**Basic Algebraic Geometry**
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**Module 11**
**Lecture 30**
**The Formula for the Local Ring at a Point of a Projective Variety**

Ok so what we saw in the previous lecture was how to calculate the local ring of an affine variety at a point ok. So the result was that if X is in affine variety and P is a point of X then the local ring of X at P is given by taking the localization of the affine coordinate ring of X at the maximal ideal inside the affine coordinate ring of X A x, because maximal ideal M P has to be you have to localize A x with respect to M P and the local ring that you get will be can be identified canonically with the local ring of X at P ok.

Now what I am going to do now is I am going to try to do the same thing for a projective variety ok and just as in the case of affine variety the local ring is given in terms of localization of the affine coordinate ring in the case of projective variety it will be given by suitable localization of the projective of homogeneous coordinate ring of the projective variety ok, so let me state that ok so here is the theorem.

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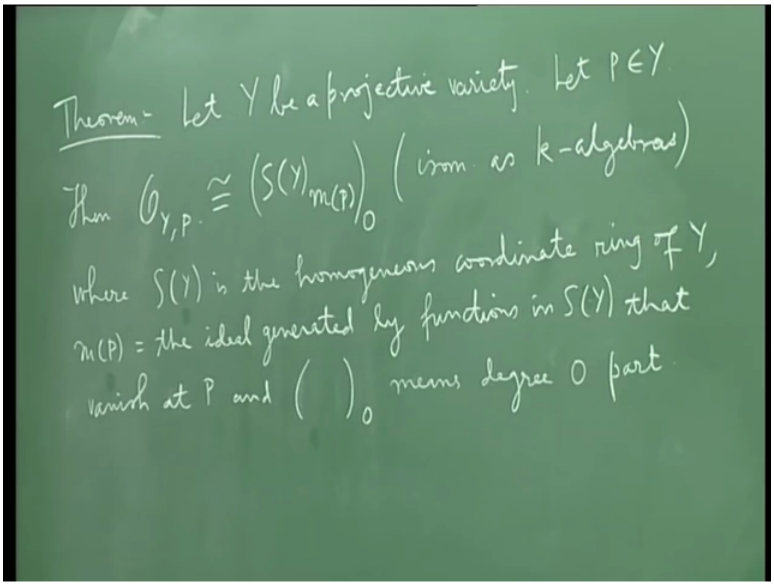


Let Y be a projective variety ok let P be the point of Y ok then O y p the local ring of Y at P is isomorphic to the homogeneous coordinate ring of Y localized at M P ok this isomorphism as K algebras where yeah I need to so I need to localize and also take degree part ok. So where

S Y is the homogeneous coordinate ring of Y M P is the ideal generated by functions in S Y which vanish at P and this notation means take the degree zero part of the graded ring ok.

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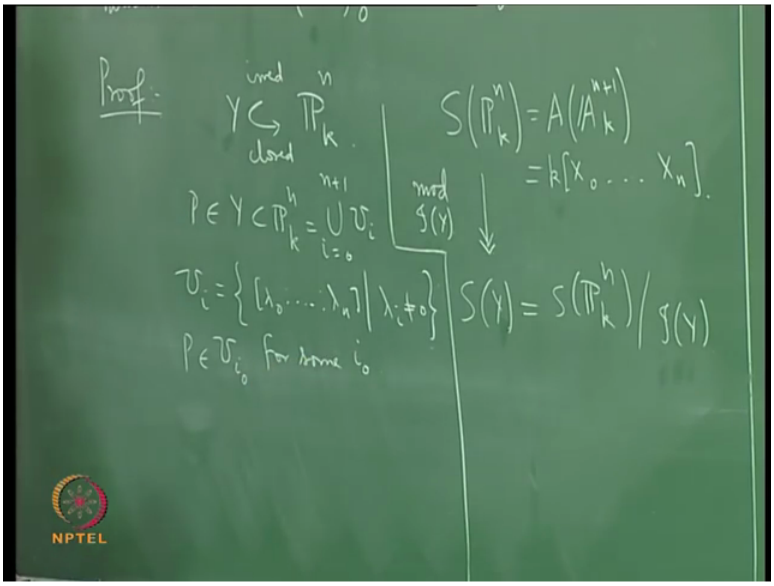


So let me explain this so you see the situation is you have Y you know Y sitting inside as a n irreducible close subset of projective space ok and then you know offcourse you have the homogeneous coordinate ring of the projective space which is just the same as the affine coordinate ring of the affine space above of which the projective space is a quotient offcourse a punctured affine space and this is going to be K X knot etc X n polynomial ring in N plus 1 variables ok and the homogeneous coordinate ring of Y is just the quotient of this by the ideal of Y.

Mind you Y is close subset of projective space so it is given by a homogeneous ideal ok and the homogeneous ideal is going to live in this polynomial ring which is the homogeneous coordinate ring of P n so this so S Y is just S of P n Mod I Y, I Y is simply all those I Y is just you know it is the ideal generated by all those homogeneous polynomials which vanish on Y ok. It is a homogeneous ideal because it is generated by homogeneous elements and therefore this quotient because you are taking a graded ring and you are going modulo homogeneous ideal the quotient also becomes a graded ring ok.

So S Y is also graded ring ok and the gradation is simply given reading Mod I Y right. Now the situation is how do we get this thing? So you know we used the (this) following facts which I said in the previous lecture. The first thing is to calculate the local ring you can go to an open subset ok that is one fact, the second fact is offcourse that if your subset is affine the you have a formula for the local ring.

