Basic Algebraic Geometry
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Module 11
Lecture 29
The Birth of Local Rings in Geometry and in Algebra

Alright, so you know we are trying to discuss the notion of focussing attention at a point which means you are trying to understand geometrically how to study functions at a point and the device for that is commutative algebraically the device is called the local ring.

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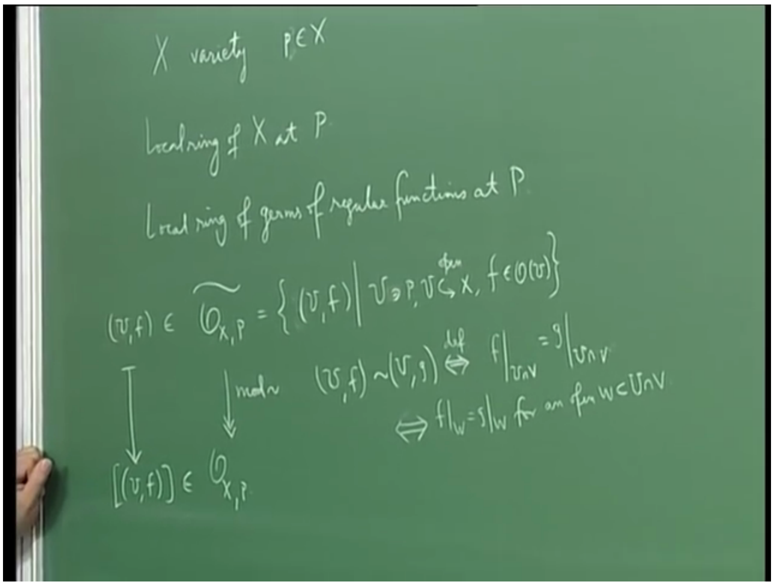

So I told you last time that if X is a variety offcourse which means that X could be either in affine variety or quasi-affine variety or a projective variety or a quasi-projective variety and you have P capital P a point of X. we have the local ring of X at P.

So where infact to expand it, it is actually called local ring of germs of regular functions of X, the local ring of germs of regular functions at P ok. So infact let me write this as local ring of germs of regular functions at P and how was this define? This was define in the following way. We took O x P tilda to be the set of all as of the form U, F such that U contains P, U is an open subset of X and F is regular function on U. So you take the set of all such pairs and then what you do is you go modulo an equivalence relation ok to get a surjection from this into the local ring and what is this equivalence relation?

The equivalence relation is well U F is equivalent to V G if and only if by definition F restricted to U intersection V is equal to G restricted to U intersection ok. So this is an equivalence relation you are just identifying functions on the intersection and offcourse you know for this, this is also equivalent to just assume requiring that F restricted to W is equal to G restricted to W for an open W inside U intersection. Because if a regular function, if two regular functions coincide on an open (sub) non-empty open subset then they offcourse coincide everywhere right.

So I need not require that the point P is in W, so this is local ring and if you start with an element here U, F its image here is written in square bracket and it is called the germ of the function F ok and the germ of the function F whenever a pair U, F is equivalent to a pair V, G then in the local ring they will give the same they will give rise to the same germ ok. The germ of F and the germ of G will be one and the same right.

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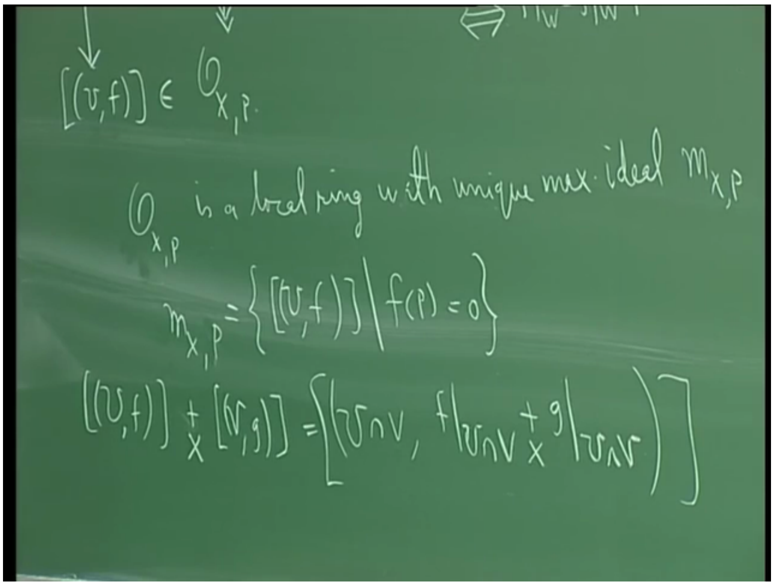

So and I tell you that O x P is local ring with unique maximal ideal M x P and M x P what is the maximal ideal, it is germs of all those functions which vanish at the point P. So M x P is all those germs U, F such that F of P is zero. The germs of those regular functions defined in a neighbourhood of the point P which vanish at P ok and so we saw that how did we show that this is a maximal ideal here, we showed this is maximal ideal here by showing that if there is a germ of which doesn’t vanish at P then it will by continuity not vanish in an open neighbourhood containing P and on that open neighbourhood it will become invertible ok.

