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Mod-13
Lecture 33
Global Regular Functions on Projective Varieties are Simply the Constants

Ok, so what I am going to do today is, first thing I want to do is to give the proof that, there are no global regular functions that are non-constant on projective variety ok, in other words the projective variety the only global regular functions are constants ok. So you know, so let me tell you couple of things, first thing is that it's this is this result should be taught of as an analogue or you know part of the general philosophy of results which say that on compact objects you do not have any the only global nice functions on nice objects are constants ok.

So for example you know, one such example is, for example you know if you take a suppose you take Riemann Sphere namely the ordinary reals Sphere S two and make it into Riemann surface which means what you do is to make it Riemann surface you just have to be able to talk about holomorphic functions at a point or analytic functions at a point and for that you uses Stereographic projection to identify the complement of a point with the plane and by taking to different points you can cover the Sphere by to planes ok by this this Stereographic projection and you can check that you can use this to define what is called a Riemann surface structure on the Sphere and then you can define what is, then you can look at global holomorphic functions on the Riemann Sphere and then it will turn out the only holomorphic functions where Riemann Sphere is constant.

The reason being that the Riemann Sphere is compact because after it is topological it is compact it just Sphere, its if for example if you want it is both close and bounded so its compact, and any global holomorphic functions on Riemann Sphere if you restricted to the complement of the point then you are then you will get a holomorphic function on the plane ok, and its image will be bounded because its image is compact ok so you get a holomorphic function and entire function which is bounded and Liouville's Theorem will constant.

So what will happen is that your holomorphic function on the Riemann Sphere if you through outside, if you throughout a point in the complement of the point it will be constant and therefore by continuity it will be also a constant at that point, so its globally reduce to a constant ok so this is for an example in (())(04:08) general philosophy of results that says that whenever objects are compact then the only global good functions are objects are constant ok.

You cannot accept non-constant good global functions on objects at compact and the fact is in at Algebraic Geometry the correct analog of compact misses projectivity in fact more generally the correct analog is called completeness ok, but I want the explaining about that probably in this course, I do not know whether I will do that but you will come to know if you take second further courses in Algebraic Geometry but I want to tell you that you must always think of projective varieties as compact as the correct analog, analog of compacts ok.

And the usual compact does not make sense, I mean does not give you anything special in Zariski Topology because you know Zariski Topology is already compact I mean that is the reason why we use the word quasi compact in fact compactness in the, in the sense that every open cover has a finite sub cover that compactness in Algebraic Geometry for Zariski Topology is renamed as Quasi compactness ok, and it just comes free of charge, I means just comes free.

So the nothing special if you define compactness to be just you know every open cover adminating a finite sub cover so you do not anything, I mean there is nothing to that is not a condition always true alright, that is the reason why usual definition of compactness no use of in Algebraic Geometry therefore it is recalled, it is designated as quasi compactness and therefore you can ask what is the correct analog of compactness, the correct analog is being projective ok at least when studies varieties.

But the more perfect answer would be that the correct analog is that you should look at what are called as complete objects ok, and projective varieties are example of complete objects. So this is one fact I am going to prove that, I mean we are going to show that on a projective variety by the only global functions are constant, then the other thing we are going to sow is, the other thing that have to be worried about is the following.

See global regular functions are constant, then how do you study the object ok, there is no global regular functions are object, then how you do the study the object now it presents a problem because you see in the case of affine variety, the global regular functions they give the affine coordinate ring and you know the affine coordinate ring is an invariant ok, it describe affine variety complete ok.

In fact if you give me affine coordinate ring you can get back the affine variety by just taking the maximum spectrum and putting the Zariski Topology on it ok, therefore for affine variety the affine coordinate ring is same as a global regular functions and that completely capuches affine variety but it is not true for projective variety, projective variety first of all global regular functions are constant ok, so there is no non-constant global regular functions.

The second thing is if you take the analog of affine ring you will get the homogenous coordinate ring, the homogenous coordinate ring is also not, will not characterize your projective variety, so in fact the homogenous coordinate ring will depend on embedding ok, if you change the embedding into a projective space if you change them, if you take the same projective variety ok and put it in a different projective space then the homogenous coordinate ring will change it will not same up to isomorphism.

