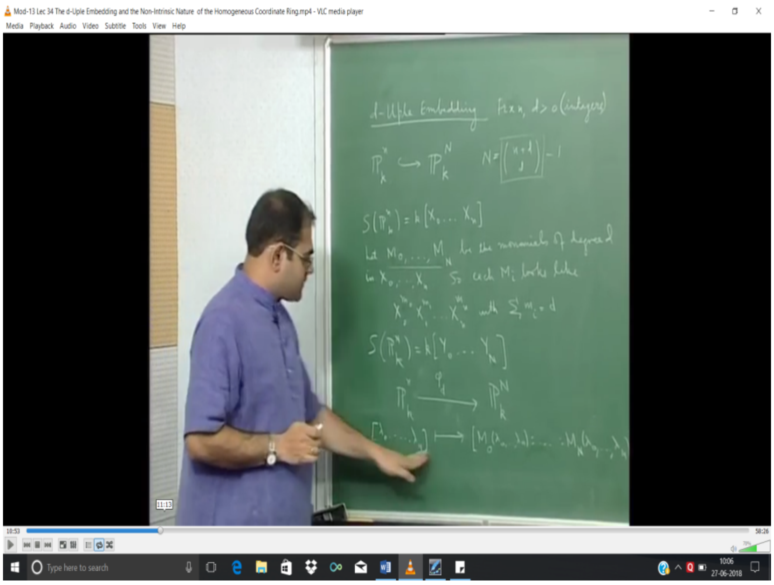

So to explain that what I am going to do, I am going to look at very simple situation so I am going to do the following thing so you know I am going take, I am going to describe what is called d-uple embedding ok so the d-uple embedding ok so the idea is following so what we are going to do is, we are going take projective n space ok and then so you fix d-positive integer ok and well I will also fix N so let me wrote fix n and d integers ok and I am going to p n and I am going to embed p n into p capital N ok where capital N is n plus d choose b minus one ok.

So what is this number so you know the well the homogenous coordinate ring of p n is polynomial ring in n plus one variables ok and now what I am going to do is at, I am going to look at monomials in these variables but all of degree d is given degree d ok so let n zero etcetera up to m sub n, b the monomials of degree d x not through x n ok.

So each Mi looks like x not to the power of m, m not to into x one to the power of m one and so on x. n over n sub m with summation of the Mi's equal to d ok, you are you are just writing out monomials of total degree d ok I mean products of all these Xi's powers of Xi's this is the powers add up d, and how many such monomials you will get you will, you can check that these monomials what will you get ok I mean n plus d choose, d is what you will get, and therefore you know if you look at the homogenous coordinate ring of this bigger projective space it will be k of y not etcetera up to y capital N ok and this n is just n plus d choose d minus one alright.

So there, the numbers of monomials will be this much ok and if you label them m not to m n you are actually getting n plus one and that capital N plus one should be this, so this is the number of monomials of degree d in small n variables ok, now what you, what we are going to do, we are going to design a very nice map so this is the this is the map from p n to it is monomials embedding so it is p n to p m, p small n to capital N, and we called this as u s about Fi d, u s like uses some Fi sub d and what we are going to do is following thing.

See any point here of the form lambda not etcetera lambda n this is how point here looks like these are the homogenous coordinates of a point in a projective space ok and what you are going to do is, it is very simple you are going to send it to this point where these small n plus one values are substituted for these Xi's in the right order each of these monomials ok. So that you get this, this you get capital N plus one coordinates which will define a point in who's homogenous coordinates will define a point in the bigger projective space so you know, let me write this m not of lambda not etcetera lambda n uhh and it gone up to m n of lambda not etcetera lambda ok so this is the map.

See you have lambda not to lambda n and then you give me monomials like this you substitute for Xi lambda ok and you do this for n each of these capital N plus monomials you will get this capital N plus one coordinates take the point and off course this is, this are the monomials in some order ok some order for example you can use lexicographic order on the powers if you want ok, you can use the lexicographer order on the variables and the powers alright you can have some suitable order after all this a finite set so yo can choose the decent order.

So this is the map ok now the question is, now the point is the following, the point is that this map embeds the smaller projective space as a close sub variety of bigger projective space ok this is a close embedding ok that in other words that this map is isomorphism of it is, on to it is image, and the image is a reducible close sub variety of high bigger projective space ok and so well so what is that, so who do you, who does one see this so first thing that one notice is that you know if you take so you know you define whatever happen here in terms of coordinates can be re interpretate in terms of competitive Algebra in terms of homogenous coordinate rings.