So it will become a unit, so in other words a germ of a function that doesn't vanish at P namely an element here which is outside here is a unit. So you have a characterization of local rings in commutative algebra which is a simple lemma you can prove it for yourself if you have a commutative ring with one and if there is an ideal such that everything outside that ideal is a unit then that ideal has to be the maximal ideal, unique maximal ideal and therefore the ring will become a local ring ok so this is what is happening here.

So you have a commutative ring with 1 and you have this, this is obviously an ideal and everything outside it is a unit so this is unique maximal ideal for this local ring and this ring is local ok. Offcourse all this are even K algebras ok, O x P is actually a K algebra. So the point is so focusing attention at a point corresponds geometrically focussing attention at a point which means trying to study regular functions at a point, that commutative algebraically bows down to studying local rings ok. Now the point I want to make is that why is this important?

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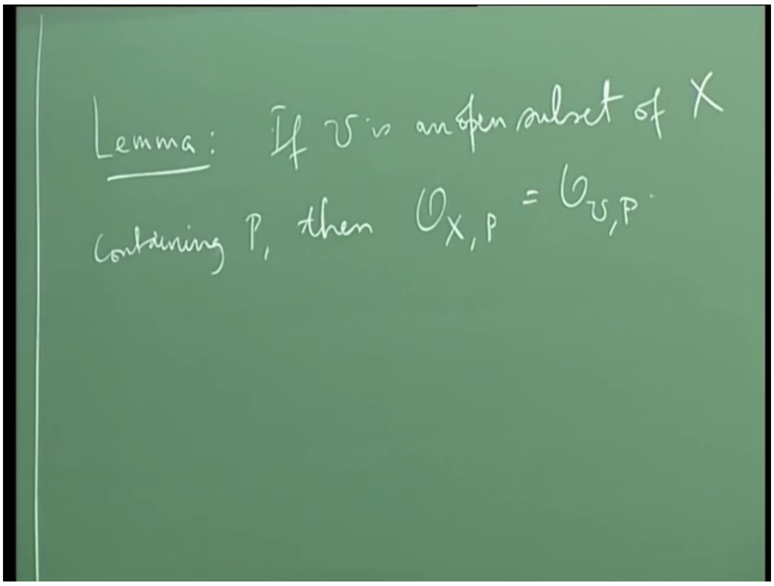

So here is a lemma so the lemma is by the way I forgot to tell you that there is an obvious way in which addition and multiplication is defined here ok. Maybe I will just recall it, you have U,F you have a germ you add it with V,G by simply adding the functions on the intersection. So you just take U intersection V and take F restricted to U intersection V plus G restricted to U intersection ok.

So you take the germ of this pair, I asked you to check that this addition which is defined like this using representatives of equivalence classes is well defined it is pretty easy to see so offcourse instead of addition I could also put multiplication and if it is multiplication I will have to put multiplication here offcourse and the zero element is given by the function zero, the germ of the function zero and unit element the multiplicative identity element is given by the constant function 1 which is also, offcourse constant functions are all regular functions ok.

So 0 and 1 are just given by the germ of the zero function the constant function zero and one is just the germ of the constant function 1, which takes a value 1 ok. So now the lemma is the following if U is an open subset of X containing P then O x P is the same as O U P, so this is the important this is the localness namely the local ring at a point of a variety depends only on neighbourhood of the point ok, it doesn't depend on the ambient variety.

So if you take so if you have a point, if the point P on your variety is lying in an open set ok then offcourse U is an open subset of a variety and therefore U and offcourse U contains a point P, so it is non-empty alright and it is offcourse irreducible ok and the point is that an open subset of a variety is also a variety ok.

An open subset of a variety is also a variety right, (whether your variety), so this is the fact that you can check using simple topology. An open subset of a variety is automatically a variety right. So offcourse if the variety is an affine variety an open subset will become a quasi-affine variety with the varieties of projective variety the open subset will become a quasi-projective variety. If the variety is quasi-affine variety then an open subset of that will continue to be a quasi-affine variety. If it is a quasi-projective variety an open subset will continue to be a quasi-projective variety ok.

So in any case an open subset of a variety is again a variety right, therefore I can think of the local ring of the point P with respect to the variety U ok and there is no difference between the local ring of the variety, local ring at P whether you are considering P as point of U or P as a point of X ok that is the process of the lemma. So well what is the importance of this lemma? The importance of this lemma is the following.

Give me any variety you know you proved last time that any variety can be covered by finitely many open subsets each of which is isomorphic to an affine variety therefore if you give me any variety and give me a point on a variety ok I can chose an affine open subset, I can chose an open subset containing the point which is isomorphic to an affine variety and therefore by this lemma I just have to calculate always the local ring of an affine variety at a point ok

That will give me the local rings of any variety at any point ok and when you calculate the local ring of an affine variety at a point there is an explicit formula for that in commutative algebra, it is simply given by the localization of the affine coordinate ring at the maximal ideal corresponding to that point ok. That is what I show next ok. So this lemma is very useful namely it tells you what the local ring looks like atleast more explicitly in terms of commutative algebra. This definition of local ring here is a very general process you could do it in very general circumstances.

You could take X to be a topological space (and in) and for functions you could just take continuous real valued functions or continuous complex valued functions or you could take X to be manifold say a domain in Euclidean space and you can take functions to be you know so many times differentiable or infinitely differentiable namely see infinity functions or you could even take X to be a domain in the complex N dimensional space and you could take the functions to be holo-morphic functions, holo-morphic in each variable and you can always do this process and this process will always lead you to a local ring ok.