So what we are going to do is we are going to do a very simple thing. We are going to take the point P of Y and offcourse the point P is going to lie in one of the U I's the U I's being the open subsets of P m there are N plus 1 of them each one of them isomorphic to A n the affine N space. So P is going to line one of them then what we are going to do is, we can just intersect, so P will lie in Y intersection U I.

But Y intersection U I is an affine variety ok and therefore it is easy to we know by the previous theorem we know what the affine, what the local ring is and we also know that the local ring that we get will be the same as local ring of Y at P, so this is what we are going to do ok. So here is a so let me draw a diagram, so there is a diagram like this, so I have the following situation.

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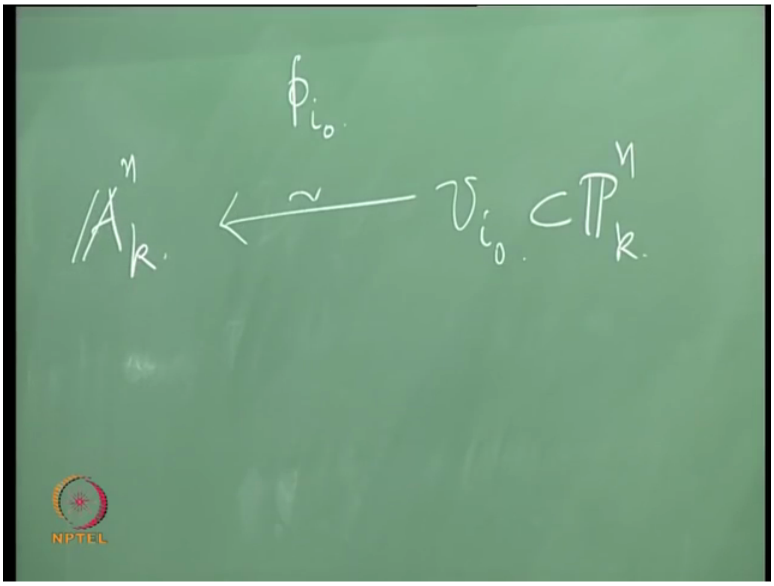

So I have P n I have U I, so P so let me write here, so P is in Y which is contain I P n which is union I equal to zero to N plus 1 of U I, where U I is the set of all points lambda 0, with homogeneous coordinates lambda 0 through lambda N such that lambda I is not zero ok and so this how so Phi belongs to U I knot for some I knot as without loss of (())(09:48) ok. So let me, so let it be like this so we do the following thing. So I have this U I knot which is isomorphic via the morphism phi I knot with affine N space ok and we saw this in the lecture before the last lecture

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No you are right I should start with zero and end with N yes-yes thank you yeah it should be U knot through U 1 there are only N plus 1 of them thank you. Yeah so it should be I equal to zero to N ok thank you for that so yeah so you see this we have already seen that this isomorphism is given in terms of rings in the following way.

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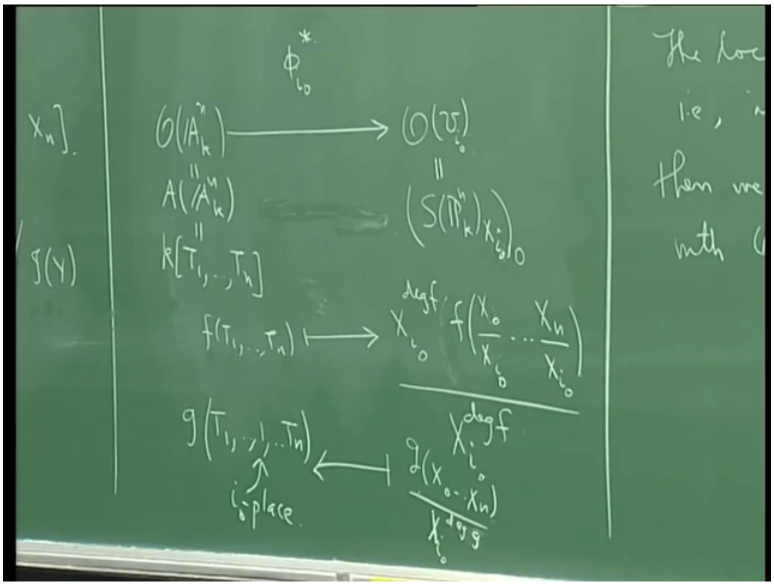

So there is a phi I knot so this phi I knot induces phi I knot star which is the which is given by pullback of regular functions that will go from the regular functions on A n to so if you have a morphism regular function the target will by composition of this morphisms give you a regular function on the source ok so I will get regular functions here to regular functions here so I will get this to this.

But you know offcourse this is just because this is affine this is just A f A n and that is identified with K T1 etc T n these are the coordinates on the A n and O u is as we show in an lecture before the last lecture it is just the coordinate ring of P n localized at Xi and then you take the degree zero part. So this is so here it is I 0, so here I also I have to put I 0 ok. So this is something that we have seen and we have also seen what this map is we also have this we know what this, what these maps in these two directions are.

So the map in this direction is given by well you give me a polynomial F of T1 etc T n then the map in this direction is given by homogenization it is given essentially by homogenization and dividing by Xi to the power of degree F. so it is going to be Xi degree F times F of X. so this Xi knot to the degree F (X) X0 by Xi knot dot-dot-dot X n by Xi knot and divided by Xi knot to the degree F, this is how this map is given ok and offcourse when I write like this I mean forget Xi knot by Xi knot alright so that you have only N of them because there are only N variables that I can fill that I can substitute for ok.