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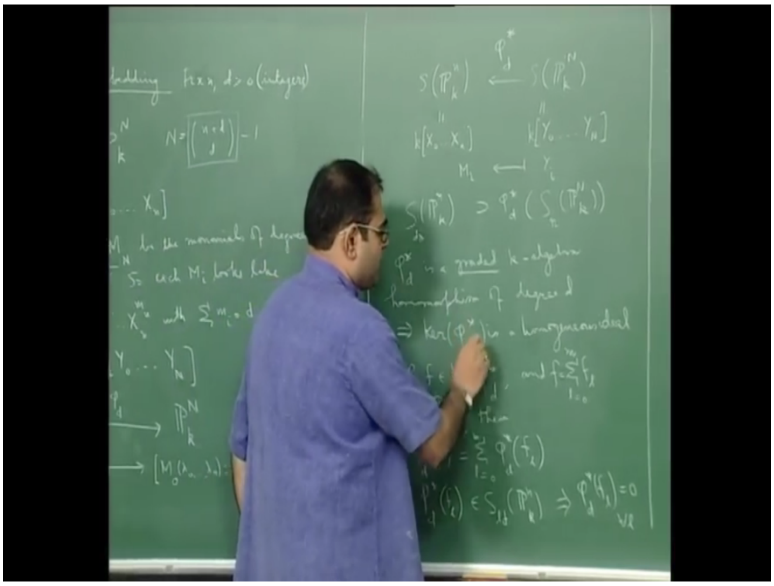

So what we have is we have for this, we have map in this direction the opposite direction and mind you this is so I called this is Fi d uple star ok because it is going to induce in some sense pull back regular functions ok we have Fi d and this is well k, y not, y, n, and this is k, x not, Xn ok and what is this map this map is very simple Yi to Mi, that map is pretty simple after all there are exactly many Yi's as there Mi's ok so send the corresponding Yi to the corresponding Mi.

So what this map dose, is that it takes mind you this Yi is degree one alright where the Mi is degree d, it is degree d monomials ok, so what this map does it takes every degree one thing to degree d alright and therefore this is called as graded homomorphism, so what will do you take the degree one piece to degree d piece ok and it will take the degree r piece to the degree r time d piece ok, you know if I instead of Yi if I put Yi power r it will go to Mi power r, and the degree Mi power r will be rd.

So this is what is called as graded homomorphism, mind you these two are graded rings we are not thinking of them as affine coordinate rings of the affine space above no, we are thinking of them, the reason why we put s, s is to remind keep reminding all sets that there is gradation is going on and this gradation is very important whenever you are working in the context of projective varieties ok, in this case we have projective space ok.

In this case we have projective spaces so Fi d, Fi d-Uple star of you know if I take s, r, degree r part p, n, k, this will land inside s, d, r, homogenous part of this small p, n, k. this is what it does it takes, so you know this is what it called graded homomorphism so it is graded beside of graded of degree d. Fi d star is a graded K Algebra Homomorphism of degree d ok this is what about it takes, it takes the any degree l piece to the corresponding l times degree l times d piece ok, and it is a fact that if you take the kernel of graded homomorphism it is always a homogenous ideal it will be graded ideal, it will be homogenous ideal ok.

So the fact that the graded homomorphism will tell you that is kernel, kernel of homomorphism always an ideal but the fact that it is graded to, tell you that the kernel is graded, it is a homogenous ideal ok, so this will imply that kernel of Fi d star it is a graded ideal, is a homogenous ideal.

I mean I am just saying you know, if a polynomial here goes to zero ok, I am just saying that if a polynomial here goes to zero which is the same as to saying polynomial runs to the kernel then every degree d, I mean every homogenous part of the polynomial also individually has to go to zero that is because you know if you take the polynomial and take any degree l part that will go to, go to degree l part ok, and therefore if the image is zero then it will you will get the each homogenous part of the polynomial goes to zero.

You know, so it is very easy to, it very easy to verify that if f is in kernel Fi d star ok, and f is sigma f sub l, l equal to zero to some m, ok, where f, l, belongs to s, l, ok, this is the breaking up of, f in to its various homogenous parts s sub l is the degree l homogenous piece of, of this big p, n, ok, then you know Fi d-uple star of, f zero will be Fi d-uple star of, f, that will be equal just sigma l equal to zero to m, Fi d-uple star of, f, l, ok.

