Basic Algebraic Geometry

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Module 1
Lecture 3
Going back and forth between subsets and ideals

Okay so let us continue with our discussion on algebraic geometry. So you know so let me recall what we have seen until last time, so broadly algebraic geometry is study of the geometry of set of common zeros over a bunch of polynomials.

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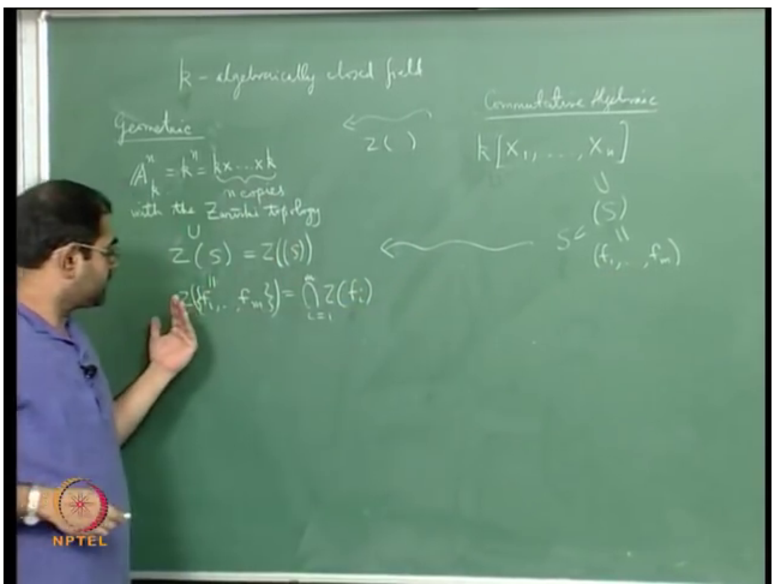

So what we do is we take k to be an algebraic closed field and of course what is a significance of this is well significance is that you have the Nullstellensatz which tells you that zeros of common zeros over bunch of polynomials this is set of common zeros over bunch of polynomials is not going to be an empty set provided the set of those bunch of that bunch of polynomials does not generate the whole ring, okay.

So that is the reason why choose an algebraic closed field and what we have is basically we have this you know that is a dictionary or we rather try to set up a dictionary on one side which is the geometric side on the other side being the competitive algebra side. So what we have in the on the geometric side is the affine space over k which is actually kn namely k cross k cross k n copies this is the Cartesian product of k taken with itself in n times of course n is greater than or equal to 1 and it is kn not just kn but it is with the so called Zariski topology.

So well if you just look at kn you would normally think of it as n dimensional vector space over k, okay but the point is and in a vector space 0 is a very special vector, okay but we want we do not want to think of it as a vector space we want to think of it as affine space. So 0 does not have the point 0 does not have any special meaning in the case of vectors it does have a special meaning because vector always needs a initial point and a terminal point and we always take the initial point with the origin, okay.

But here in affine space any two points are literally the same except that only the code (())(4:11) have changed, okay. So but more importantly what is what we have is not just a set it is not just a Cartesian product this Cartesian product is consist of n tuples of elements of k, okay it is not just the Cartesian product as a set but there is a so called (())(4:31) what is Zariski topology it comes from the it rather comes from the commutative algebra side so let me put that on this side is the commutative algebra side on this side we have the polynomial ring in n variables over k, so X1 through Xn are n indeterminate or n variables and of course this is the polynomial ring in n variables, okay.

So these are polynomials in these n variables with coefficients coming from the field k, okay. And how do you get this Zariski topology? You get this Zariski topology by prescribing of course any topology is given by either prescribing collection of closed sets of collection of open sets and these two are complimentary to each other because a close set is a complement of an open set and vice versa. So the approach to this Zariski topology is by specifying a collection of closed sets and how do you specify this collection of closed sets what you do is when you take any subset of this polynomial ring which means you are just taking the bunch of polynomials in n variables with k coefficients and what you do is associate to that subset the set of 0's of the set of common zeros of that subset.

So this set this subset here will be all those n tuples which satisfy each and every polynomial in this collection in this subset, okay. And the we declare sets like this to be the closed sets, so they give a name they call algebraic sets because they have the common 0 locus of bunch of algebraic equations, okay solutions to bunch of algebraic equations you can think of the equation corresponding to each polynomial as a polynomial being equated to 0 and then the solutions are nothing but the 0's of the polynomial, okay.