So the process is very general. So the fact that you are getting a local ring is very beautiful, it is something that tells you that (gomet) local rings come out of geometry by focussing attention at a point. But the question is what is this local ring? If you ask the question, what this local ring is? Can you write it down? As some nice ring that you know in terms of rings that you know, then the answer to that comes from this lemma, for example if you choose U to be an affine open subset of X there is an open subset of X which is isomorphic to an affine variety then what I am going to say later on, will tell you, how to write down that local ring ok.

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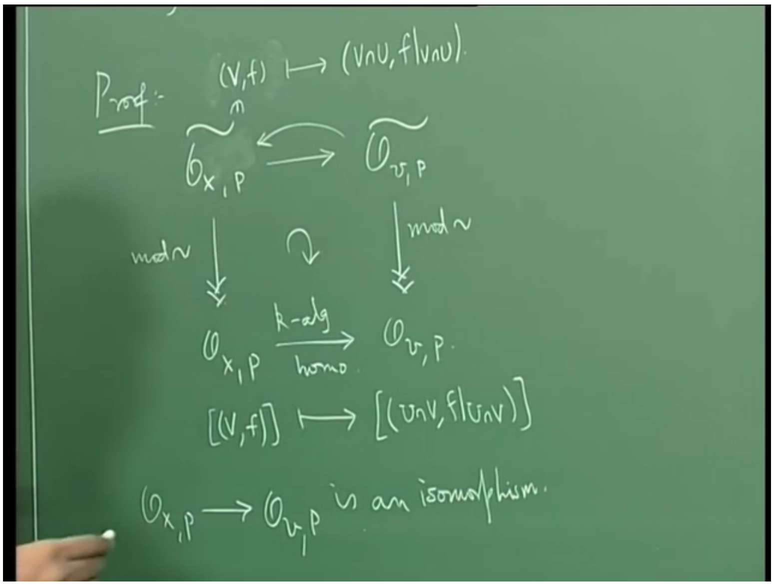

Alright so you see, so lets prove this is very easy to prove, it is pretty easy to prove, so proof is rather simple so you know you have O x P, so you have O x or rather I should say O x P tilda ok and there is O U P tilda ok and the quotient O x p tilda by the equivalence relation as I have defined here is going to give me the local ring of X P and similarly if I take the quotient by this equivalence relation here I am going to get the local ring over U P ok and what I want to say is to say that these two are isomorphic it is very-very simple.

So what I do is I give amp from here to here ok, so namely you give me U, F here you give me an element here so this consist of a pair which has a function F which is regular on the open set U which contains the point P, maybe I should not use the same U let me use something else, let me use V because U is already fixed, let me used V.

So if you have a pair V, F then I have V an open subset of the point P, F is a regular function on V ok. Now after all I can simply send it to I can just restrict it to V intersection U, so I can just V I can just take V intersection U, F restricted to V intersection U I can do this. See restriction of a regular function to an open set is continuous to be a regular function because the notion of a regular function is local ok, that is one thing. Then the second thing is any two open sets any two non-empty sets will always intersect, because of irreducibility ok.

So all our varieties are irreducible. So any two non-empty sets will intersect any non-empty open set will be irreducible it will be dense ok, that is the, that is a coarseness of the Zariski topology right. So there always V intersection and offcourse the intersection will contain the point P because P is already in U, P is also in V.

So V is also in V intersection U and so this is a well-define map like this alright and what you must understand is that this map also, this map will also respect the equivalence namely if two functions on X, if two regular functions a neighbourhood of the point P coincide in a smaller neighbourhood they will also do so when you restrict them to when you restrict these functions to U, the intersection with U ok.

So this, so what this will tell you is that this will induce a map like this so that this diagram completes ok you will get a map like this ok and it is easy to check that this map is actually an isomorphism ok infcat you can check that this map is, it is a natural identification ok. So what you do is, so the fact that you need a map like this (())(19:00) to checking that this map respects the equivalence relation ok. So which means that if two things are equivalent here then the image there are equivalent.

Once you have that this map goes down to a map like this ok and then which means that this map is not just a map of sets but it is also a map which preserves the equivalence relation on these two sets ok. So it goes down to a map of the equivalence classes ok. So you have a map like this and then what I want to say is that this map is you can (remember) you can explicitly check that this map is both injective and surjective because you know see if I have a germ of a function V, F well the map that is going to be here is going to be I am going to send V,F to the V U intersection V,F restricted to U intersection V and germ ok, this is going to be the map here alright and because this map is induce by this map alright.

Now you see now it is very easy, why is this? It is very clear that this is going to preserve both addition and this map is going to preserve your addition, multiplication and so on. So this will be a K algebra homo-morphism. It is going to preserve addition, multiplication it is going to take zero to zero, zero element to zero element ok and it is going to take one to one ok. So this map is if you are almost see that you know the germ of F at the point P where F lives on V is going to be literary the same as a germ of F restricted to U intersection V.

Because after all is the same function you just restricted it and it should define literary the same germ so infact this should be thought of as an actually identification but if you want to formally say it you have to say it is a K algebra a homo-morphism which injective and surjective. So the fact is why is it injective, this is zero of K. if this germ is zero it means that the function F restricted to U intersection V is equal to the zero function on a smaller open subset of U intersection V which contains a point P.