But you see this Fi d-uple star of, f sub l, this will belong to s, sub l, d, of this p, n, small p, n, ok. And as l changes this are all in different pieces and this so you know this are all pieces in various different degrees and if the sum is zero then each piece has to be zero that's what direct sum each, so whenever polynomial is zero then every homogenous piece has to be zero, so and these are the different homogenous pieces of the image Fi d-uple star of, f ok.

What this tell you is that Fi d-uple star of, f, l, is zero for every l, this is what it will tell you and what you are saying is that if Fi d-uple star kills f, the Fi d-uple start kills any homogenous component of f, and that is precisely this same as saying that f, belongs to capital Fi d-uple star then every homogenous component of f, also belongs to kernel Fi d-uple star that's another you are saying kernel Fi d-uple star is homogenous ideal ok.

So and you also notice that the kernel is also prime ideal because the image is a domain after all so moral of the story is that, this kernel of Fi d-uple star is prime ideal, it is a homogenous prime ideal and therefore it is zero set will define a projective sub variety close sub variety of the bigger projective space and the claim what is that close sub variety of the bigger projective space it is actually image of this ok.

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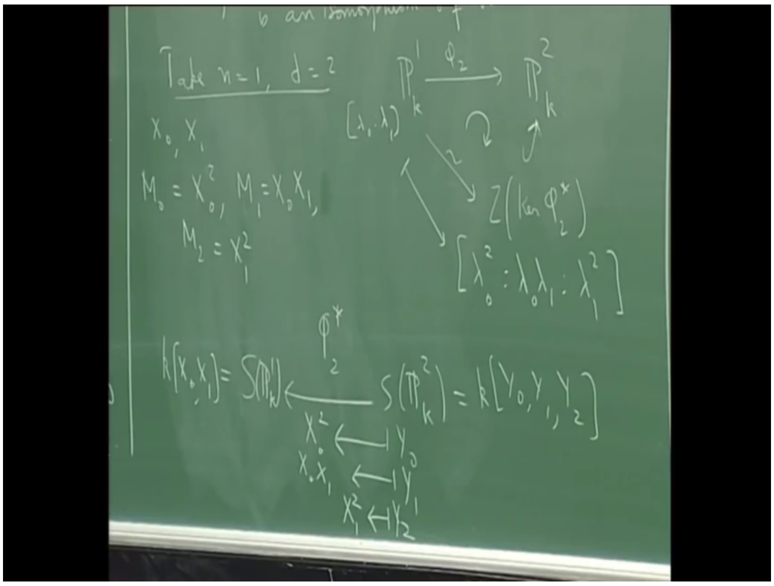


So further kernel Fi d-uple star is prime since s, in small n, k, is a domain article domain ok, so zero set of kernel Fi d-uple star inside the big projective space is a sub variety it is a projective variety, it is a reducible close subset ok, and the claim is that the image Fi d, is exactly this, the claim is the image of Fi d, Fi d is exactly z of kernel Fi d-uple star this is, these are the claims and the second important claim is Fi d from this, it will p, m, to you are restricted to this image is an isomorphism of varieties so this is the claim.

The claim is that this Fi d there so called d-uple embedding it maps this variety projective space isomorphicaly on to a close sub variety, and that close sub variety is nothing but the zero, I mean zero set of the kernel of this homomorphism ok, and so this is a claim alright. This require little bit of computation ok but you know before we try to settle this claims, I just wanted to look at the case to, small n is one and d is too, this is a simplest case and the reason we often look at that case is to tell you that you can have two projective varieties which are isomorphic but there homogenous coordinate rings are not isomorphic ok.

So take n equal one d equal to two ok, if you take n equal to one d equal to two then you have p one, k, you have Fi d this is p two in to, p well here I am going to get, so I will get three choose three choose, two which is same as three choose one, which three minus one this is two, you want to get two so it is p minus at p two ok, and you know and I have this z of kernel of Fi two uple star inside this.

So you know I just want you to, I just want to verify this for the case n equal to one d equal to two is a simplest case alright, to verify the general case will need a further calculation but there is a lesson to be learnt even in the simple case ok, so you know how does one insure this so you know what is the map, the map is lambda zero lambda one goes to monomials in you are looking at monomials of degree two in two variables ok.