So the algebraic sets are sets like this and they are if you declare them as closed then you see that they satisfy the schemes for closed sets and therefore this sets of subsets of this kind taken as close sets does define a topology on this set kn and along with this topology we call kn as a affine space n dimensional affine space over k we give the special symbol and we call this topology as Zariski topology, okay in honor of Oscar Zariski who is whom you could say is founding father of algebraic geometry from the viewpoint of commutative algebra, okay.

So and then I told you that they so, yeah so there was a so on this side we have a collection of nice sets namely the algebraic sets which are the closed sets but on this side you seem to only have subsets of the polynomial ring but then a subset does not make sense as a sub object of the polynomial ring and the right sub objects are ideals, okay and of course these everything here is commutative the k is of course always commutative field we are in this course we are only worried about commutative rings.

So and therefore you know ideals are always two sided, okay. So the point is if you want the right objects on this side more interesting than the subsets are the ideals and how do you pass from subset to one ideal this is the very general yoga it is a general philosophy but you can always take the smallest sub object which contains your subset in any mathematical structural if you have a subset which is not a sub object then how do you get a sub object you just look at a small the smallest sub object which contains that subset it is called the sub object generated by the subset, okay.

So in this case you can take the ideal the sub objects which we are interested in our ideals, of course you know you can think of subrings as sub objects, okay but subrings are not going to help because the point is the moment this set contains a unit then the 0 set will become empty, okay because a unit has to be just a constant polynomial which is non-zero, okay and that constant polynomial is never going to vanish anywhere so the 0 set will become empty.

So certainly you are not interested in sets like this which contain 1 or which can generate 1. So basically you are only interested in so that tell you that you are not interested sub rings and what other objects sub objects can you think of you can think of ideals and why ideals because ideals are really nice sub objects because the nice sub objects are the ones which can give you quotient objects, okay. So if you have a ring and you have an ideal then you can get the quotient ring, okay.

Whereas of course if you have a ring and a subring I am not going to get any quotient, okay. So from the point of view of a sub object being the right one if it gives a decent quotient object you see that ideals are preferred, okay ideals are the right choice and subrings are not but anyway we do not want subrings because this is going to end up as a null set if you going to take a subring, okay. So (())(10:26) up short of all these is that you look at the ideal generated by this subset which I put as bracket S and then you see that if you take the 0 set of the ideal generated by S, okay that turns out to be same as a 0 set generated by S I mean the 0 set of S the 0 set of the ideal generated by is the same as 0 set of S.

So what it tells you is that if you replace the set by the ideal it generates you are not going to change anything here, okay. So the advantage of all this is on this side on this side you have closed subsets, on this side you have ideals, okay and not just subsets and what is the prescription though we started out with subsets we always take the ideal generated by the subset and work with that, okay and that does not change anything on this side, okay because of this equality.

Now so as I told you the you know the whole propose of algebraic geometry is to look at this these two sides of the picture, okay and go from one side to the other and keep translating what properties on one side mean on the other side. So geometric properties on this side should mean they should give rise to some ring theoretic properties which means they give rise to some ideal theoretic properties more generally they might give rise to module theoretic properties on this side and conversely some module theoretic or ideal theoretic properties or ring theoretic properties on this side will correspond to geometric properties on this side and our and aim of algebraic geometry is to discover this relationship, okay.

So the first thing I wanted to say is you know well but the two points I have to mention this is the first point is so let me again recall you know an algebraically closed field is a field in which if you take a polynomial of 1 variable in variable over that field and if it is non-constant then all its 0's are there in that field, okay. So normally field theory tells you that if you have polynomial over a field then you may have to go to an extension field to find the 0's of the polynomial, okay and in fact field theory gives you what is called as a spitting field for a given polynomial which is a kind of a smallest field extension over which the polynomial completely splits into linear factors but you do not have to worry about such things I mean if you are working with an algebraically closed field because a definition of algebraically closed field tells you that if you have a polynomial if you already split into linear factors that means all the which is equivalent to saying that all the 0's of the polynomial already elements of this field, okay.

But the story does not end there the important thing is the when you define the algebraically closed field is only for one variable, okay but when we do algebraic geometry in the general sense we are worried about polynomials not in just one variable we are worried about polynomials in several variables and then the question that arises is if you give me a you know a subset of polynomials or for that matter the ideal it generates and look at the 0 set what is the condition that this 0 set is non-empty, okay and the Hilbert Nullstellensatz tells you that this will be non-empty so long as this ideal is really a proper ideal so long as this ideal is not the whole ring or the unit ideal, okay.