But then by but if two regular functions equal on an open set then they are equal everywhere, so it will tell you that F itself is identically the zero function on V ok. So the moral of the story is that if something goes to zero then this is zero that tells you that this is injective ok and then how will you get surjectivity. If you give me uhh anything here actually the beautiful thing is anything here is also an element here. So actually there is a map in this direction also namely you give me any if you give me any function defined on an open neighbourhood inside U of P that is also an element here ok.

So there is a reverse map ok and that will induce a reverse map here alright. So that is a reason why it is surjective. So the fact is that O x P to O U P is an isomorphism this is an isomorphism. So you know so it is obviously an isomorphism but you know we tend to treat it as an equality we train to treat it as an equality with the understanding that you are literary taking the germ of the same function but restricted to a smaller neighbourhood ok. So that is the proof of this lemma ok.

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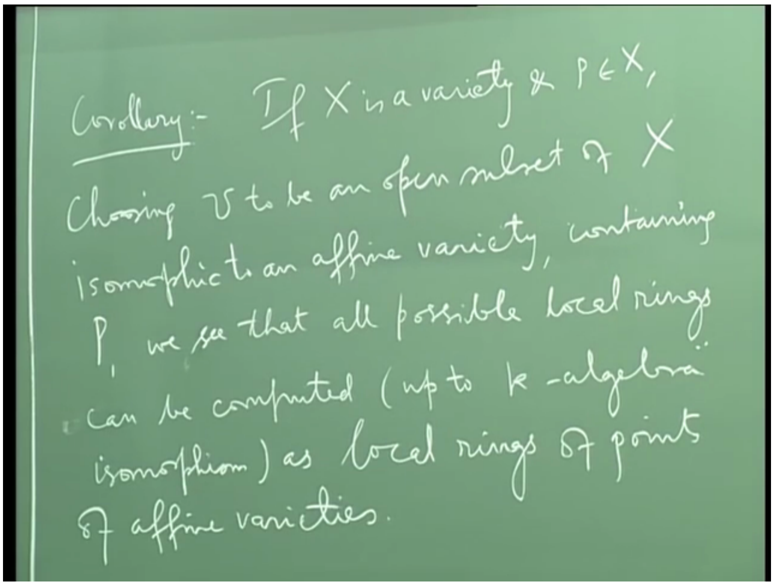

Now so what is the corollary to this, very important corollary to this is if X is a variety and P belongs to X choosing U to be an open subset of X isomorphic to an affine variety containing P we see that all possible local rings are can be computed upto K algebra isomorphism as local rings of points of affine varieties ok. So this uses the important fact that I stated in the previous lecture that any variety affords a finite open cover a cover by finitely open sets each of which is isomorphic to affine variety ok, it uses that fact.

So finally so there is another thing that has gone into in between the lines which I should have said but I will say that now, so I want to say the following thing, I want to say that this local ring this is an invariant if the point on the variety ok. So what I am trying to say is that it doesn't change under isomorphism's.

So you know if you have a variety and you have point then you have it the local ring at that point, if this variety is isomorphic to another affine another variety ok and under this isomorphism this point goes to another point then this isomorphism will automatically induce an isomorphism of the local ring of the source variety at the source point with the local ring of the target variety at the target point ok. So you will get it automatically because the isomorphism will be induced by the pullback of functions ok.

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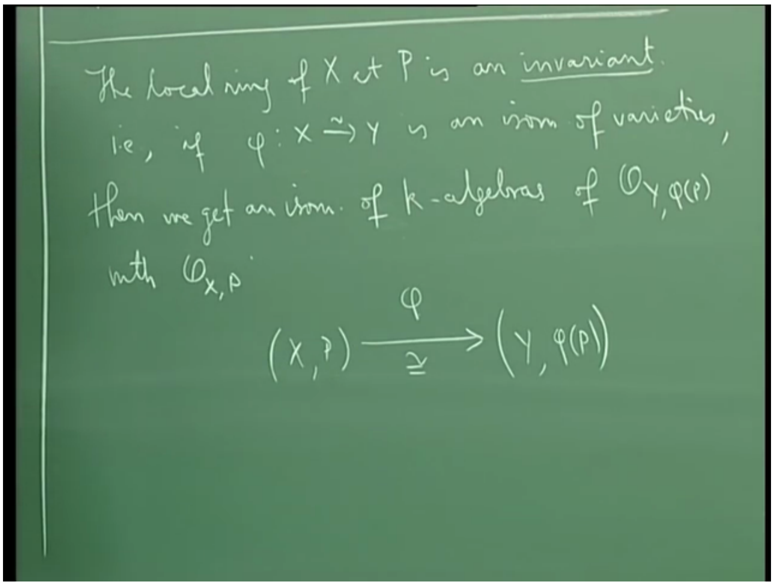

So let me write that maybe I should say that before this, so let me say that the local ring of X at P is an invariant. So what is an invariant? An invariant is something an invariant of certain object, geometric object or an invariant of mathematical object is something that doesn't change or changes only upto isomorphism, if you change the object upto an isomorphism ok.