So you cannot keep trace over the projective variety so usually just either using regular functions because there are not any non-constant regular functions and you cannot use the homogenous coordinate ring to track your projective variety ok, therefore the only way to studying the projective variety is studying embedding into projective space ok, and this leaves to study of isomorphism into projective space the so called classical it was done by studying what are called as linear systems and then things like that in modern language we study line bundles and things like that ok.

So but these are all things you will gone cross probably in second course of Algebraic Geometry but so what I want to say is that the fact that, there is no non-constant regular functions on a projective variety and fact that the homogenous coordinate ring or projective variety dose not characterize as projective variety leave you to study embedding of projective varieties into different projective space and the embedding are they need to study carefully ok.

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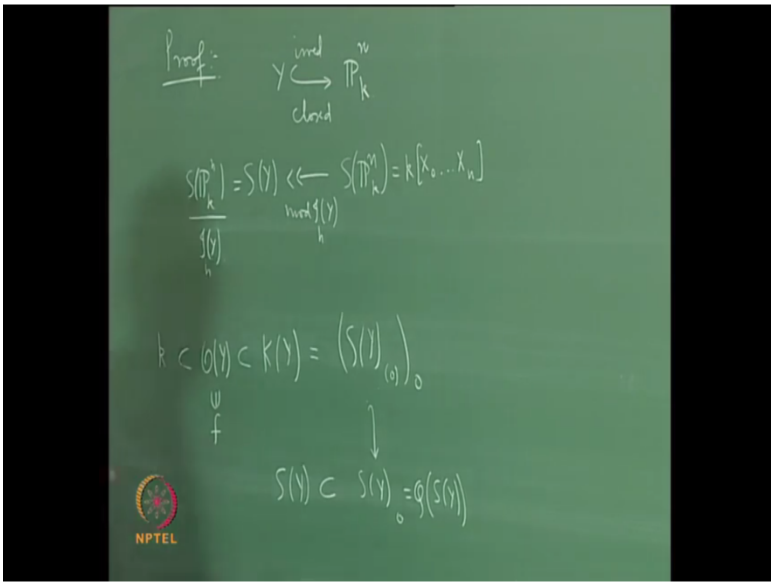

So any way with that preamble let me start with the Theorem that we want to prove so here with Theorem if Y is a projective variety then O of Y is equal to K, so in other words every global function on Y is a constant ok, when I write K, I mean every element of K is constant and it define it thought of as constant regular function, constants of regular functions ok, so what is the proof, so the proof is you know few ideas from competitive Algebra and module theory but I will explain that, I will explain that.

So your situation is like this you have Y which is setting as reducible close subset of some projective space off course we are always as usual working over small k which is Algebraically closed field ok and you know what then you also have so this diagram if you go to homogenous coordinate rings it translates to a quotient so close subsets always corresponds quotient in Algebraic Geometry and off course moving to open subsets corresponds to going to union of localization if you want ok.

So this is the polynomial ring N plus one variables you know this is the affine coordinate ring of the affine space which, who's for which a corresponding punchered affine space sits above this projective space this is the quotient of the punchered affine N plus one space ok, and that affine space have affine coordinate ring equal to this and that affine coordinate ring is define to be projective homogenous coordinate ring of projective space and then this closet YB is close subset of that correspond to a quotient.

You go the module of O ideal of Y that why put a sub script H this sis the homogenous ideal of Y namely it is all those homogenous polynomials which vanish on Y ok and then you get SY, so SY is just SPN mode IY and off course the fact with the all these things is at everything is graded ok so that’s gradation here you know the gradation here just corresponds to every polynomial being broken down into its homogenous parts ok and so and therefore you also get a gradation here, and the gradation here it just, gradation when you read it mode IY ok.

So well, well now what we have understand we already seen that you know if you take, if you take the so, so you know you have, so we have a, we have picture like this, we have the, we have OY this is the ring on regular functions on Y, global regular functions on Y and off course K inside this as constant as a constant functions, because every scalar is being thought of us as a regular function which is equal to that constant scalar alright and then OY goes into KY, KY is the function field of Y ok.

That is what we discus and studied the last lecture and what is the function field of Y, this is actually the calculated for a projective variety it would be show that if you take SY ok, you take homogenous localization at zero ok, zero is a prime ideal ok, the zero ideal is a prime ideal because it is a integral domain even here you take the zero ideal here it is a prime ideal because this is an integral domain you have only one module O.