So you know, so you have x not and x one, so this small n so you have x not x one and then you are looking at the monomials of degree two in x not and x one so you get m, m zero which is x not square, you get m one which x not x one and you get m two which is x one square these are the three monomials you will get alright and therefore what is this map, you going to same lambda not lambda one to lambda not square, lambda not lambda one, lambda one square that is, what this simple map is ok. And what is this Fi two star, this Fi two star is, is the map as going to go from s of p two which is just identify with k, f, y not y one, y two, to s of p one which you know is k, x not x one and you know what this map is, just going to send y not to, well according to this definition you are going to say y not to m not is m not square, y one will go to x not x one and y two will go to x one square ok, so are able to see that right.

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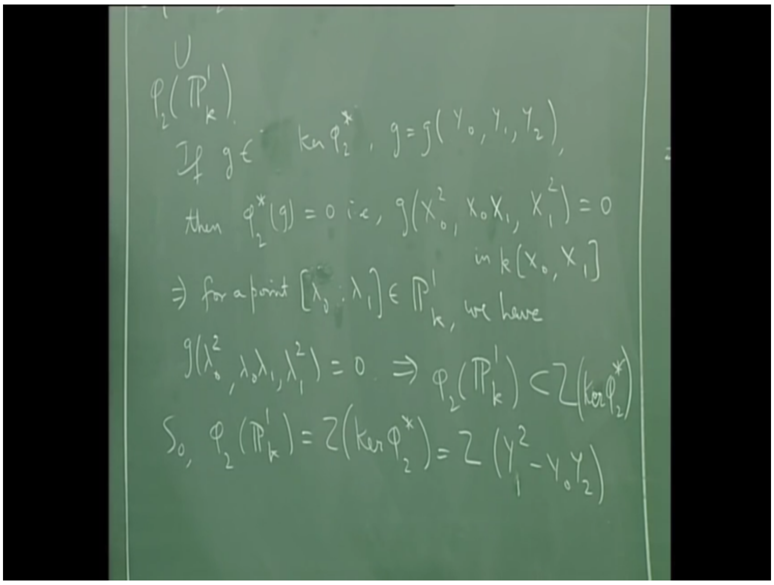

And what is a kernel of, lets calculate kernel of Fi two star, what is the kernel of Fi two star well kernel of Fi two star you know you can see this y zero is going to x not square y one is going to x one's square (())(25:56) everything here so y one is going to x not x one sorry and y two is going to x one square ok.

You can easily see that y one square minus y not y two goes to zero ok, so this contain the ideal generated by y one square minus y not y two, ok, because this is going to zero and if some element is seen in an ideal then ideal generated by that element is also mind you this is degree two element so it's a homogenous element so the ideal that generated its homogenous ideal contains this, this contains in this and you know in fact if you use some competitive Algebra you can show that this is exactly this kernel ok.

So see the reason being that this is reducible y not square minus y not y two is zero and reducible polynomial therefore the ideal it generate prime ideal ok and this is homogenous prime ideal alright and the height of this ideal is going to be one that's because of you know, this just because of Krulls principle theoremok.

So Krulls principle ideal theorem says that you take noetherian ring, and you take the an element in the ring which is neither as zero device nor a unit then any minimal prime ideal that contains that element will have height one, so if you take any minimal prime ideal that contains this it has to be equal to this because it is already prime and therefore its height is one and since its height is one the zero set of this will define hypo sophist ok.

Which is one dimensional object ok, whereas this one have also same height ok, so what this should tell you is that the zero set of y one square minus y not y two will contain the zero set of kernel Fi two star you will have this and this guy is one dimensional, this is one dimensional object ok this is one dimensional object that's because you know if you just look at the zero set of this in the affine space over this projective space you will get affine space over this projective space is three dimensional ok, and there I am having a single a reducible polynomial therefore it zero set above in the affine space will give me one dimensional less sub variety.

The co dimension sub variety, so I will get two dimension sub variety. But then but when I will remove origin and come down to through the projective space I will cut down one dimensional therefore you will get only one dimension object, so therefore this is one dimensional reducible close sub variety of projective space ok and that contains this ok but then, but this is also reducible close subset of a projective space but the point is that this contains FI two of p, y, by definition it contains p two of p one because you know you take you see take any element, you take any point in p two of p one it si of this form ok and the fact that the way we have define p two star will tell you that this is contain inside this ok.