So in other way this should not contain a unit we all know that if it contains a unit if S contains a unit it is very clear that this is empty, okay and the Hilbert Nullstellensatz tells you that is a only case when it is empty so long as this does not contain a unit and of course it is very important that you know well I should correct my statement a little this may not contain a unit but this might generate a unit, okay.

So to be very strict this ideal should not be this ideal should not contain a unit which is a same as saying this ideal should not be the whole ring, okay. In that case and then and only then will the 0 set be non-empty, okay and that assurance is given to you by the Hilbert Nullstellensatz, okay that is one important thing then the other thing is the other important theorem that comes is the Hilbert Basis theorem or Emmy Noether theorem, okay.

So you see you already see the impact of results on this side which mean something on this side you see the fact that the Hilbert Nullstellensatz is basically a result which comes from this side of the diagram, okay. If you want to think of commutative algebra also as involving field theory for that matter because you know fields come because you know you start with a ring basically you start with rings which are integral domains and then if you go to the quotient field or the field of fractions then you know studying things over that already leaves you on field theory, okay.

And so you know Nullstellensatz is a kind of result that you know comes on this side of the picture but geometrically it means that you know the 0 set you are working with gives you conditions when the 0 set you are working with is not empty, okay that is already a translation from a result on this side to this side, okay and but the way I have given it I have already given it on this side, okay I have not told you the commutative algebraic version of the Nullstellensatz which I will do, okay.

And the other statement that I am worried about is that I want to talk about is about the Hilbert Basis theorem or the Emmy Noether theorem, what does it say? It says that if you start with the ring R a commutative ring of course you know you must always remember that we always work only with commutative rings with 1 when we always assume that all rings homomorphisms take one to one, okay. So we start with a Noetherian commutative ring, okay which means ring satisfies a property that every ideal is finitely generated. Then if you take a polynomial ring and finitely many variables over that ring the polynomial ring also becomes Noetherian, okay.

So if you take for the ring a field you know a field is always Noetherian because it has only two ideals namely 0 ideal and the full field which is unit ideal so it is Noetherian and now if you apply Hilbert Basis theorem or I mean Noetherian theorem you get this ring is Noetherian what it means is therefore that every ideal is finitely generated, okay. Now what is the importance of saying that an ideal is finitely generated the importance is that every element in this ideal can be written as a finite linear combination of a fixed number of elements with ring coefficients that is what it means but what it really means is that you can take for this generating set only a finite set of polynomials and what it means therefore is even though you start with a set which is probably infinite, okay or you start with an ideal which is infinite, okay.

In fact an ideal will be infinite because even if it has one element then it will contain all multiples of that element by ring elements and this is an the infinitely many elements here because k is any algebraically closed field is infinite that is a result from field theory, okay and this polynomial ring is also infinite so all these ideals are all going to have infinitely many elements and you when you look at the common 0's of elements in the ideal you seem to be looking for common 0's for infinitely many polynomials.

But then what the Hilbert Basis theorem tells you is that that is not what is happening really what is really happening is you are just looking at the 0 set of finitely many polynomials. So what it tells you is that even S is infinite in any case any ideal like this will always be infinite that ideal is the same as the ideal generated by finitely many polynomials, okay and therefore the 0 set is just the common 0's of this finitely many polynomials and in fact this is the set this is same as intersection of the 0 sets of the individual polynomials, okay Z of fi is just the 0 the 0's of fi, okay and then if you take the intersection you will get the points which are 0's of all FS which is exactly what this means this is the set of common 0's of all the FS.

And of course this tells you that you are always only going to solve finitely many equations, okay and why this is important is it is also important for computation, okay because once you are finitely many things to deal with you can have inductive procedures based on some ordering, for example you can do lot of computations in commutative algebra using software and all this is possible just because of this result that you are only dealing with finitely many polynomials at a time, okay.

So that is the importance of these two very basic but very important theorems one is the Hilbert Nullstellensatz and the other is the Hilbert Basis theorem, okay fine. So that is the setup now so what I want to do is I want to give you something that goes in both directions, okay. So already I have this Z which associates to every subset or every ideal on this side the common 0 locus on this side, okay. I also want to give something on this in this direction, okay I want to give something in this direction. So when I go from this direction, so from how do I go from here to there?