IY and IY remind you is a homogenous prime ideal ok, and the reason Y, IY is homogenous prime ideal is because Y is reducible alright so since you have gone module O prime this is still a domain the zero ideal is prime ideal so you take the homogenous localization at zero, so I want you to a, I want you to distinguish between this and the homogenous, non-homogenous localization so this SY sub-zero this zero here I do not out a round bracket around that zero.

See this zero is you invert everything out side zero ok, this is actually quotient field of SY, there is a quotient field of SY ok, so this consist of literally taking a quotients of two polynomials in SY. Off course by polynomials SY, I mean every polynomial in SY is some polynomial here red mode IY ok off course here IY is same as IY sub H, I keep putting this H some times to just remind you its homogenous ideal alright it generated by homogenous elements.

So this is a quotient field of SY but this is different this is homogenous localization so what you do is you do not here you invert everything that is not zero but here you invert only those things that are not zero and which are homogenous ok and the reason why you do that is, when you do it like that then this consist of elements of the numerator by denominator, the numerator coming from SY and the denominator being a non-zero homogenous element ok, and therefore if the numerator is also homogenous since the denominator is also homogenous the difference in the degree of homogeneity will give degree so this will become graded, and for this grad you take degree zero part that is precisely function field of KY that what proved in the last lecture alright.

So I am just asking that just recalling that this is KY the function field of Y is actually this homogenous degree zero part of the homogenous localization at zero alright which seems inside in huge field this is a huge field this is a huge quotient field of SY ok and off course you must remember that this so you know you see SY a sitting inside the off course this is the after all this is a quotient of the integral domain this sitting inside that and you know what I am going to follow, I am going to following things. I am going to start with an F here, I am going to start with global regular function on Y.

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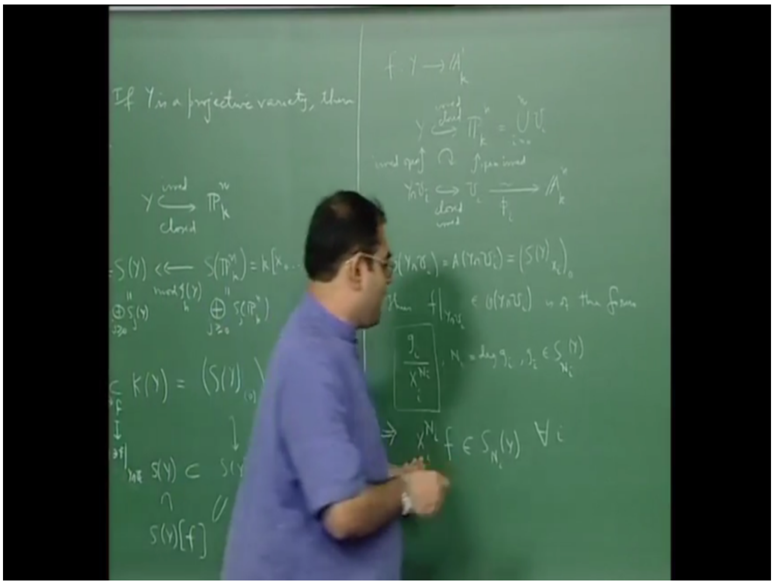

So you know F is global regular function, F at, F is from Y to A one isomorphism a global regular function is regular function is just a isomorphism to A one ok we have seen that, before so this F is a global regular function it is define on all of Y and takes values in A one it is isomorphism into A one and I am going to show that this F is a constant ok. I am going to show that this F is a constant and how I am going to do that, I am going to do a following thing, I am going to show that F satisfies I am going to show F satisfies is a monic polynomial with quotient in K ok.

I am going to show F satisfies a monic polynomial with quotients in K ok, that means if you consider everything as being inside this field this is a field extension of K this K is a field this is also bigger field and this is a field extension of K, and this is an element there and beautiful thing is it this Algebraic over K, if I prove that F satisfies a monic polynomial with quotients in K, I am just saying that F is Algebraic over K but K is Algebraic close therefore F has belong to K and that is how I prove F equal to some constant ok. So this is how I am going to do it and how I am get a monic polynomial that F satisfies with quotients in K.

What I am going to do, I am going to look at this way, so I am going to look at SY, I am going to look at polynomials in F ok which is also sitting inside this, I am going to look at this ring this is a polynomial I am writing polynomial in F with SY quotients ok I am going to study this field ok, so this is the this is the broad idea of proof alright so now you let's get detail's.