I think that it is probably pretty easy to see just a minute I think, I just have to write it down so you know if g in z of kernel Fi two star g is say g of, so g is a polynomial invariables y one y not y two then Fi two upper star of g zero that is g of instead of y not If I out x not square x nit x one x one square is zero this is what it means sorry g is just in kernel Fi sorry suppose g is in kernel ok not the zero set.

Suppose g is the kernel write g is the polynomial of these three variables and then so if it is seen in kernel Fi two upper star of g zero so that's means g of this is zero mind you this is polynomial this is happing in x not x one that's how this map is define ok, so this implies that you know if you take a point for a point lambda not squared for a point lambda not comma lambda one of this p one we have you know g of lambda not squared, lambda not lambda one, lambda one squared is zero ok.

If g of some polynomial is zero then for the polynomial whatever variables that also should result in zero so this implies this calculation actually tells you that Fi two of p one the image of Fi two, image of p one under p two has to be in the zero set of this kernel because everything in the kernel vanishes on this ok.

So the so you know now what you must understand is that you know this is already one dimension ok this is already one dimension mind you Fi two is a topological Fi two is a injective in fact topologically you can check p two is a homomorphism ok so Fi two is injective actually, Fi two is injective its topologically homomorphism ok these are all thing that you can check so since Fi two is a homomorphism, Fi two of p one which is an image of Fi two is topologically isomorphic to, p one and p one is one dimensional therefore this is one dimensional ok.

So that's means this is already one dimension ok this will be one dimensional but this also one dimensional and this is a close subset ok in finite dimensional noetherian topological space ok if you have a close subset of the same dimensional then the close subset has to everything in other words if you take if you go to a proper close subset the dimension has to fallif you go to a close subset and the dimension does not fall then the close subset has to be everything ok this happens in a finite dimensional noetherian topological space which is the case with all our varieties ok.

So this will tell you that this is equal to this, this equal to this, this is equal to this ok so moral of the story is so you know if you use little bit of topology then you will get that Fi two of p one is the same as the zero set of kernel of Fi two star and that is the same as zero set of this ok, and in fact what I want to say is that a see this Fi two actually so therefore this Fi two actually gives an isomorphism of p one with this zero set ok which actually the zero set of y one squared minus y one y two this is a same as the zero set of y one squared minus y one y two ok.

These two are equal for dimension basis and this is an isomorphism this with that ok and in fact I am saying this even an isomorphism of varieties this thing is an isomorphism of varieties and you know the one way to check it by you know to check that isomorphism is has several property it's enough to check it on a cover, a cover suitable cover on a target and then you pull back that cover to get a cover of the sources and restricted to the morphisms each of these members of the cover and this morphisms has a particular property for each member of the cover.

Then it has a property thorough out for example so you know if I want to show that this from p one to the image, I want to show that this is an isomorphism it's enough to show it on a cover and what cover will I use, I use usual cover of p two  which p two has cover (())(36:21) u one, u two, u three I mean u not u one, u two which are the three a two which cover p two so you know I can make a computation involving that ok.

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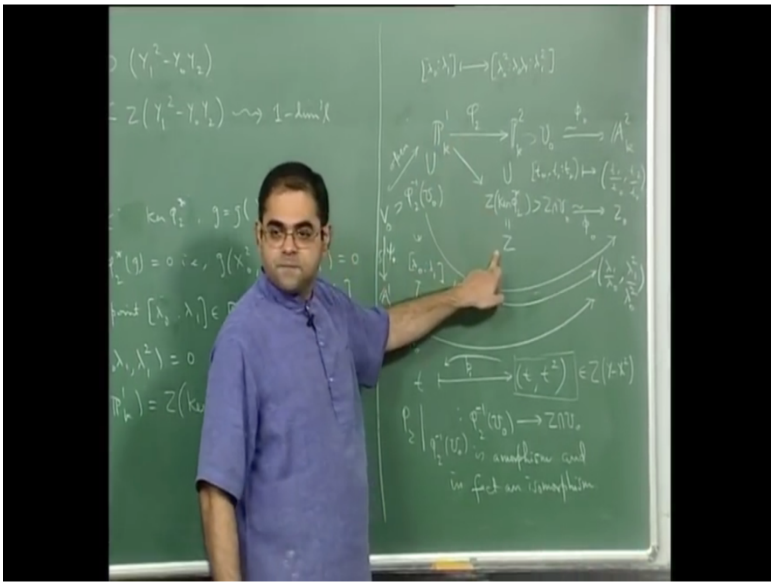