So what I am trying to say is that you know if you change X and P upto isomorphism then the local ring will change only upto an isomorphism ok. So that is if Phi from X to Y is an isomorphism of varieties then the we get an isomorphism Phi of K algebras of O Y Phi P with O x P ok. So you know the situation is that you are having a uhh you have X, P this is called a pointed variety, that is a variety with a point ok and you have this map Phi it is an isomorphism that carries X, P to Y, Phi P ok.

So when I write a pair like this, that is a variety and a point on it ok and the map is just not suppose to respect the variety I mean it not suppose to just go to this variety to that variety but it is also suppose to take this point to that point which is what it does and the point is that the local ring at of X at P will be isomorphic to the local ring of Y at Phi P because of this isomorphism and the answer is the reason is very-very simple.

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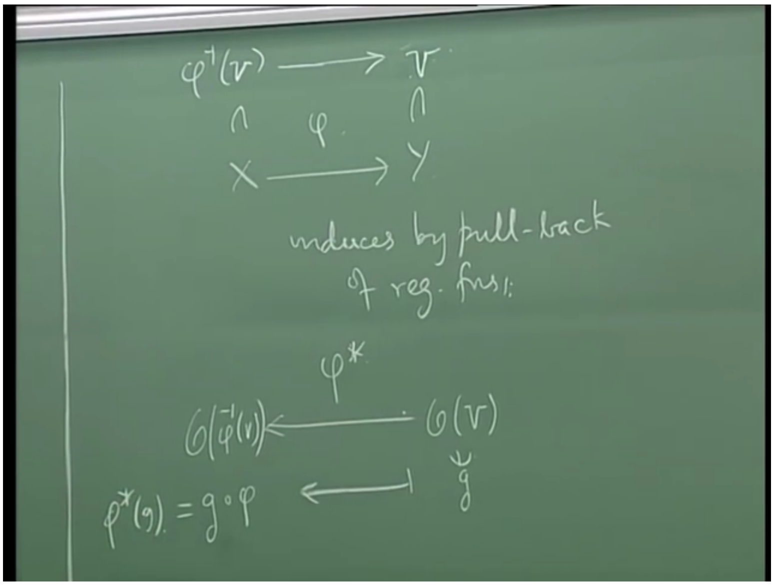

See because you see what is going to happen is you have X you have Y ok and you have Phi, so this Phi will induce by pullback of regular functions. So you know Phi is a morphism of varieties basically Phi is offcourse an isomorphism but Phi is just even a morphism of varieties it supposed to pullback regular functions to regular functions ok. So phi is going to induce a pullback so the pullback is going to be Phi upper star and what it is going to do is that you know it is going to go in this direction. You give me regular function on an open subset Y of V ok then I am going to get a regular function on the open subset Phi inverse of V of X ok.

So if you have a Y a V open subset of Y the Phi inverse V is going to be an open subset of X ok and phi will restrict to phi inverse V and take it into V this Phi restricted to phi inverse V will be a morphism of the variety Phi inverse V into V ok and give me ant regular function on V by composing it with Phi I, I will get a regular function on Phi inverse V that is the pullback map so this map is just if you give me a G here I am going to get the pullback which is just G followed by first apply Phi then apply G this is the pullback ok. This is part of the definition of a morphism, the definition of a morphism is a continuous map which pulls back given any target given any open subset of a target and given a regular function on it.

Namely given a regular function on an open subset of the target then it pulls it back to give you a regular function of the inverse image of that open set on the source ok.

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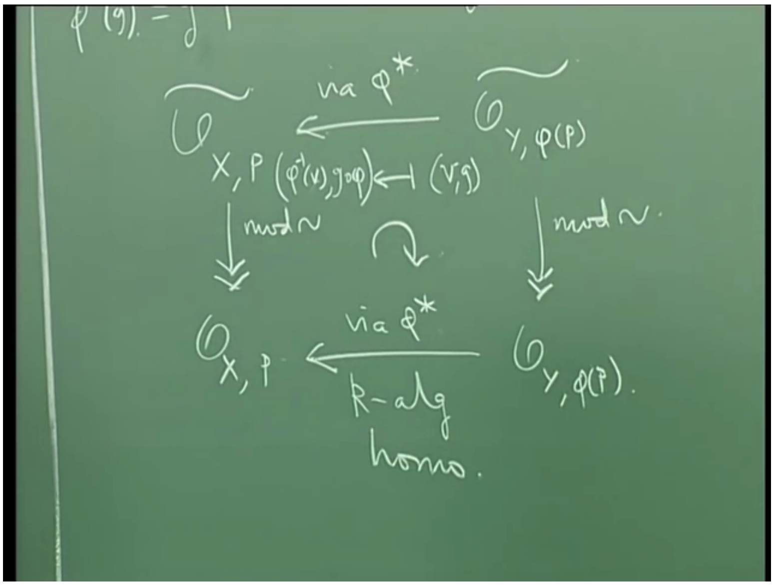

So this is what happens and you can see that once you have this then you know O, you have O Y Phi P tilda whose quotient by the equivalence relation as I defined earlier is going to give you the local ring over Y Phi P and you have a O x P tilda this is whose quotient by the corresponding equivalence relation is going to give the local ring of X at P ok and the fact is that there is map like this, this is induced by Phi upper star ok.