So this how this map is given and we also know what the map in this direction is that is also something that we have seen if you give me any element here will be of the form G of X knot etc X n this homogeneous polynomial of certain degree ok divided by Xi knot to the degree of G this is how an element here will look like ok because you are localizing at Xi knot ok which means you are dividing you are inverting powers of Xi knot but then you want the degree zero part which means that the numerator degree and denominator degree should be the same which means that the numerator should be first of all a homogeneous polynomial of certain homogeneous degree.

And the denominator should be the same degree as the numerator so the denominator should be Xi to the power of to that power which is equal to that degree of the numerator polynomial. So it is going to be like this and what you are going to get here is just simply you know you substitute Xi equal to so you just substitute so you divide you take X knot etc upto Xn divided by Xi knot ok and omit the Xi knot by Xi knot term you get N terms and you just no that was the map in this direction.

So what you do is you just put Xi equal to 1 ok and the n for the remaining X knot to Xn leaving out Xi you put T1 through T n that is what it look so, so the map in this direction T1 blah-blah-blah 1 T n where this 1 is in the I knot place ok so that you now get you have substitute for N plus 1 variables N plus 1 variables and offcourse you essentially you are substituting Xi knot is equal to 1 ok. So the denominator is going to just vanish I mean it is not going to vanish I mean it is going to be 1 so it is not going to show up here right. So this is how this isomorphism is. This is the commutative algebra version of what is happening here in geometry in terms of regular functions everything here is terms of regular functions.

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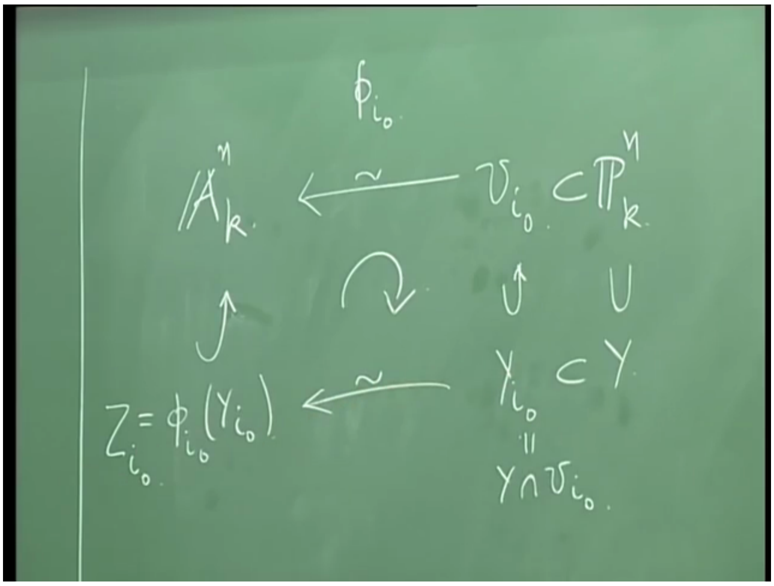

Now what I am having is I have inside P n I have this Y which is a closed subset of P n irreducible close subset of P n and then I get this Y I knot which is just intersection of this is just intersection of Y and U I knot ok. So what happens is that Y I knot is an irreducible sub variety of U I knot because you see Y is irreducible closed Y is first of all closed in P n therefore Y intersection U I knot is closed in U I knot so this is certainly a closed subset of that and then this is a Y I knot on the other hand is also irreducible, why?

Because Y I knot is Y intersection U I knot which is an open subset of Y and an open subset of something that is irreducible continuous to be irreducible if it is non-empty therefore mind you P the point P is in the point P is here the point P is in U in knot and it is in Y so the point P belongs here ok. So this is a non-empty open set so it is irreducible, it is non-empty open in Y so it is irreducible on the other hand it is also closed in U I knot so this becomes a irreducible closed subset of U I knot which under this isomorphism is going to be carried to Phi I knot of Y I knot which is well I have to give this some name let me call it Z I knot right and so this isomorphism carries this irreducible closed subset into an this irreducible closed subset but then this is the variety here ok and so what is going to happen to so this isomorphism so offcourse this diagram come here, here also diagram there is no diagram here I am going to draw it in the next board.

So this isomorphism can also be you can also write down what this isomorphism is ok, so what this isomorphism is, is well lets write it down it is going to be the following.

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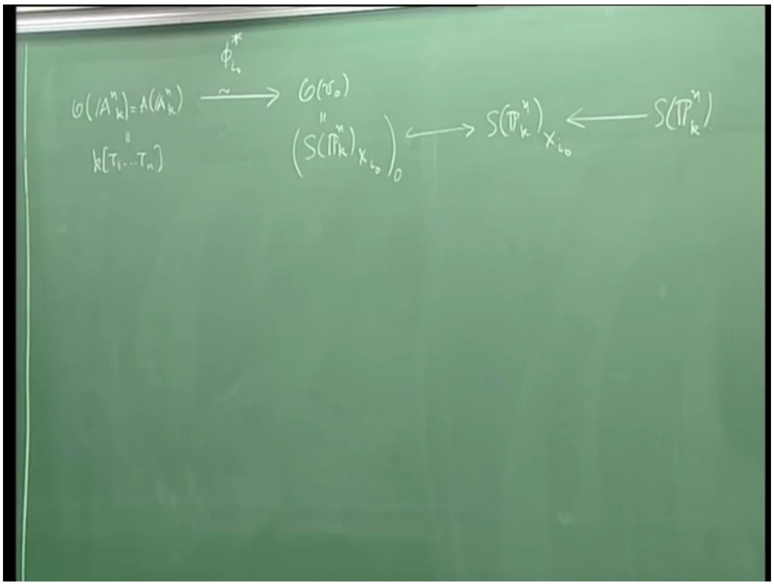

So what is happening here so let me write this part alone there so I have O of A n which is Af Am which is A T1 Tn and I have this which is Phi I knot star to O U knot which is S P n localize at Xi knot take the degree zero part this is what it is. So this is just translation of this arrow into commutative algebra.