So well let's try to do that computation for a movement so you know if you look at so you know I have this so I have this p two here and I have this suppose I take u two zero then I have this so I have this z of kernel p two star then I will take z of kernel p two star intersection u zero so I get this diagram this is just intersecting u zero, so and you know this is well u zero is identify with a two by Fi zero you know, and u zero corresponds the place where y zero is not zero alright and therefore under this isomorphism this will be identify with close sub variety of a two ok, and if I take so I have p one to, p two I have this p two.

If I take Fi two inverse of this beside of kernel or I simply take Fi two inverse of u zero it will land inside so you know it will land inside this ok, so I have u zero, u one, and u two the three a two's that cover p two and I am working with u zero and I am taking the inverse image of u zero taking the inverse image of u zero and if you take the same as inverse image of u zero intersecting z get cable kernel Fi two because z cable Fi two is actually image of that what we have already seen it goes like this the map factors like this ok.

And mind you sytheortically this map is you must understand that sytheortically this map is injective and it is sytheortically surjective ok why it is sytheortically injective because you know if you have lambda not lambda one and you have lambda not prime lambda one prime suppose they go the same thing ok the fact that you have this you have the defect product here.

First of all you will get lambda not square is equal to lambda not prime squared, and you will get lambda one squared, lambda one point squared the only problem is that you might get it when it take square root you have taken different square root but then the fact this lambda not lambda one is also equal to lambda not prime lambda one prime insure that lambda not lambda one you started with should be equal to lambda not prime lambda one prime.

You can easily check injectivity and you can get you can check surjective because you know any point here is actually here, any point here is actually here, so the square of the middle coordinate is equal to the power first and the last coordinate ok, therefore you can take the square root of, square root of the first coordinate and you can take the square root of the last coordinate that will be the point here it will go to that ok if you take the correct square roots.

Therefore it's both injectivity and surjective is a very easy sytheortic checking only thing is that you can take the square roots because you are in Algebraic close filed you always will have square roots of elements ok that why there you need Algebraically close field off course otherwise for all over are given some varieties we are at the back always semi Algebraically closeness ok. So this is very easy to check this map is y jective it's very easy to check that this map is homomorphism ok that very easy to check.

So but you know the big deal is I am looking at this map ok what is this map is going to do see it's same map it's lambda not, lambda one going to after all let me write the map above its lambda not colon lambda one going to lambda not square colon lambda not lambda one colon lambda one square this is the map ok and the movement you are going to I mean that means that by lambda not squared is not zero ok, and lambda not squared is not zero means lambda not (())(41:16) will not zero.

The point this is affine variety ok the point this is affine variety here I need a larger diagram it will write again you know so I have this p one I have this p one, f, e, two Fi two and I have two zero here which is identified by Fi zero it a two and you know what this map is this is, this map is just something that times while, t zero, t one, t two, this map to, t zero is not zero on u zero so I will see t one by t not, t two by t not this what this map is ok, and what is this map this map is lambda zero lambda one going on to lambda zero squared.

Lambda zero lambda one, lambda one squared that is what FI two is ok and taking and mind you this map actually factors through this zero set of Fi two star inside this ok which I mean let just call it z easy to write down so and if I intersect with you know I will get, I will get z intersection u not and that under isomorphism Fi not will go into z not ok and if you follow it up so you know if I take Fi two inverse of u not that is an open set here in p one alright and if I know write out this map all the way to z not ok the map will be lambda not colon lambda one going to well you see first of all lambda not comma lambda one will go to this ok and this will go to the deviation by lambda not squared.

So this map finally land in if go all the way to z not you know the point I will get is, I do it by lambda not I will get lambda one by lambda not comma lambda one squared by lambda this is a point I get too ok, when I follow it all the way up, and mind you since I am taking inverse image of lambda not squared is zero, is not zero which implies lambda not is also not zero which means actually this thing is actually u not corresponding to this projective space this p one is also cover by two a one's namely u not and u one.