Namely what is does is if you give me a V, G it will send it to Phi inverse V, G circle Phi ok, it is going to send an pair here to a pair here ok and what is going to happen is that this is going to offcourse respect the equivalence relations ok and therefore you are going to get a map like this ok. So this is via Phi star ok and this also via Phi star ok. So what this diagram shows is that the moment you have morphism of varieties then you know and you take a point P if it goes to a point Phi P then you immediately have a K algebra homo-morphism of local rings in the opposite direction ok. So this is the K algebra homo-morphism, this diagram first completes so does this ok.

So you know you must think of this as a infinitesimally version of the following fact. See whenever you had morphism of from one affine to another affine variety it induced a pullback from the regular function on the target to the regular function on the source ok. When you have a morphism from one affine variety to another affine variety. It induced a pullback (function) pullback of regular functions from the regular function on the target to the regular functions on the source.

If you take the infinitesimal version what the infinitesimal version is? Is going to be this, is going to pullback germs of regular functions on the target at a point to germs of regular functions of the at the source at the point of the source which corresponds to the target point ok. So it is very simple you have a regular function on V you compose it with Phi you get a regular function on Phi inverse V. So the germ of a regular function of on V at Phi P is going to after composition with P is going to give you eventually is going to define the germ of a regular function at P itself ok.

So a regular function in a neighbourhood of Phi P is after composition with Phi is going to give a regular function at P ok, regular function in a neighbourhood of P and therefore you are going to get a map on germs, you are going to get this but the point is that if Phi is an isomorphism ok then you also have a map in the other direction and you can check that these maps are inverses of each other ok. So whatever I have written here for Phi I can also write for Phi inverse because I can do that if I know Phi is an isomorphism.

So whatever I wrote here I can write everything for phi inverse so I will get amp like this ok and then you can check that these two maps are inverses of each other ok. These two maps inverses of each other and therefore these two that will give you an isomorphism of this with this ok.

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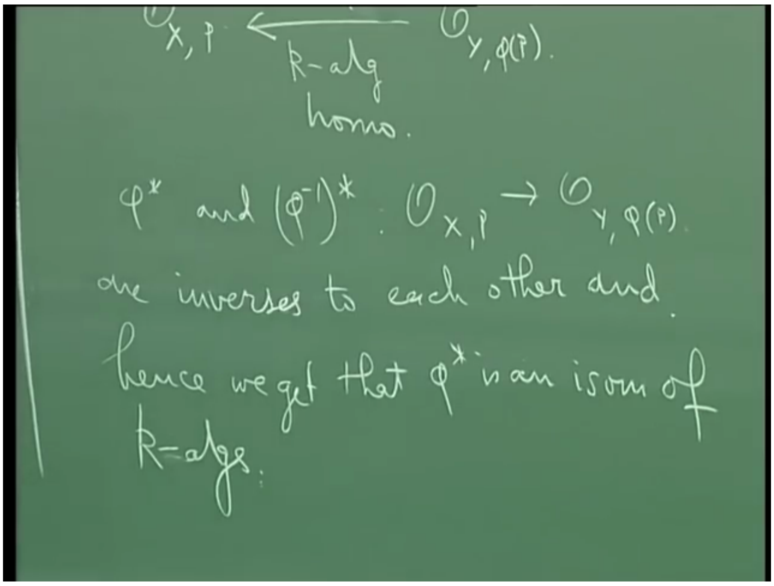

So Phi star so you know this map via Phi's by Phi star I will call this as Phi star this is Phi star going down to the point Phi P. So I will call this via Phi star ok let me not complicate the notation. So phi star and Phi inverse star which is from O x P to O Y so phi inverse is going to be in this direction. So it is going to be from O x P to O Y Phi P ok are inverses to each other and hence we get that Phi star is an isomorphism of K algebras ok.

So this is the fact that I mean this is what this remarks says that you know the local ring is invariant, if you change the variety upto isomorphism the local ring will not change, it will change only upto a K algebra isomorphism ok and offcourse we don't distinguish between we try not to distinguish between our varieties that are isomorphic in the geometric sense and we don't try to distinguish between K algebras which are isomorphic in the commutative algebraic sense ok.

So well alright so, so this is what I am trying to say here , if X is a variety and P is a point of X and you choose U to be an open subset of X which is isomorphic to an affine variety then under that isomorphism the local ring of X at P which is the same as the local ring of U at P according to this lemma will be isomorphic to the local ring of that affine variety to which U is isomorphic at the point which the isomorphism carries P2 ok.

So it so somehow this has to be thought of before one can understand that properly. Now I come to the so the big deal is how do you calculate the, what is the formula commutative algebraically for the local ring of an affine variety at a point ok.