Now going to do this so you see when you go to an irreducible closed subset the homogeneous coordinate ring is given by a quotient because S Y is going to be given by S Y is a quotient of S of P n ok and so you know so well maybe I will try to use this part of the board as well let me rub this so that I can extend this diagram to the right so you know this P n here so there is U I knot sitting as an open subset P n so this is an open subset and that corresponds to this localization of S of P n.

So this is a localization followed by well so infact this is sitting inside the localization as the degree zero part which comes out which comes as a localization of S of P ok. So this is the localization which is sitting inside the zero part and this part of the diagram corresponds to this ok and what is happening is that Y is an irreducible closed subset of P n and its homogeneous coordinate ring is a quotient of P n.

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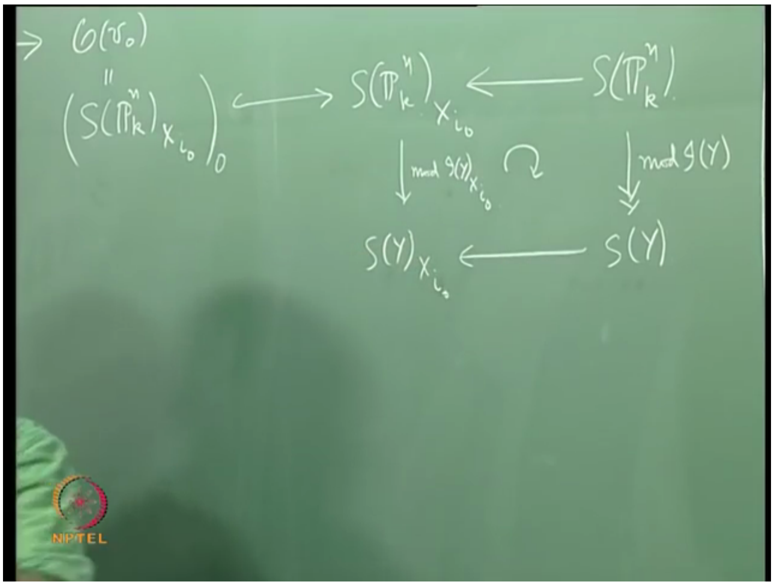

So what is happening is that you have here you go mod I Y you get X of Y this homogeneous coordinate ring of Y. So this you know corresponds to this here which is irreducible closed a closed subset corresponds to a quotient ok and so this corresponds to this and then you know this thing which is the intersection Y with this open U I knot ok that will correspond to something what is that? it is going to be well it is going to be it is just U localize so you know this is a very important property in commutative algebra that localization commutes with taking quotients localization is what is called localization is exact ok.

The process of localization is exact ok so localization transforms exact sequences to exact sequences and you know a quotient fixed to an exact sequence so you take a quotient and then localize is the same as localizing and then taking a quotient and these are all since standard from a first course in commutative algebra so if you go by that so here is so what I can do is I can take S Y and I can localize it X I knot ok. So this is the localization alright.

On the other hand what I can do is I can on the other hand first localize at X I knot and then go Mod I Y localize at Xi knot ok. So this is not only can you localize a ring you can also localize an ideal in the ring infact you can localize a module ok.

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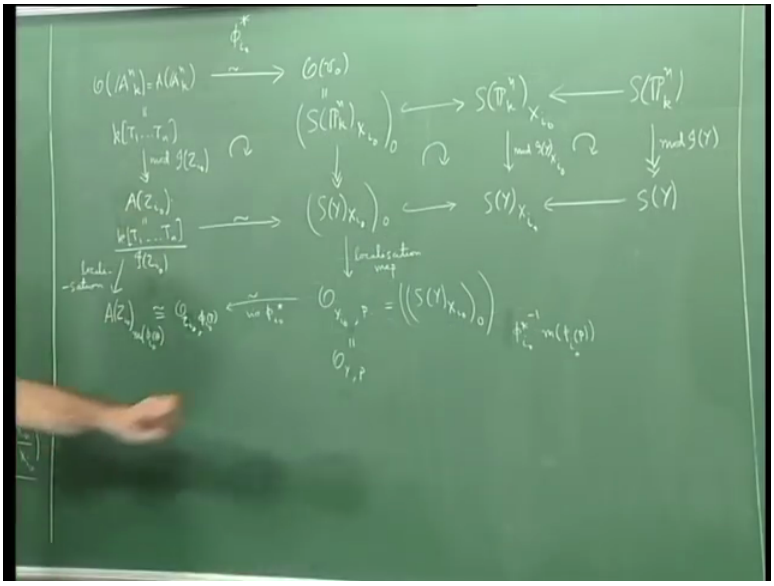

So I Y is an ideal there ok and you can therefore localize it ok, this is the localization and then so you know then here what you will get is just S Y localize at Xi knot and the degree zero part which is again it will be a quotient here ok and it is this quotient and this diagram commutes and it is this quotient that corresponds to this irreducible closed set ok.