So the first thing I want to say is, well you see Y is you know you know Y is contain, Y is off course inside PN and you know PN is the union of all this Ui's I equal to zero to N these where each Ui you know inside PN, this correspond to the coordinate Xi not vanishing ok so this is the this affine piece is isomorphic to AN so in fact you know that there is a isomorphism Fi I of this with AN you have this then I have Y intersection Ui inside this so well this is open, this is open, this is closed, this is closed, this diagram commutes well this is reducible, well this is also reducible ok, off course here also this also reducible that's also reducible ok, so this is your diagram and the whole point is that you know, we know what, see you know, we know.

So this isomorphism is how we showed that Ui is actually affine we proved that Ui is, actually isomorphism affine space therefore Ui is affine and this is a reducible close subset of A therefore this is affine variety ok, and therefore you we have that O of Y intersection Ui is just a same as A of Y intersection Ui, because Y intersection Ui affine and what is this, this is just we know what is this, it is just Y well you, you localized at Xi and then take the degree zero part this is what this ok.

So this is, this is affine coordinate ring of Y ok, so you take SY you localized at Xi ok when you because you localized at Xi means you invert powers of Xi ok, and Xi is off course homogenous of degree one therefore you invert power Xi there is natural gradation of this, then I am take this degree zero part, which means just looking at some homogenous polynomial in SY ok, mod Xi to the power of degree of that polynomial that's what you are looking at, that's what this thing is ok.

So now what you should realize is that you see thus, F if you see if you take F and restricted to Y intersection Ui this will belong to O of Y intersection Ui, because you know you take a regular function and restricted to an open set you will get an regular function alright, but O of Y intersection Ui is this so this implies is of the form well it is an element here so it will be it look like Gi divided by Xi to the power of Ni, where Ni is equal to degree Gi, Gi belonging to SY sub Ni this is the homogenous part of degree Ni in SY ok.

Mind you see I want to let me again recall see this is a homogenous, this is a graded ring so this is a direct sum of J greater on equal to zero S PN K J where S J consist of homogenous polynomial of degree J and S zero is going to be K homogenous polynomial of degree zero alright, and this is just standard fact that any polynomial can be broken down into its homogenous component's and this homogenous components are unique and each homogenous component is a homogenous degree ok.

So you know the polynomial has first degree homogenous component which is an consist term then there is a degree one homogenous component which is linear term then you have degree two homogenous component which is the quadratic term and then the cubic term and so on and that is the cut this decomposition alright and this decomposition also gives decomposition here ok, you are also going to get decomposition of this ring ok, the only thing is that you read, you read everything module O this ideal is a homogenous ideal that's a whole point so in particular you know if you take a polynomial here if you take a polynomial IY ok that will go to zero here.

So if you take a polynomial IY which is homogenous of some passive degree it will, suppose it has degree J it will be in SJ but if you go to the quotient it will become degree zero because you read it, you have to read whatever you get you have to read it mod IY ok, so you have induced gradation here alright, so I think by notation I should use not S, so I use S sub Ni ok so every element here, every element here, is going to look see without the zero it is going to look like some element of SY module O some power of Xi and off course this Gi is actually GI is actually being red mod I, IY mind you ok.

If you want actually I should out Gi bar but I will not do it Gi is just the image if Gi here and I have tread mod IY alright so it is going to look like this and the point is since I am taking degree zero part that Gi is homogenous is degree Ni and the denominator is also may returns so that you know the power of Xi because Ni so the denominator is also homogenous of degree Ni, so the induced degree is going to be zero that's how it is adjusted alright, now this is how F is restricted to U, Y intersection Ui looks like alright in fact you know if you, I am just trying to think of Gi as a degree Gi homogenous polynomial ok then if you take this quotients I have the degree Ni homogenous polynomial divided by another degree Ni homogenous polynomial it is a quotient of two homogenous polynomials.

So this is certainly a regular function and this regular function will leave on Ui this regular function leave on Ui because I should not say the whole projective space leave on Ui alright so I want to make the following statement you see O, you see this OY so F is in OY but you know so let me try it do something here so you know this is contained in O of Y intersection Ui ok because you know the this is something that we have already seen if you have regular functions on open set then you can restrict them for further smaller open set ok and the restriction map from a regular function from a larger open set to a regular function on a smaller set is a injective map.