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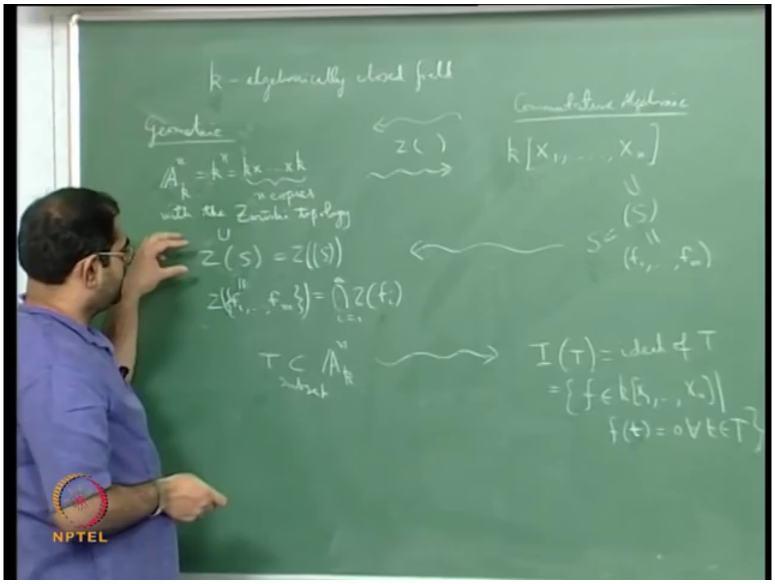

So you know if so let T be a subset of An, okay and mind you this is just a subset is just some collection of points, okay I am not requiring the radius closed or open or something like that, I am just taking any subset and want I do is from here I associate to T I of T which is called the ideal of T, okay this is the ideal of T, okay and what is ideal of T this is the set of all polynomials in the polynomial ring such that f of, okay so let me write it like this f of T equal to 0 for every T (())(22:29) in capital T, okay this is called the ideal of a subset, okay.

So there are two set I want you to understand first thing is I am defining a set and I am calling it an ideal, okay in general that is not correct I can define a set then I have to verify it is an ideal, okay. So the fact is that but if that if you really look at these definitions obviously it is an ideal because you see if you take two such f's say f1 and f2 then if f1 and f2 both vanish at every point of T then so does a sum, okay and of course 0 vanishes to every point of T so 0 is there so this close on addition this as 0 and of course for that matter if f vanishes at every point of T so does minus f, okay and also if f vanishes at every point of capital T then multiplying f by a g will also vanish at every point of capital T so this is an ideal.

So it is in fact an ideal and this is called the ideal corresponding to the subset, okay. And so you see now we have so basically what is happening is that to a subset here we associate a set here which is actually closed, okay it is an algebraic set and in fact for an ideal here also we can associate a subset here which is actually closed, okay and in this direction given a subset you associate an ideal, okay.

So you can see more or less that on this side you are worried about closed sets, okay and on this side you are worried about ideals, okay. So let us explore the properties of these two associations many of them are quite you know quite straight forward, so the first thing is so let us look at the association in this direction, okay that associates to every subset the 0 set of that subset. So the fact that I want to put is that this association is actually inclusion reversing, okay on this see on both sides you have subsets only thing is here this special subsets we are interested in a closed subsets and here the subsets we are interested in are ideals, okay.

But never the less inclusion makes sense on both sides as a partial order, okay and what I want to say is that both of these associations they just invert the they are not order (())(25:30) a preserving but they are order reversing, okay.

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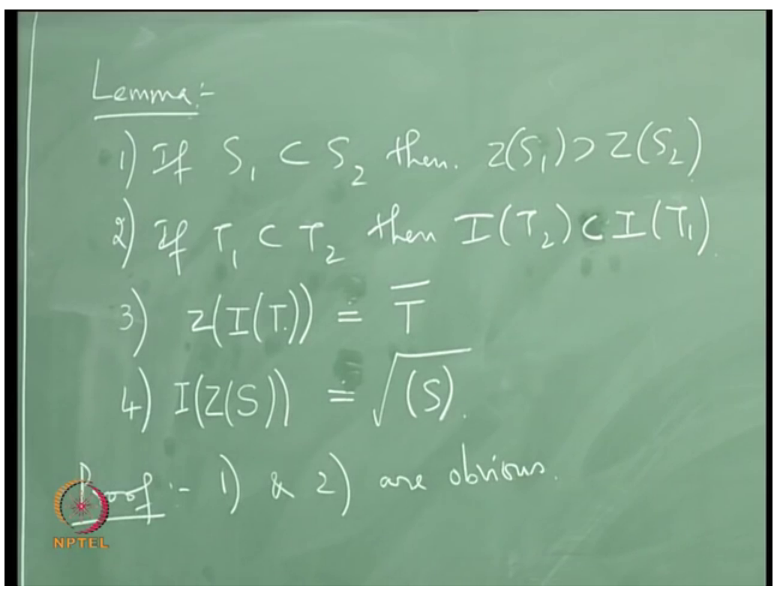