So you know, I will just called v not and v one's I do not confuse with u not here ok so in fact this is this fellow inside u, sorry this inside u sorry this is inside v not ok where this v not in this projective space is opens this is the open which is identified with a one by Fi not here is identified to a one ok and do not confuse this Fi not with that Fi not ok maybe I can so that you know I, let me not call this Fi not I will call Si not this is Si not identification of v not with a one.

V not is locus where the first coordinate x not does not vanish ok and you see and under this identification this point here you will go to what is the point you will go to, you will go to the point lambda one by lambda not ok and therefore finally if you take this and Fi two inverse u not is an open subset of v not so its image here will be an open subset of a one ok and if you finally write out the morphisms.

The morphisms is this mind you this is not a, you should not out a square bracket its round brackets here also its, so are coordinates of affine space ok they are round brackets and they are coordinates separated by comma's ok, you use a square bracket for homogenous coordinates, so you know finally when I write out this map for Fi two if I take the inverse image of u not from Fi two inverse u not two, u not if I write out the morphisms after going to affine piece here and the affine piece the corresponding affine piece there the morphisms is lambda one by lambda zero into lambda one by lambda zero comma lambda one, lambda one squared by lambda zero squared, that is the map.

So the map is simply p going to t comma t square that what the that is the map the map is just t going to t comma t square and its map from a one to a two so when I come to this map Fi two from p one to, p two ok in, on this affine piece, on this affine a one where the first coordinate this p one is first homogenous coordinate in this p one is not zero and on the affine piece here where the first coordinate piece the first coordinate piece in this p two is not zero which an a two then when I translate this map finally I am getting the map t going t comma t square, isn't that a morphisms?

Isomorphism because projection on each coordinate gives me a regular function so position of the first coordinate which may identity position of the second coordinate give me t square ok, so this is a regular function so the moral of the story is that Fi two inverse u not see if you take so Fi two restricted Fi two inverse u not from Fi two inverse u not to u to its image which is z intersection u, u not is a morphisms ok this is a morphisms alright and actually the funny thing is actually an isomorphism this is an actually isomorphism because you know if I take this map t going to t comma t squared that's an isomorphism because you know if, there is a inverse morphisms which is given by the projection of the first coordinate.

IfI take the map t going to t comma t squared ok and if I project on a first coordinate I will get, the projection of the first coordinate will send t comma t square back to t ok therefore projection on every coordinate certainly a morphisms alright, so the moral of the story is that I have an inverse morphisms say said that this forward by this is identity ok, and where is inverse morphisms define I am looking at the graph of I am actually looking at this points of this format.

What are these points the are the points of the parabola y equal to x squared this is line, see these are all points on z of y minus x square after all t comma t square it is a parametric representation of the parabola and z of y minus squared is just it's a conic it's a conic it's a plain conic it's a conic in a two parabola in a two and all the point in t comma t squared is, if you were a t you are simply going to all the points on this conic only thing is t is not zero ok.

I can have lambda one zero so, lambda not zero is not zero but lambda one can be zero so this t can be zero as soon as so, the moral of the story is that this map is not just K morphisms actually an isomorphism and in fact an isomorphism ok what we have proved is Fi two if you take if you restrict to Fi two to Fi two inverse u not to u, to its an image that's an isomorphism I did not this for u not you try it and write it down do it for u one and u two it will be an isomorphism.

So what you done is you checked down a cover that Fi two is an isomorphism and therefore it is an isomorphism so Fi two actually use an isomorphism p one on to sit ok so I repeat what I proved is Fi two, from Fi two inverse u not to z intersection u not is an isomorphism, ok similarly I want you to check that Fi two from Fi two inverse u one to z intersection u one is an isomorphism you can write it out and Fi two from Fi two inverse u two to z intersection u two is also an isomorphism ok and z intersection u one, z intersection u two, z intersection u not form a cover of z because u not u one u two form a cover of, p two and what you have done is your check that morphisms is an isomorphism on a cover and therefore it is an because of property of being an isomorphism where the local property ok.