Because the injectivity is because of two regular functions coincide on some open set, they coincide everywhere ok therefore this is contain inside this and you know I started with an F here ok and what is the image of this F, the image of F under this inclusion F is precisely F restricted to Y intersection Ui, ok I need some more space here so it write it correctly ok, so F goes F Y intersection Ui, but the point I am going to do something now what I am going to say is, I am going to say, I am going to identify this two together, I am going to identify these two thinking that everything is be happening here so this is big filed where everything is happening all things are happening here ok.

So everything is happening in this big field which a quotient field of SY ok so there is no deference between this and this after all because this is, this element is F of F restricted to Y intersection Ui, is just coming from F which is a subset of this and when I consider everything here mind you this is my universe this is a big filed where everything leaves so you know I am going to identify this with this it is correct alright.

Now what I want you to understand is you know if F restricted to this with this then Xi power Ni multiplied by F will can be identify with Gi which is here ok so I am going to write this Xi power Ni ok times F belongs to S, Ni Y. ok, this is two for true for every (())(28:02) alright. And in fact you know what actually, what I should write is if you want Xi to the Ni F restricted to Y intersection Ui, ok that what I should write but I am identifying F with F restricted to Y intersection Ui, because everything is sitting inside this huge field and then this element goes to that ok alright.

Now so you know but what you should understand is that you know you see if I instead of Ui if I take Uj ok then I will get Y intersection Uj right and well the F will also go to F restricted Y intersection Uj but then the fact is both of them this F restricted to Y intersection Ui and F restricted to Y intersection Uj will define they will correspond to the same element here that something you should not forget ok.

Even is the "I" changes these things change but the all come from the same F so they all, if I changes the various F restricted to Y intersection Ui are all one and the same element here I am calling them just as F ok . So this is a small thing that you have to sat iritic think that settle you have to notice ok.

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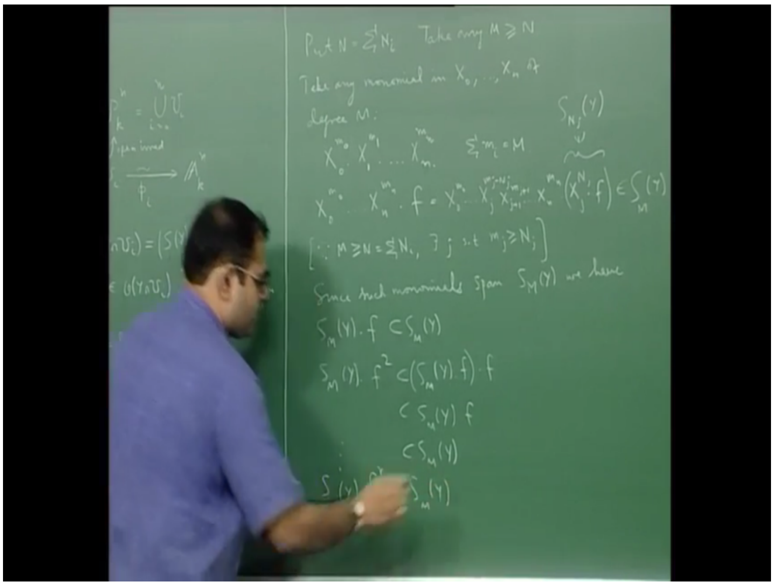

Now ok so I have this now I am going to play with this I am going to say the following things I am going to say that you see you know basically idea is F is a regular function see a regular function on a projective variety or quasi projective variety is just a quotient of two homogenous polynomials of the same degree.

So essentially it is degree zero you see you must understand that it is degree zero if you think it as locally as quotient of two homogenous polynomials which is what is it, it is a degree zero object and this is so you know this is correct aspect you take a degree zero thing you multiplied with degree Ni the resulting thing is of degree Ni it lands in the piece with which consist of homogenous degree Ni so the statement is correct alright I mean it is believable.

Now what I want to do is out N is equal to sigma Ni take the sum of all those Ni's, ok, consider take any M greater than N, greater or equal to N, ok, take any monomial in X not dot dot dot, X not up to so on Xn of degree M ok so this monomial will look like X not power M one into X not power M two and so on X not power MN with sum of all the Mi's equal M, this is how monomial in all the Xi's of degree M will look like alright something wrong ok I am this should be one should be N may be this should be zero my numbering is bad ok alright.