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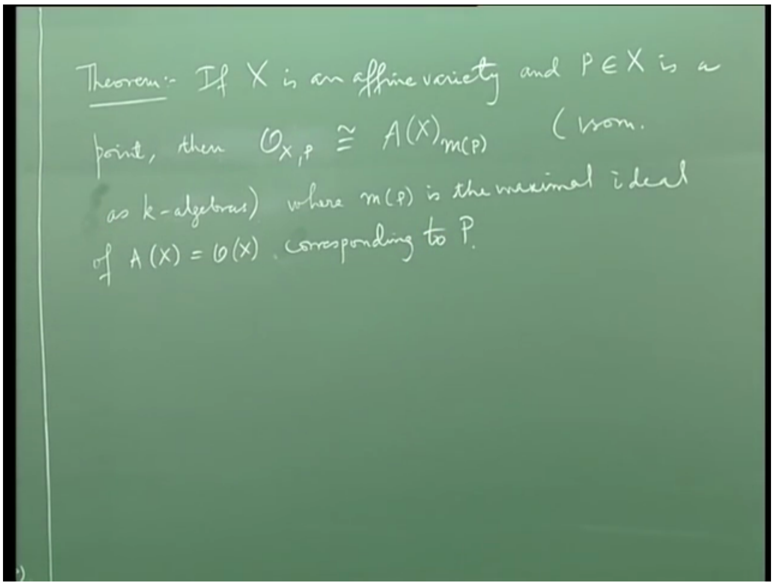

So this is the so here is the theorem (if F) if X is affine variety and P X is a point corresponding to ok so let me make a small then O x P is canonical isomorphic as K algebras to A X localize at M P as K algebras. So this isomorphic as K algebras, where M p is the maximal ideal of A x which is the same as O x corresponding to P ok.

So here is the formula for the local ring and a point of an affine variety, so local ring is just gotten by taking the affine coordinate ring of the affine variety which you know is also the same as the ring of regular functions ok, these two are the same for affine variety and then you localize at the maximal ideal that corresponds to this point ok. So offcourse localizing at an ideal means that you should (to) for this to make sense you have to localize at either certainly at a prime ideal because localizing at a prime ideal or a maximal ideal means you invert everything outside all the elements outside the prime ideal or the maximal ideal.

So this means take the ring A x and invert all the elements which are not in the maximal ideal corresponding to the point P ok. Then the other thing is offcourse so I now I want to you recall that you know according to Nullstellensatz ok the points of X the points of the affine variety X are in one to one correspondence with the maximal ideals in A x ok and this is just because of you know the Nullstellensatz that is more general that can be more generally applied to the affine space in which X is embedded in ok.

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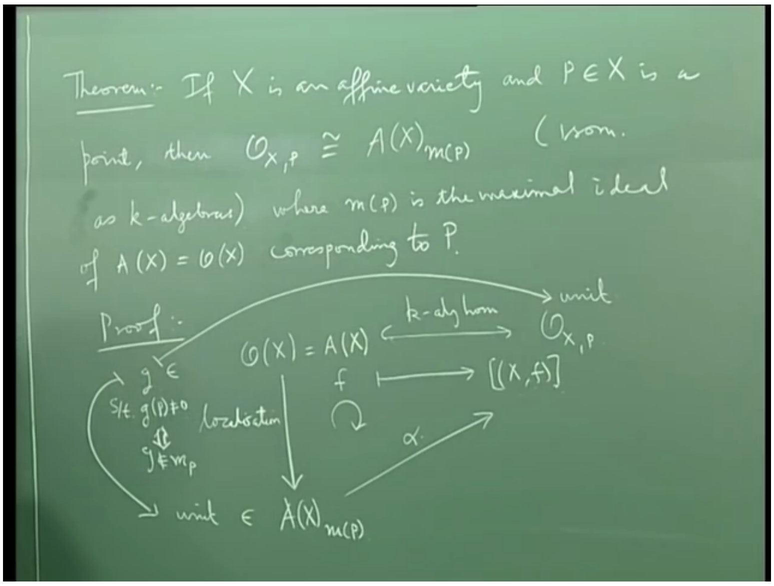

So here is a proof of the theorem which is also pretty easy, the proof is well so you have O x which is A x ok and from O x you have the following map into O x P, so here is my map. You give me any regular function F you just send it to the pair X, F and the germ of that, so simply send every regular function to its germ ok. So F is in A x means that F is a regular function on all of X, A x is same as O x ok. So whenever you write a pair U, Phi, Phi has to be a regular function on U ok. So here F is a regular function on X. So I can it makes sense to write the pair X,F and then I am taking the germ that is why I have put the square bracket ok.

This is offcourse a K algebra homo-morphism this K algebra homo-morphism that is easy to check and well you can also see that it is injective because you see you know if the germ of F at the point P is zero it means that F coincides with this zero function in the neighbourhood of P but then it has to coincide with zero on all of X ok. So if this is zero then this is zero so this is an injective homo-morphism ok then you also have the localization map. So you have the localization map is the localization of this ring A x at the maximal ideal corresponding to the point P ok.

So offcourse you know M P is all those functions in A x which vanish at P that is all ok, it is a maximal ideal right and so you know the fact is that you get a map alpha like this because of the universal property of localization ok. See if you start with G in O x such that G of P is not zero ok. Suppose you start with a regular function global regular function on X G an element of O x which is same as A x and suppose G is not going to vanish at the point P then this is equivalent to saying that G does not belong to M P.

So this is, this implies that you know G does not belong to M P and if G does not belong to M P its image in the localization will become unit because the localization has its property that it inverts all the elements which are outside M p and since G is not and M p is going to be inverted so G under this localization map goes to unit in M p and so you have this fact alright and you should see that G will also go to unit there if I take the image of G in O x here also it will go to unit why, because you see G will go to the germ of X, G ok but the germ of X, G will not belong to M x P which is a, because M x P is suppose to be the germs of all those regular function which vanish at P.