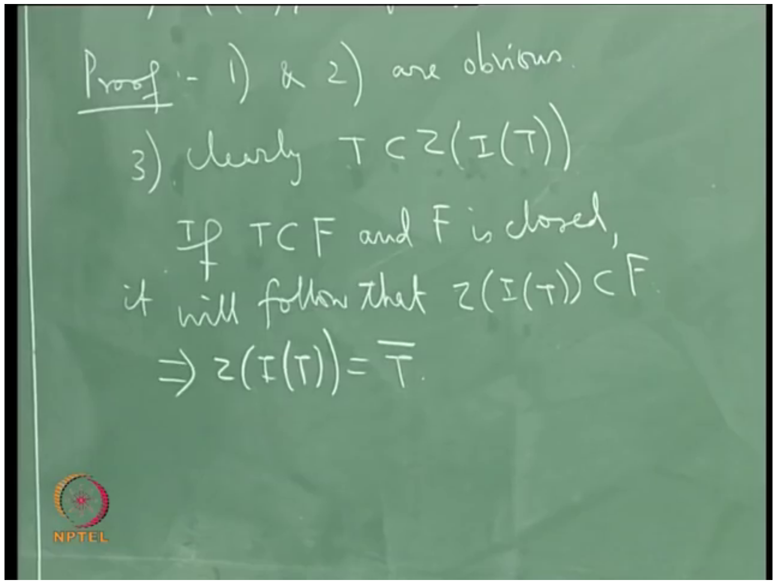

So what it means is that you know if S1 so let me write that down as the lemma it is pretty easy to see if S1 is subset of S2 then well Z of S1 contains Z of S2 second thing is the corresponding result on this side if T1 is contained in T2 then ideal of T2 contains ideal of T1, okay.

So it is if you look at it it is pretty easy to understand you see if you start with S1 inside S2 probably I have I make a mistake somewhere, okay I am probably I am making obvious mistake. So let me talk through this, okay so you see S1 S2 has more equations than S1, okay so a solution a point of the affine space which satisfies every equation S2 will always satisfy every equation S1 so this is correct, right? I think there is not anything wrong there and look at this if T1 contains T2, okay then if you take a function which vanish at every point of T2 then it will vanish at every point of T1 therefore probably this has to be the other way around, okay.

So probably that was a mistake you where pointing out, yeah. So if T1 is contained in T2 then the ideal of T2 is continuing in ideal of T1 because the way I first wrote it it needs to be order the serving which is not correct, okay fine. So this is quite obvious but what is not directly obvious is a following thing, you see what happens if you go and come back, okay so if I start with a T on this side, okay then I take the ideal of that and then I take the 0 of that, what do I get? So this is something that one has to worry about and then the other one is the other way around if I start with well a set a subset here I take the 0 set of that then I take the ideal of that, what do I get? Okay. So this is what we want to we want to investigate and the answer to that is well the answer to this is that you get the closure of T, okay mind you I start with the set T the set T need not be closed but when I take I of T it becomes an ideal and when I take Z of y of T it is a closed set because Z of anything is closed by definition.

Therefore what I am going to get is a closed set which contains T, T is of course going to be here it is very clear any point of capital T will be common 0 of all the functions which vanish around all of T and therefore it is going to be here, so it is clear that this contains T and it is a closed set containing T but the fact is that it is the smallest closed set which contains T and therefore it is T closure, okay and then as for as this is concerned this is literally the involves the commutative algebra version of the Nullstellensatz.