So how monomial of degree looks like and the point is you multiply monomial of degree M with M greater than N with F and you will again land inside degree M piece because of this observation now you see X not to the power of M not Xn to the power of Mn times F, if I calculate ok, see this is you must understand that diary since M is greater than N, which is sigma Ni ok see there axis J such that you know Mj is greater than or equal to the corresponding Mj this has to happen.

Because every Mj small mj is less than slightly less than capital NJ then the sum of all small mj which is M as to be slightly less than the sum of all the N's, NA, NJ which is N by whereas it seems the greater than or utm so this is a an obvious has to happen, so then you know I can write the X not power M not then you know when Xj comes I will put Mj minus Nj then I will write Xj plus one power Mj plus one and go on up to Xn, Mn I will take out this Xj power Nj times F and I know this Xj power Nj times F is in S sub Nj Y this is in S sub Nj ok.

So this is a degree Nj homogenous object ok, it is degree Nj homogenous polynomial and what is left out is homogenous polynomial of degree M minus Nj so the moral of the story is this whole thing going to lay in SMY ok this is what happen because this fellow lies in this belongs to S NJ Y, that is because of this observations ok. So this part is homogenous of degree Mj the remaining part is monomial of which is homogenous of degree M minus NJ ok therefore when you multiply it you will get the total degree will be M minus NJ plus NJ so you land in M ok.

Now this happens for every monomial of degree M ok but all these monomials of degree M go to span this span precisely SM ok after all SM is the space of all homogenous polynomials of degree M off course red mod IY, red mod IY because you are in SY, you are not in S of PN ok you are in S of IY therefore the moral of the story is that since such monomials span SM Y we have SM Y dot F goes in to SN Y ok you have this.

You take an element of SM Y multiply by F you end up inside SN Y ok because any element of SM Y is just a linear combination of K linear combination of such wanna finite linear combination of such monomials and each monomial is going to push F into SM Y alright so you are going to get this and know from this what you can get is that SM Y of F squared will goes into SM Y of F dot F which will go into SM Y dot F which will go into SM Y, and if you continue by induction you will get SM Y will take F power R into SN Y ok so multiplication on the right by powers of F pause non negative powers of F is going to push SM Y into itself ok.

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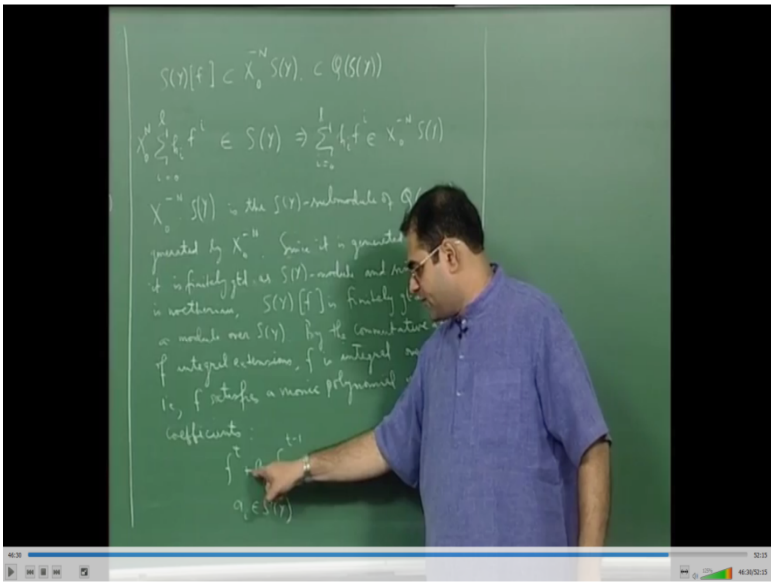

So you have this so you know in particular you know, what I want you true notice is that if you take X not power so I want to here is here is the observation that is very important for us the observation is that SY F, ok I want to say SY F is contained in X not to the power of minus N times SY so here is the important observation. I mean it is a that is the result of this actually ok see and why this is all happen this is all mind you all this is happening in the quotient filed of SY the big huge field where everything is contained it is happening there.