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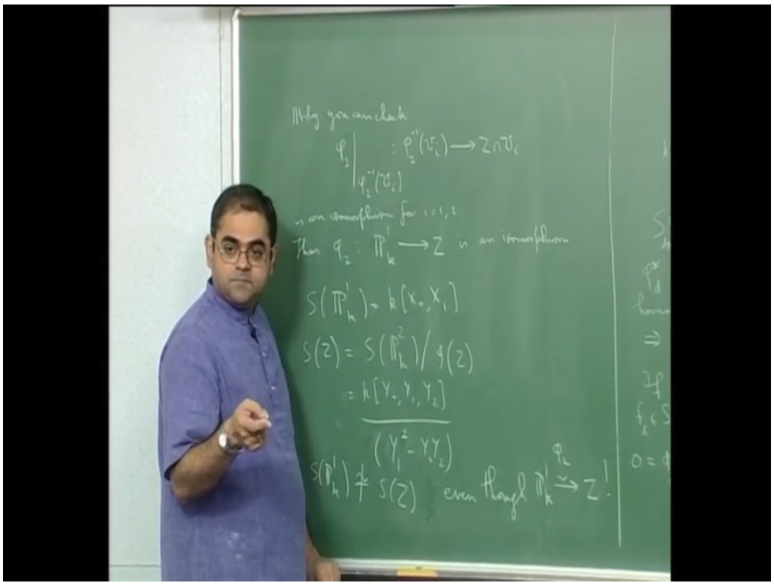

So what is tell you is that Fi two is an actually isomorphism of p one onto z ok so let me write that here, and now I want to state the important point similarly you can check Fi two restricted to Fi two inverse of Ui from into inverse of Ui to z intersection Ui is an isomorphism for I equal to one two thus Fi to one to z is an isomorphism because z is a union of z intersection u not, z intersection u one and z intersection u two ok.

So this is why you get that the image p one here is this under its an isomorphism of the image but now comes in deep this p is also p one ok and that is actually restricted p one you know if you want to think of p one is a projective line ok then image here you seen locally it is a conic it has been twisted into parabola so what is happen is this this line p one has been mapped isomorphicaly on to conic locally ok.

But the beautiful thing is u into this isomorphism projection variety you should calculate their coordinate rings, coordinate rings are the homogenous coordinate rings are not isomorphic ok, that gives the lesson that the homogenous coordinate ring of a projective variety is not as well behaved as the affine coordinate ring of affine variety it is not a invariant ok so you know if you calculate the homogenous coordinate ring p one what you will get is this is just k, x not x one, and what is the homogenous coordinate ring of z, it is the homogenous coordinate ring of the target p two module o, ideal of z, and what is that, that is just k, y not y one y two module o what is the ideal of this z, it is just the ideal generated by y squared minus y not y two ok, and these two are not isomorphic even though ok.

So the, see this is a polynomial ring in two variables alright and this is a polynomial ring in three variables and you are going module of degree two homogenous polynomial these two rings are not isomorphic you can try to proof the exercise that this ring cannot be isomorphic to this ring because actually you know this ring will have the full universal property for polynomial rings in two variables this will not that's how you check the the this is not isomorphic this cannot be isomorphic to this because the polynomial ring has a universal property.

This is a polynomial ring in two variables so it has a universal property where is that property will not hold you cannot find an analog of the property for this ring that will show this cannot be isomorphic to that ok, so I want you to check as an exercise that these two rings are not isomorphic ok as K Algebra is not isomorphic and therefore the homogenous coordinate rings p one and z are different but yet p one z are isomorphic as projective varieties.

So that gives us the lessons that the homogenous coordinate ring of projective variety is highly depended on embedding ok if you take the, so the embedding decides how your homogenous coordinate rings going to look like and the homogenous coordinate rings can change even though your variety up to isomorphism does not change so this is the problem that for projective variety you cannot keep track of them by just looking at the homogenous coordinate ring.

So to sum up this and in that previous lecture what I want to tell you is that unfortunately that projective varieties neither can you use global regular functions, the ring of global regular functions, for the simple reason it just constants nor can use the homogenous coordinate rings because it is not invariant and this is really in sharp contrast with the affine varieties where ring of global regular functions, is same as affine coordinate ring and that is an invariant.

You can completely recover affine variety from its ring of regular functions same as it's affine coordinate ring ok but you cannot do with projective variety so this necessity it's that you this problem that you have in which is intrinsic higher Algebraic Geometry that you take a projective variety to study you have to study all of its embedding into various projective spaces and look at that geometric properties of that embedding of those embedding and try to extract more information from this embedding's just looking at the homogenous coordinate rings with respect to embedding will not help ok you need to extract some more information ok so that and usually this kind of study is accomplish by studying so called line bundles and linear system sound which are connected with embedding namely morphisms of a variety into projective space and this is usually done in a second course in Algebraic Geometry ok so with that I will stop.