So what it says is this is just the radical of the ideal generated by S, okay and so we so as for the proof I think 1 and 2 are obvious, okay they are quite straight forward, okay the question is with 3 and 4, alright? So if you look at 3 so let us prove 3 clearly T is contained is Z of I of T, right? That is very clear because every point of T is a common 0 of all those functions which vanish at every point of T that is what it says I of T is all those functions which vanish at every point of capital and Z of y of T is all those points at which all these functions vanish, okay.

And therefore T contained in Z of y of T is obvious but more importantly if p is contained in F and F is closed, okay the what will happen is that you will see that it will follow that Z of y of T is contained in F, okay what this means is that Z of y of T because Z of I of T is already closed it means that Z of y of T is smallest closed set which contains T it is a closed set which contains T and Z of y of T is a closed set which contains T and whenever some other close set contains T that closed set also contains Z of y of T.

So this implies that Z of y of T is equal to T bar because by definition the closure of a subset is the smallest closed set which contains that subset and which you can obtain set periodically as intersection of all the closed sets which contain that subset, okay. So I think it is so I have written it will follow that it is something that you can very easily check, okay.

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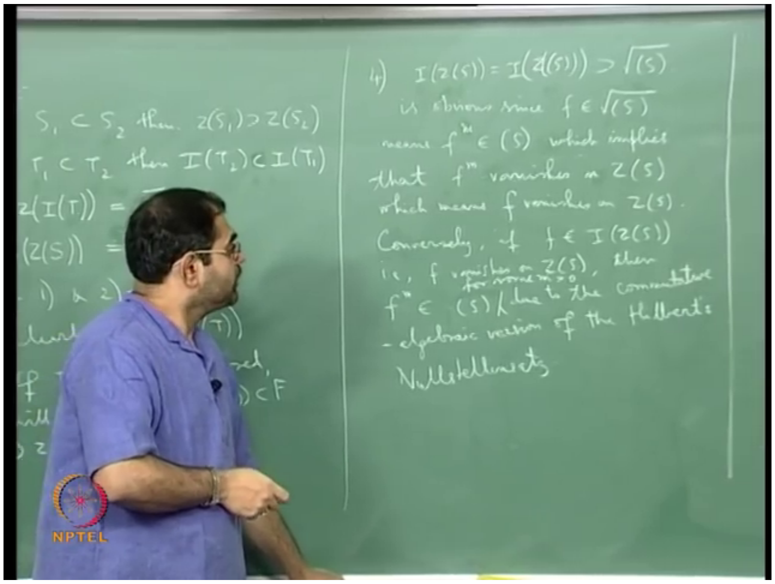