See X not to the minus N make sense there in the quotient filed of SY ok so everything is working there everything is living there right. Why is this true that is this is because you see you take any homogenous piece here you take any element here and multiply by F and you further multiply by X not power N the result is going to SY ok see any element here is going to look like sigma HI F power I, I equal; to zero sum L, this is how something going to look like and you see this each HI in HY ok but now I take if I multiply outside by X not power N you have multiply outside by X not power N then this thing this X not power N into HI will push each homogenous component of HI into degree greater than or equal to N ok.

See each HI is in SY alright, each HI in SY so each HI breaks down into homogenous component's and it compress will it could have compress from degree zero onwards up to some value finite value but multiply by X not power N hikes all these degrees homogenous degrees of these homogenous component to make even the minimum to be greater than or equal to N ok, therefore when I multiply X not this goes into SY because of this observation whenever you take any homogenous degree greater than or small n, I mean degree greater than to capital N polynomial and you multiply by any power of F you again get degree greater than to equal to I mean you again get a homogenous polynomials mod IY.

So this goes into this therefore this is in X not power minus N that, it implies at any polynomials in the F with quotients of SY this belongs to the X not minus N SY ok, so that is how you get this in here ok now now the nice thing is so the moral of the story you see this fellow here is actually contains what I have got is at this fellow actually contained in X not to the minus N SY which is also contained here ok so this see this object this polynomial ring F with quotients in SY this is caught inside this and we are moral is done you know why.

The fact is because you see this fellow here this is now think of everything as an SY module this is a finite this is a SY module generated by the single element X not to the minus N, X not to the minus N SY, is the SY module is the SY sub module of the quotient filed of SY generated by X not to the minus N ok.

So it is a module which is generated by single element so it is a finitely generated module ok and SY is what, SY is noetherian ring, it is a finitely generated module over noetherian ring and this is a sub module of that therefore this is also finitely generated ok, so you see SY since it is generated by one element it is finitely generated over S, as SY module and since SY is noetherian so this is where we are using the fact the polynomials rings is noetherian and SY is just a quotient of the noetherian ring, a quotient of noetherian ring is also a noetherian.

Therefore SY is a noetherian ring and you have a finitely generated, any finitely generated module over noetherian ring is also noetherian and every sub module of finitely generated module over a noetherian ring is also noetherian is also finitely generated ok, so the final conclusion is that SY of F is finitely generated as module over SY this is what I want ok.

Now you know, now we go into a little bit of competitive Algebra you see the fact that SY F is finitely generated module SY is equivalent to saying that F is integral over SY ok, that is the F satisfies a monic polynomial with quotients in SY ok, by the competitive Algebra of integral extension SY F, SY F is a integral extension of SY, F is actually integral over SY and F is integral over SY, so this is some competitive Algebra which you can I mean you can easily look up in a book on competitive Algebra for example standard reference is RT and McDonald's introduction to competitive Algebra , you look at the chapter on integral extensions and you will find this, and it is very easy to prove, it is just an argument that involves some determinants ok.

So by the competitive Algebra integral extension F, over integral extension of SY that is F satisfies a monic polynomial with SY quotients so you know you will get something like this you will get F power T plus A T minus one up to the T minus one plus one plus A not equal to zero ok, where the AI, actually SY and this is all happening in the quotient filed of SY, this is all happening in the quotient filed of SY, SY is a huge extension filed of K, it is a huge extension filed of K, in fact you know this (()):(45:32) is a huge extension of K because you know actually this is a quotient filed of Y, it is transcend degree over K is actually equal to the dimension the variety Y.

So if Y, see Y inside PN ok so Y can have at most dimension, and if it has dimension it is all of PN otherwise it will laser dimension, suppose Y has dimension R then this quotient that transcend it means that this field extension the transcend they the quotient field of, or the function field of Y it is transcend degree over K is equal to dimension of Y so this is a huge field extension ok this contains lots of transcend elements.

So this is a huge field (())(46:12) is a huge field, field extension, the transcend extension and this is a much more huger one ok, such a everything happening in a huge field ok now you see, now each of these A sub I's they are in SY so they have homogenous components and what I want to say is that you can write out, you can replace each of these with the degree zero homogenous component ok, and now what you do multiply this whole thing throughout X not to the N.