So this irreducible closed subset corresponds to this quotient and this irreducible closed subset corresponds to this quotient ok and on the other hand what is going to happen on this side? On this side Z I knot which is phi I knot of y I knot is an irreducible closed sub variety of the affine space so it is going to be given by its coordinate ring is given by the coordinate ring of affine space modulo the ideal of Z I knot ok. So here it is going to be just here this irreducible closed is going to corresponds to quotient here and that quotient is just K it is just going to be K T, so let me first write it as A Am A of Z I knot this is what it is going to be and this is just this is A this is just gotten by going Mod of ideal of Z I knot ok.

And so which is just so it is going to be identified with K T1 T n modulo of Z I ok and you are going to get an isomorphism like this, this diagram is going to get commute and this isomorphism is precise to the isomorphism of it is just this isomorphism Phi I knot that is carrying Y I knot to Z I knot ok so is what is going to happen.

Now so you know the point you see if you take the point P the point P is here it goes to the point Phi I knot of P which is a point here ok and you know that because you have a n isomorphism of these two varieties the local ring of this at this point is going to be the same as a local ring of Z I knot at Phi I knot of P ok. So what you are going to get is you are going to get local ring of Y I knot at P is going to be isomorphic via Phi I knot star to the local ring of E Z I knot at phi P phi I knot of P ok you are going to get this.

Because we have seen that isomorphism of varieties is going to identify give rise to an isomorphism of local rings. But then you see the local ring of Y I knot at P is the same as a local ring of Y at P because Y I knot is an open subset of Y ok because we have also seen that the local ring to get the local ring at a point you can restrict attention to an open subset which contains the point and Y I knot is an open subset of Y which contains a point because it is Y I knot is just the intersection of Y with this ambient open subset or the projective space.

So this is well this is the same as this guy is the same as O Y P ok this is what we want we want to calculate this and this is isomorphic to this right and what is this? See this is this as we have seen this is the local ring at a point of an affine variety and therefore it is given by this is isomorphic to you take the affine ring affine coordinate ring of Z I knot and localize at the maximal ideal corresponding to phi I knot of P, this is what we have already seen. We have seen this theorem the previous class that previous lecture that the local ring of at a point of affine variety is just the affine coordinate ring localize at the maximal ideal corresponding to that point.

So this A , so this local ring of Z I knot at Phi I knot of P is just the localization of the affine coordinate ring of Z I knot which is A Z I knot at the maximal ideal m OF phi I knot of P corresponding to the point phi I knot of P ok and offcourse you know this map is the localization this is the localization map from the ring of coordinate functions to its localization and similarly this will also be the localization map it will be a localization map and the question is that you need to know what localization map this is? So what is this going to be?

This is going to be this ring S Y X I knot ok and then you have to further localize it I mean this is take the degree zero part ok and then further localize it at the maximal ideal that you get which is the image of this maximal ideal here ok. So it is going to be localize it at Phi I knot star inverse so this (())(29:57) map like this and that is going to pullback this maximal ideal Phi I knot P it is localizing you have to localize at this ideal ok, this is what it is going to be alright.

If you just follow the commutative algebra this is what it is going to be and you have to show that this thing is the same as that ok right yeah so one has to really write this down. So let me do the following thing.

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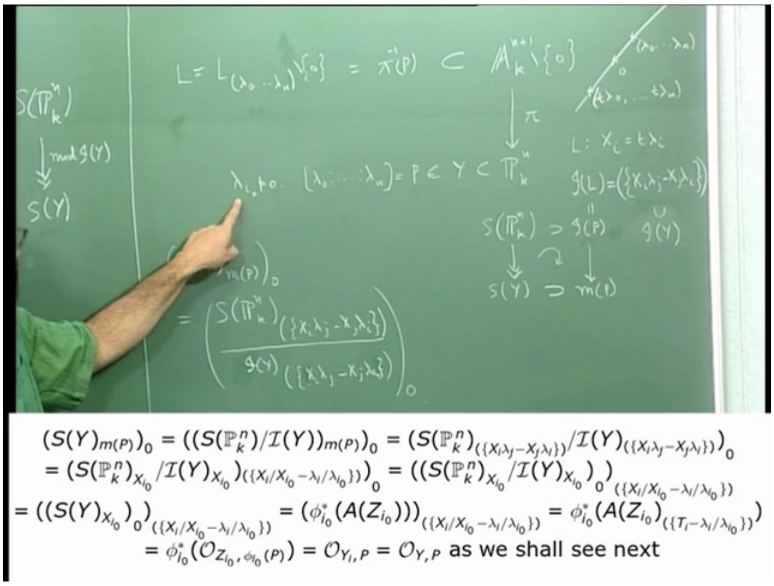


So you have, see you have this projective space and you have this Y and you have this point P ok. Suppose P offcourse P is in U I knot ok. So you know this P has a homogeneous coordinates lambda knot blah-blah-blah lambda N ok and lambda you know lambda I knot is not zero ok because it is in U I knot and now you look at the projective space as a quotient of the affine space above I mean the punctured affine space above and then the inverse image of P will be just the line in affine space through the point lambda knot lambda N minus the origin ok, this is what you are going to get and you see what is the equation of this line in the affine space?

Ok so you see the picture is that you know you have something like this you have the origin and then you have this line L ok which is this line L and offcourse there is a point on it which is given by lambda knot etc lambda N and any other point on it given by T lambda knot etc T lambda N that is how any other point on this line is given by and therefore if you look at the ideal of functions that vanish on this line it will be a homogeneous ideal ok we have already seen that whenever a polynomial vanishes on a line it has to have no constant term and then every homogeneous component of that polynomial also has to vanish on that line.