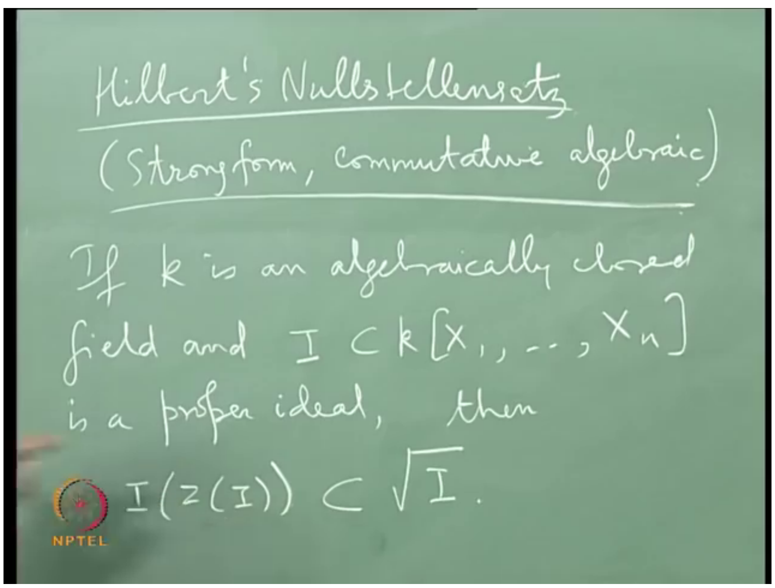

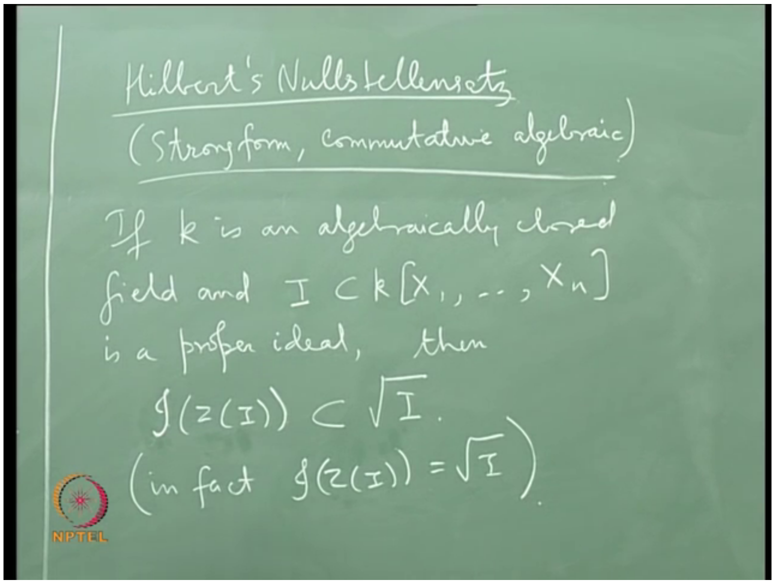

And the as far we proof of 4 well so one way is kind of obvious and is the other way which is not obvious and which involves a Nullstellensatz, okay.

So you can see that I of Z of S is same as I of Z of ideal generated by S because Z of S and Z of ideal generated by S are the same, okay and I think it is very it should be very clear that this contains a radical of this should be very clear because you see you want is the element of the radical of S radical of ideal generated by S, it means it is an element whose power is in the ideal generated by S, okay and that but any element the ideal generated by S will be in this ideal, okay and therefore it will be essentially in element that will vanish at every point of Z of S, okay.

And therefore the element whose you started with will also vanish at the points of Z of S, so this will be obvious, okay. What is more difficult is the other way around namely if we started the function which vanishes at every point of Z of S then some power of the function is actually in the ideal generated by S that is the non-trivial part and that is precisely the commutative algebra version of the Hilbert Nullstellensatz, okay.

So let me write that this is obvious since f belong to root of I mean ideal of radical of ideal generated by S means f power R belongs to f power m belongs to ideal generated by S this implies which implies that f power m belongs to I of Z of S, okay and so in fact I should say more importantly not only this what it means is that f power m belongs to, okay so f power m vanishes on Z of S, okay which means f vanishes on Z of S, okay.

So this means that I mean this means that if we start with an f in the radical of the ideal generated by S it has to be in the ideal of Z of S, okay. So this is kind of obvious, what is not obvious is the following conversely if f is an I of Z of S that is f vanishes on Z of S then f power m belongs to S for so let me add for some m greater than 0 due to the commutative algebraic version of the Hilbert Nullstellensatz, okay.

So what is commutative algebraic equation let me expand on that so here is Hilbert Nullstellensatz so this is the what is called the strong form commutative algebraic, if k is an algebraically closed field and I in k X1 etcetera Xn is a proper ideal, okay then the ideal of Z of I is contained in radii this is the commutative algebraic version, in fact we often say it as I of Z of I equal to radii, okay but that fact that I of Z of I contains radii is something that is obvious, okay but what is not obvious is that I of Z of I is contained in radii, namely if there is a function which vanishes on Z of I then some power of the function is in I because radical of an ideal is just all those elements some power of which some positive integral power of which is in the ideal and you can check that is the bigger ideal in fact, okay.

And what this says is that if a function vanishes on the 0 locus of an ideal, then some power of the functions has to be in the ideal which means that function has to be in the it may not be in the ideal but some power is in the ideal therefore the function is in the radical of the ideal, okay. So you think of the radical as trying to expand the ideal by trying to take all the possible nth roots of elements in the ideal, okay. So this is the Hilbert Nullstellensatz this is the commutative algebraic form, okay oh yeah, so as one of the student rightly points out maybe, okay.

So let me make the correction right from here let me call this let me use a script I, okay for the for going from this direction to this direction so I will change this script I, okay so that thing become far better so this changes to script I and this changes to script I so does this, yeah so it is lot more helpful to have this kind of notation which does not confuse so I will change it to everywhere, yeah of course the statement here that f power m is in the ideal generated by S for some m greater than 0 is another way of saying that f is in the radical of ideal generated by S which is what we want for the other inclusion, okay.