So but G does not vanish at P therefore the germ of G in the local ring corresponds to an invertible element a unit. So you see every element which is outside the maximal ideal M p is going to unit here and therefore this there is a universal property of localization which says that whenever you have a ring homo-morphism from a given ring to another ring with the property that it inverts all the elements that are inverted by the localization process then that ring then that homo-morphism has to factor from the localization, the localizing the localized ring is in some sense the initial object ok. So this is called the universal property of localization.

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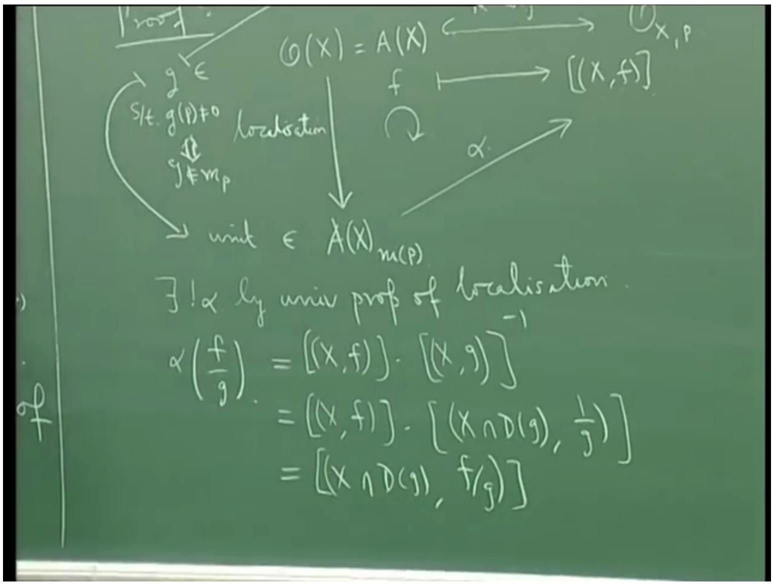

So there exist alpha there exist a unique alpha by universal property of localization and you know it is very easy to write down what that alpha is see alpha see an element here will look like F by G ok an element of the localization will look like F by G where F is a regular function, G is a regular function but G is not in M P that means G does not vanish at P. That is how elements here will look like ok and alpha of F by G is going to be very simple, it is going to be just you just take the image under this, so you get the germ of F times you see the germ of G is invertible.

So it will be this times germ of G inverse ok, this is what it will be, and what is this? This is you know if you think about it this is just you know this is just X F germ of F times germ of G inverse is what? I mean G doesn't vanish at the point P so G will not vanish in a neighbourhood of the point P which means the D G the set of all points where G doesn't vanish inside X. So D G will contain P and it will be an open set which contains P ok and you can write one by G as a regular function on D G so this will be just X intersection D G, one by G this make sense, this is what it is and so and you know so this is how the map is defined.

So infact if you, now if you take the product you have take the product from intersection so it will be just X intersection D G F by G, this is the map. Because you see F by G is certainly a regular function on X intersection D G and it is a germ of this function so this is the map ok. Now you can see very easily that this map is both injective and surjective ok. Why is it injective? Well you know if F by G vanishes ok, if the regular function F by G vanishes in a neighbourhood of the point then you know see, what is going to tell you is that yeah so what will happen is that it will tell you that F by G and zero function they are equal on this open set X intersection D G which is a you know dense open subset of X ok.

So that will tell you that F has to be zero ok. So the moral of the story is that you easily see that this is both this is injective and why it is surjective is because well you start with germ of a regular function at a point P ok then in a neighbourhood of the point P it is going to be it is going to look like F by G and it is going to be the image of under this the same F by G under alpha. So it is trivially surjective just by the definition of a regular function, regularity at a point ok.

What is an element here? An element here is germ of a regular function at the point ok. But what is regular function at a point? Is it suppose to be given by it is regular function is something that has two locally look like a quotient of polynomials. So regular function at this point is in a neighbourhood of that point is going to look like F by G ok where F and G are regular functions F and G are polynomials infact ok polynomials functions on affine space in which X is embedded ok.

So but then F by, but then that F by G will also be here. It makes sense of an as an element here and alpha will carry it onto your germ. So it is surjective it is injective it is surjective so it is an isomorphism. So clearly alpha is an isomorphism. So it is just a moment's thought will tell you that alpha is an isomorphism. So in other words you have this formula ok. So the moral of the story is that whenever you want to calculate whenever you want to go down to commutative algebra and try to focus attention at a point, all you will be doing is always trying to look at this ring which is you know you take you will have some finitely generated K algebra which is (())(50:45) domain.

These are how affine coordinate rings of varieties look like and then you will have to localize it at a maximal ideal ok and so these are the rings that you have to study in commutative algebra if you want to get what happens geometrically at a point on a variety. So the study of the you know geometrically the study of functions in a neighbourhood of a point on a variety that is reduce to the commutative algebra of trying to study localizations of finitely generated K algebra which are interval domains.

Such rings which are affine coordinate rings at maximal ideals ok. So this is what one has to do ok. So this is how local rings enter into geometry ok and but interestingly enough the local rings were produced by geometry ok, they were defined by geometry. Here is how they come commutative algebraically ok atleast in the case of algebraic geometry ok so I will stop here.