So the ideal of this line is simply gotten by eliminating T from the equation general point on this line. So this line is given by you know Xi equal to T lambda I ok this is how it is given where T is a parameter ok and if you, you know if you eliminate T you will be writing Xi by lambda I is equal to T is equal to X J by lambda J but then you cross multiply it because some lambda I or lambda J maybe zero.

So you will get that the ideal of this line in affine space in of polynomials that vanish on this line is just the ideal generated by Xi lambda J minus XJ lambda I, it is just this ideal ok and you know this ideal will also be the ideal the homogeneous ideal of the point the image of this ideal first of all this ideal will be a homogeneous ideal offcourse because it is generated by all this elements which are all homogeneous of degree 1 ok and what is the zero set of this homogeneous ideal in projective space?

It is just this point ok so this will also be equal to the ideal of the point in projective space ok and therefore so this ideal corresponds to this point in projective space and you know what it will corresponds to as a point of Y, it will correspond to the image of this ideal in S Y ok. So you see this ideal is here alright. It is a homogeneous ideal here and if you take its image here, here it will correspond to the point P considered as a point of S Y, in other words I am saying if you take the image of this ideal here you will get exactly M of P namely the ideal of the ideal generated by all homogeneous functions homogeneous polynomials which vanish at P vanish at the point P ok.

So I am just saying that this is the same as M of P inside so its quotient is M of P. So this sits inside S Y (())(35:10) and then you have a quotient which is S of Y and its image there will be just M of P ok this is what you will get. So this is what M of P is alright it is just (the image) quotient of this ideal ok. Mind you this ideal will contain the ideal of Y ok, this will contain I of Y ok and there is reason why mod I Y it also defines an ideal alright and it corresponds to the point P in Y alright.

Now so you know if you go and try to calculate what this quantity is, this quantity is just so you know S Y localized at M P zero is going to be you know you see what I am doing is I am taking a quotient ok I am taking a quotient by an ideal and then I am localizing alright but I have already told you that localization commutes with quotients. So what you can do is this is the same as you take the S P N ok localize it with respect to that ideal the ideal generated by Xi lambda J minus X J lambda I ok and then what you do is, you localize at this and then go Mod that is right.

You go I of Y localized at this ideal and then take the degree part ok this is what it is right but then the trick is that you know the point is the following, the point is that you know since the point lies in U I knot this lambda I knot is non-zero. Let me write it down, so the point here is lambda knot lambda N ok with homogeneous coordinates let us go into the point here which is going to be lambda knot by lambda I knot lambda N by lambda I knot that is going to be the point here ok.

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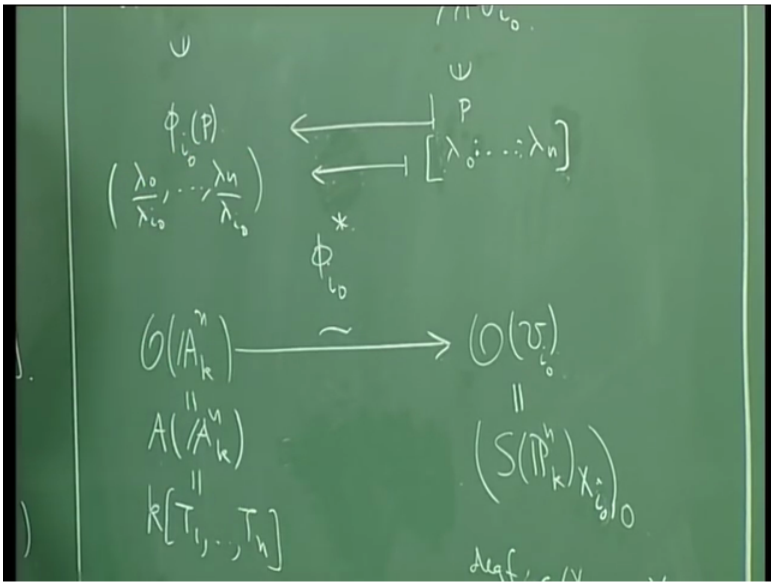


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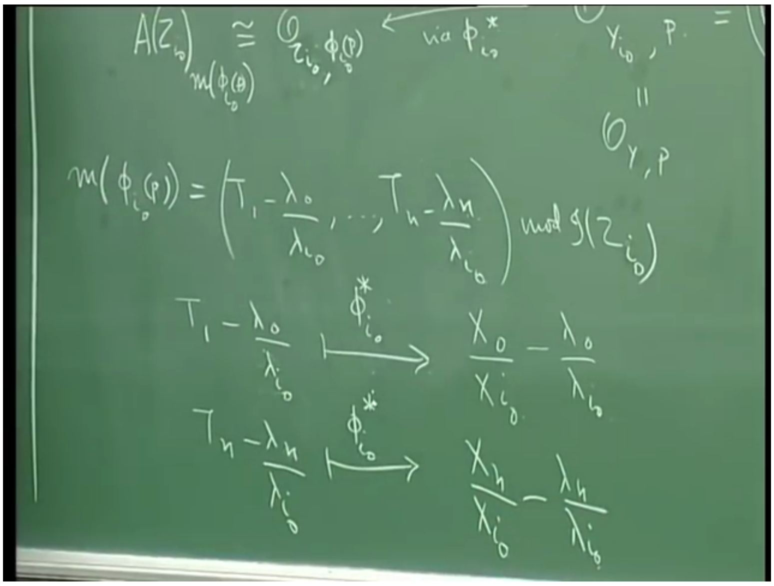


Therefore if you look at this calculations you will see that the maximal ideal corresponding to this point is going to be the maximal ideal of Phi I knot of P is going to be just (T) T1 minus lambda knot by lambda I knot and so on T n minus lambda knot by lambda I knot.