And here let me let me put script I here so that it becomes better to read, okay. And you know you see I gave a so called weak form of the Hilbert Nullstellensatz, okay what was the weak form? The weak form was the assurance that so long as an ideal is not the unit ideal the 0 set defined by the ideal is going to be non-empty, okay and that weak form can be reduced from this strong form as follows, if an ideal is not the unit ideal then that translates to the fact that the radical of the ideal is also a unit ideal, okay an ideal is a you can check that the ideal is a unit ideal namely the whole ring if and only if the radical of the ideal is also a unit ideal and the reason is because if because of the fact that if a power of any element is a unit then that element itself has to be a unit, okay.

So I not being the unit ideal I being the unit ideal is same as radical of I being the unit ideal, okay and then of course you see if EZ of I is empty, okay if Z of I is empty, okay if Z of I is empty then the ideal of the empty set is the whole ring, okay. So if you assume that I is not the unit ideal then we will get a contradiction from this if Z of I is empty so you should read it in fact you should read it with equality, okay because the other inclusion is obvious.

So this strong form of the Hilbert Nullstellensatz does ensure that you know so long as I is not the unit ideal the 0 set is non-empty, okay. So that is how the strong form gives the so called weak form, okay. So what one needs to know is you know since you have two associations going in two directions you would like to make this into you would like to see this as a bijective you know equivalence and it is obvious that you will have to restrict the subsets here in the subsets there and what we going to do next is to go towards that, okay.

So the see the problem is on both sides the arrows are not injected the associations are not injected, for example you know if I take an ideal I and if I take the radical of the ideal I they both go to the same thing here, okay. So if I take I here which is contained in its radical every ideal is of course contained in its radical, okay then both of these things they go to the same thing the 0 sets are the same, okay.

So you have two different things going to the same thing here, okay. So to avoid this what you will expect is that on this side you should replace ideals by radical ideals, okay not just look at all ideals as well not look at all subsets you pass from subsets to ideals and then do not just look at ideals look at radical ideals, okay. Then you see that you can expect that this kind of a thing does not happen, okay so on this side you put radical ideals, right? And on this side you put closed sets then the then it is fact that this is a it is a bijective corresponds.

So radical ideals on this side and closed subsets on this side is the first (())(45:34) bijective corresponds, okay. And the other important thing is you know this statement about inclusions being reversed what it tells you is that as the ideals become bigger the 0 sets become smaller, okay. So in fact when I say ideals become bigger it is with respect to inclusion and you know the biggest ideals with respect to the inclusion on trivial ideals with respect to the maximal ideals.

So what you can imagine is that the biggest ones on this side as a maximal ideals and they will correspond to the smallest sets on this side and what you expect them to be they would be points, okay. So what will happen is that the biggest ideals here the maximal ideals they are correspond to the smallest sets here which are the points and it is again a corollary of the Hilbert Nullstellensatz and all this machinery that we have built up that that the set of points in affine space can be simply identified with the set of maximal ideals in the in this competitive ring, okay.

And the beautiful thing is so therefore you know you are able to see the set of points here this is geometric as again a set of points there but these are not actually points here you have to form another space called the maximal spectrum and the maximal spectrum of ring is a set which contains all the maximal ideals of the ring and the fact is if you take the maximal spectrum of this polynomial ring what you get back is affine space, okay but this is getting it back as a set the truth is it does not stop there the truth is that on this maximal spectrum here there is Zariski topology and if you take that Zariski topology and give that topological structure to the maximal spectrum then that topological space becomes homeomorphic to this so it is not just a bijective correspondence but it is a topological homeomorphism.

So you see in conclusion what is happening is that you are finally managed to completely rub off the geometric side and obtain it completely using commutative algebra. So you see your affine space along with the Zariski topology can be completely forgotten and you can recover it only from the commutative algebra side by doing what by taking the so called maximal spectrum of this commutative ring which means take the set of all it maximal ideals and on that maximal spectrum impose the so called Zariski topology there is a Zariski topology on this side there is something called a Zariski topology on any ring on any commutative ring which you would have come across in course in commutative algebra but any why I will recall it and if you the fact is if you put a topology on this on the maximal spectrum amazingly you get back your affine space.

So the beauty you see the beauty of the dictionary is that I am able to see the affine space on this side without ever going to that side. So you see this is the kind of you know translation that one is able to do and then there are many more things that happen on this side that can be seen here and vice versa, okay. So this discussion will proceed in that direction, okay so I will continue in the next lecture.