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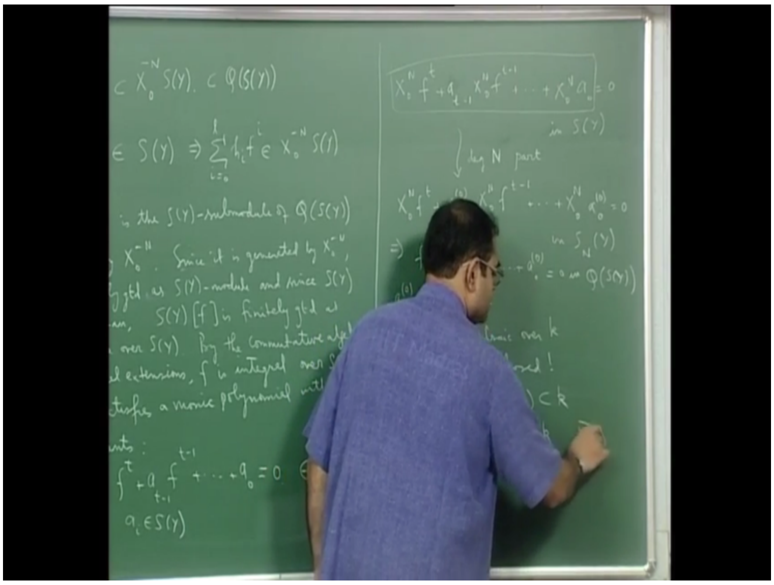

So you will get X not to the N F plus X not I mean AN minus one AT F to the T, A plus AT minus one X not to the power of N, F, T minus one plus dot dot dot, X not to the power N times A not is zero now this is, this certainly make sense inside, so this make sense inside SY ok, that's because you know X not multiply F by any homogenous polynomials of degree greater than or equal to capital N pushes into SY that's the whole point here you multiply any power of F by any homogenous polynomials or even any polynomial with every to that condition with every homogenous component as degree greater than capital N then the result will land inside SY so this happens in SY ok.

Now what is this happen in SY, SY has a gradation if something is zero then everything graded piece zero so if you take the degree zero part so you know you take the degree X not power N part of this which is the lowest degree, if you take the degree X not power N part then what I will get, I will get this, I will get X not power N FT plus AT minus one and you know I am going to put zero here to tell you that A T minus one zero is the degree zero part of A T minus one.

See each AT minus one is in SY and since SY has a gradation AT minus one breaks down into various parts of homogenous parts of various homogenous degrees and I am taking the degree zero part then I will get X not to the N, F to the T minus one plus and so on and then finally last time I will get the X not to the power of N A zero upper zero, this A zero upper zero is a degree zero part of A zero I will get this also equal to zero ok, and so and this will happen in SN Y this is will happen in the degree zero piece, and the degree N piece ok.

This is because mind you when you whenever you have a graded ring an element in that ring is zero if and if only every homogenous piece is zero so this is element on the left side is an element of SY which is a graded ring the fact that is equal to zero means, each of its homogenous piece of degree zero and what is the minimum degree homogenous piece, the minimum degree homogenous piece is capital N ok.

So I am taking the degree I should not say degree X not to the power of N I should say degree N, I am taking the degree N part ok the lowest degree so I end up with this alright now what I will do is, I will cancel this X not to the N, and when I cancel this X not to the N I have to go out of this but I still lie, I will still in the quotient field of SY so this implies that you know FT plus A zero T minus one F to the T minus one plus plus A zero, zero is equal to zero in the quotient field of SY.

So this make sense, because I am just multiplying by I am simply multiplying by X not minus N which is, which throws me out of SY, certainly keep me inside this quotient field of SY because X not to the minus N leaves there alright but after all what are these gies, what are all these AT minus one zero what are all those things, they are scalars, they are all constants they are degree zero polynomials, see they are degree zero AJ zero belong to S zero Y, this is a degree zero polynomials, red mod IY this is just K.

This is just K and so this implies that F is Algebraically over K but K is algebraically closed so F belongs to K so this implies O of Y goes into K, which means that O of Y is equal to K and that's end ok. So finally I end up showing that every fellow is here actually here I end up showing that every regular function small f, if OI is actually in here so that means this is actually equal to this ok,

So the only thing that you will have to look up in competitive Algebra is that integrality which I think very easy to understand its small exercise you can look it up very clearly in hour Atia McDonald's book introduction to competitive Algebra any way the moral of the story is that there are no non-constant regular functions on a projective variety ok so will stop here.