Offcourse I omit lambda I knot by lambda I knot ok, it is this ideal mod this is this ideal mod ideal of Z I knoot this is what the maximal here is ok, it is a maximal ideal of this point, the maximal ideal of this point is just the maximal ideal of this point in the affine space modulo this ideal of is a (())(39:28) ok. So this point you see this point corresponds to a point in affine space so it corresponds to a maximal ideal here that maximal ideal is this ok and that maximal ideal mod I of E Z I knot will give me the maximal ideal in A Z I knot to which the point corresponds to in E Z I knot ok.

So this is what it is alright and you know if I take each of this elements like T1 minus lambda knot by lambda I knot ok if I take any such element and if I take its image under this Phi I knot star the map induce by Phi I knot star what I will get is you know what this map is, this map is just homogenize with respect to A X I knot ok so what I will get is I will get you know I will simply get X knot by Xi knot minus lambda knot by lambda I knot ok this is what I will get when I apply this map I am suppose to homogenize it with respect to Xi knot and then divide it out by the same power of Xi knot ok.

So this is what I will get and this is clearly an element of and similarly you know if you take the element T n minus lambda knot by sorry this is, this should be lambda N T minus lambda N by lambda I knot that will go to under this Phi I knot star it is going to X n by Xi knot minus lambda N by lambda I knot this is what it is going to go to. Because after all the map from here to here is you just send F to homogenization of F divided by and then you remove this power of Xi that you put to homogenize it ok.

So if you apply this, this is what you will get therefore what this calculation tells you is that it tells you that this ideal ok is actually it is no, Xi knot by Xi, X knot by Xi knot is degree zero and lambda knot by lambda I knot is also degree zero this is a difference of two degree zero elements so it is degree zero. It certainly X knot by Xi knot is certainly in the degree zero part of this ok. So you know if you do this what you will get is you see all this elements are here this elements they are all here ok and these elements generate a maximal ideal and it’s that maximal ideal is precisely this maximal ideal ok at which you have to localize this to get O Y I knot P.

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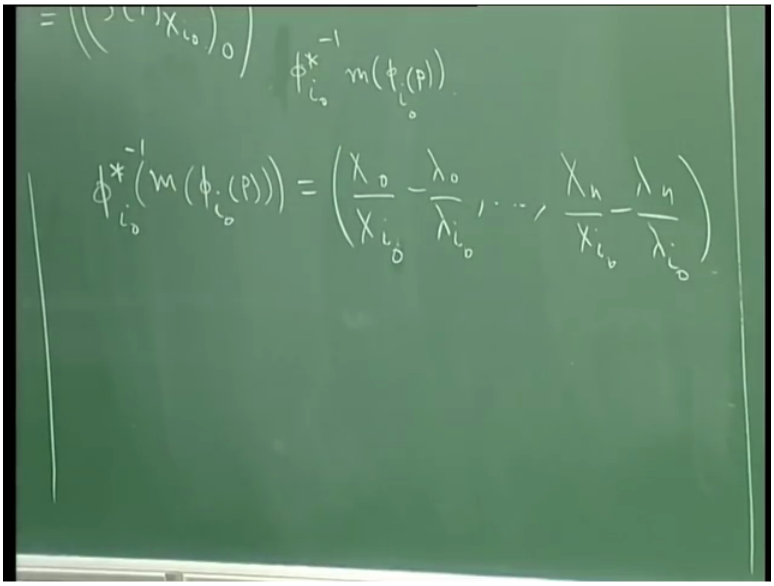


So you know so Phi I knot star inverse of M of Phi I knot of P is actually the ideal generated by all this guys X knot by Xi knot minus lambda knot by lambda I knot dot-dot-dot X n by Xi knot minus lambda N by lambda N offcourse you know I able to write all this because lambda I knot is non-zero that is why I can write all this and therefore this ring what you get is just.

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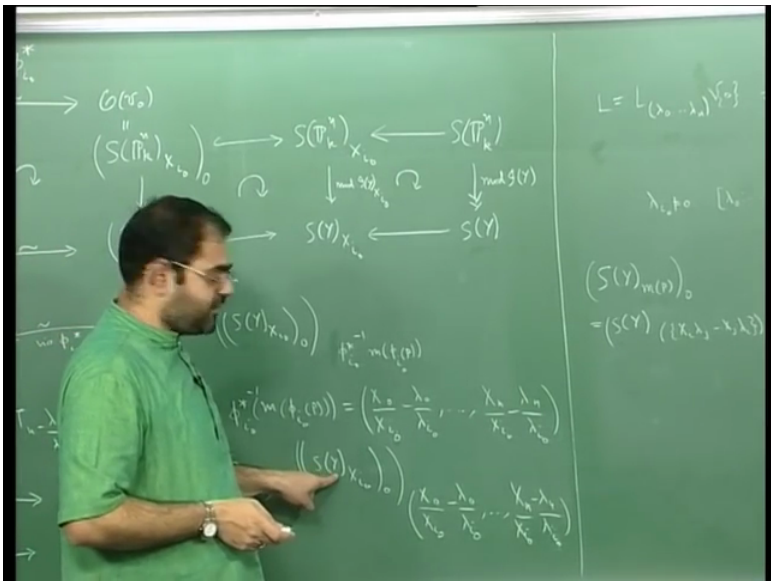

So O Y I knot P is just S Y localize at Xi knot take this degree zero part and further localize at this guy X knot by Xi knot minus lambda knot by lambda I knot and so on Xn minus by Xi knot minus lambda N by lambda I knot, this is what it is.

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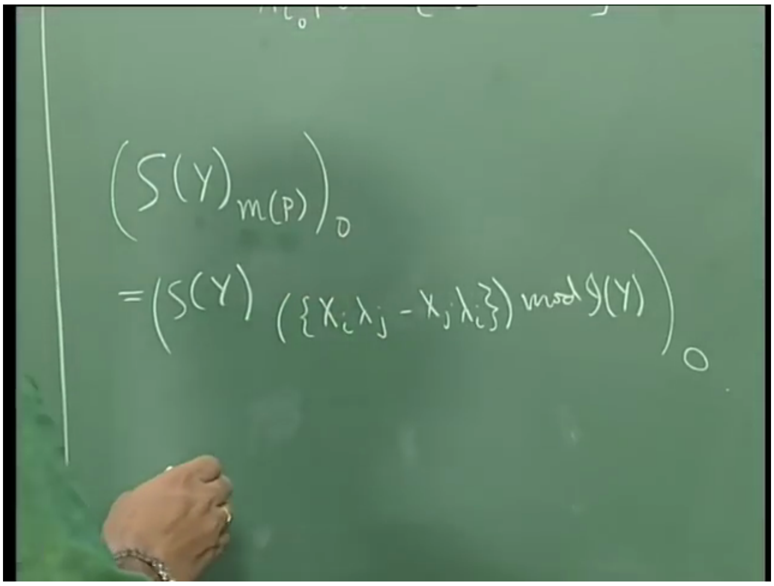


So let me write this what I am getting here is this is S Y localized at M P, M P is just the this ideal mod I Y ok so it is just S Y this ideal, ideal generated by Xi lambda J minus X J lambda I mod I Y the localize at zero ok but you see what I want you to understand is that these two are the same you see what I have done here is I have taken S Y I have inverted Xi knot ok then I have taken the degree part and then I am inverting everything outside this ok, but then you know if you think of it naively think of this Xi knot as going out and getting multiplied with these ok see if you already inverted Xi knot ok and you want to do this ok it is the same as multiplying this thing throughout by Xi knot.

So what I want to say is that this is the same as S Y localized at this and then take the degree zero part which is exactly that ok. So I am claiming that these two are the same ok you have to just satisfy yourself that, that is true purely it is a purely commutative algebraic thing. See what you must understand is in S Y localized at Xi knot you know Xi knot is a unit mind you Xi knot is invertible therefore you know to localize this here it is just enough to localize this at (Xi) X knot minus you can multiply everything by Xi knot ok.

Because Xi knot is after all a unit so it is enough to localize this at X instead of localizing at this ideal it is enough localize it at X knot minus Xi knot times lambda knot by lambda I knot that is you can multiply by Xi knot throughout ok but then localizing but you know X knot minus Xi knot lambda knot by lambda I knot is the same as upto multiple of lambda I knot it is just X knot lambda I knot minus lambda knot X I knot which is of this form. So these two are one and the same ok.

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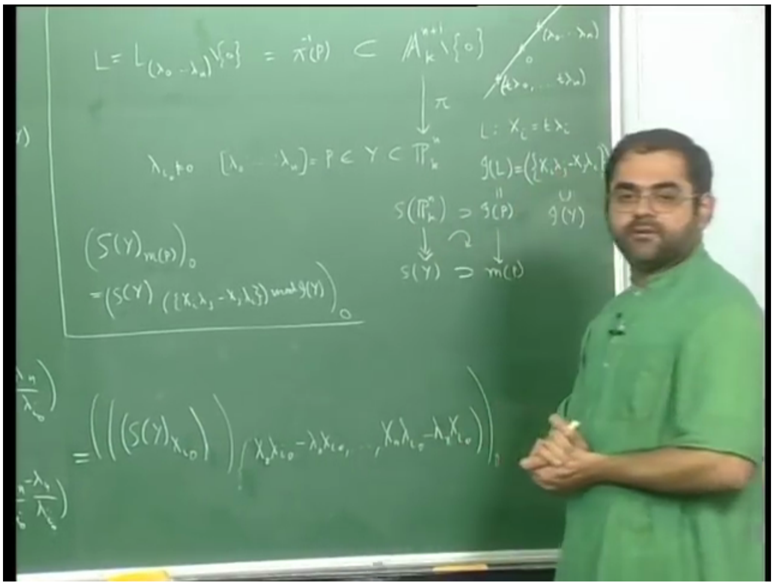

So let me maybe continue here and say that you know so let me erase this line and write that this guy here is you know it S Y Xi knot localized at zero and then further localized at see now I am going to multiply throughout by Xi knot because Xi knot is a unit alright. So it will become X knot so infact I can also multiply by lambda I knot because lambda I knot is also a non-zero element so I multiply throughout by lambda I knot Xi knot .

So what I will get is I will get X knot lambda I knot minus lambda knot Xi knot blah-blah-blah and then I will get X n lambda I knot minus lambda N Xi I will get this so the way to do it is you remove the zero here then you take the zero ok you can do this. So I will leave it to you to check that you know this guy here and this guy here are one and the same ok. It is just a matter of some commutative algebra ok to check that this is the same as this ok. It is an easy check you will have to check whether this is also equal to this alright but I leave it to you to check that this is equal to this ok and I will stop here