Basic Algebraic Geometry
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Module 11
Lecture 28
Doing Calculus without Limits in Geometry

Ok so this is a continuation of the previous lecture so we have just seen that projective space is covered by N plus 1 copies of open sets which are isomorphic to affine N dimensional space. Now what I wanted to say is that from this the first thing that we should notice is that you know if take any projective variety the projective variety is an irreducible close subset of projective space then that projective variety if I intersect with U I, I will get a projective variety and take its image I will get a projective (variety) I will get an affine variety in A n ok and therefore from this you will get that any projective variety is a finite union of affine varieties ok. So that is the, so these are all corollaries of this construction.

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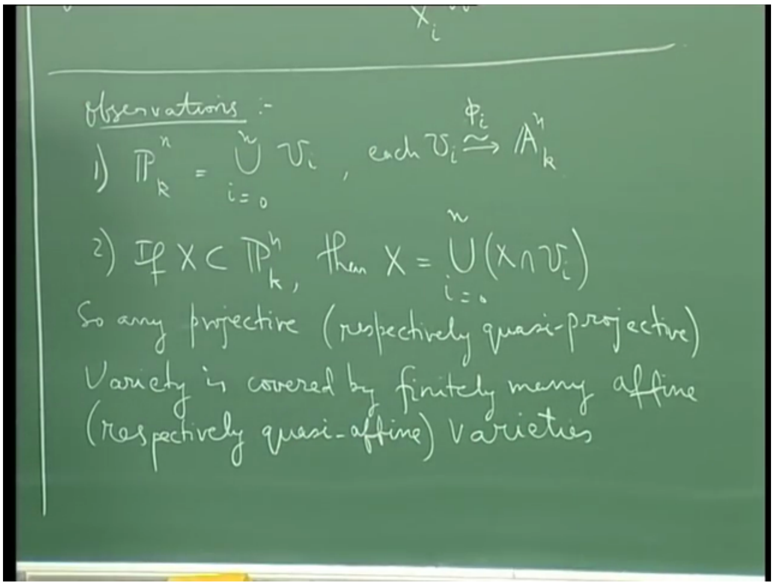

So these are all observations which are very important for us, number 1 the projective space itself is a union of all the U I's I equal to 0 to N each U I isomorphic via Phi I to A n, so you have this. Two, if X is a subset of Pn then X is a union of from I equal to 0 to N X intersection U I, so any (affine) any projective respectively quasi-projective variety is covered by finitely many affine respectively quasi-affine varieties ok and the most important thing is that any variety as therefore is a finite union of open subsets each isomorphic to an affine variety ok.

So for that if you take a Pn or if you take a projective variety it is very clear that the that by the first two remarks that it is certainly a finite union of affine varieties. The only case one has to cover is the case of quasi-projective or quasi-affine varieties ok. So what I want to say is that if you take first of all I want to say that if you take a quasi-affine variety that also can be covered by a finite union of affine varieties ok.

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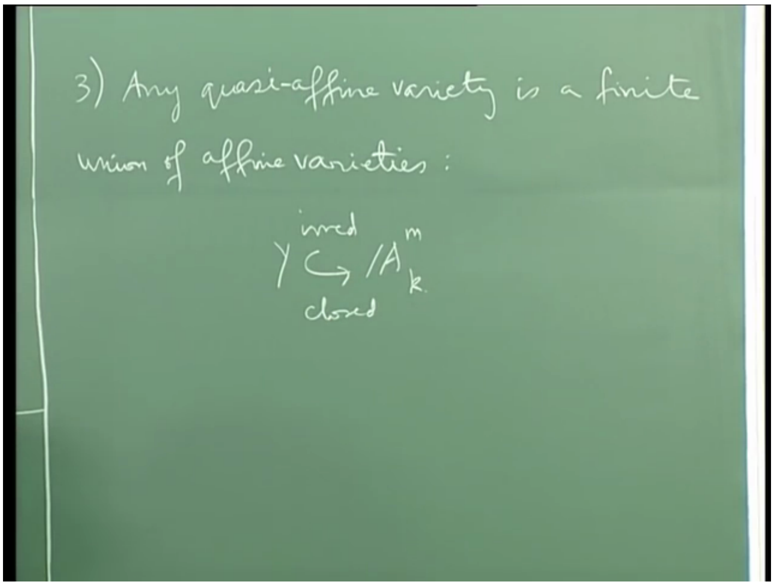

So three, any quasi-affine variety is finite union of affine varieties, why is this so? Because you see you take Y inside some A m this is irreducible closed which means that Y is a closed sub variety of a affine space so it’s a Y is an affine variety.

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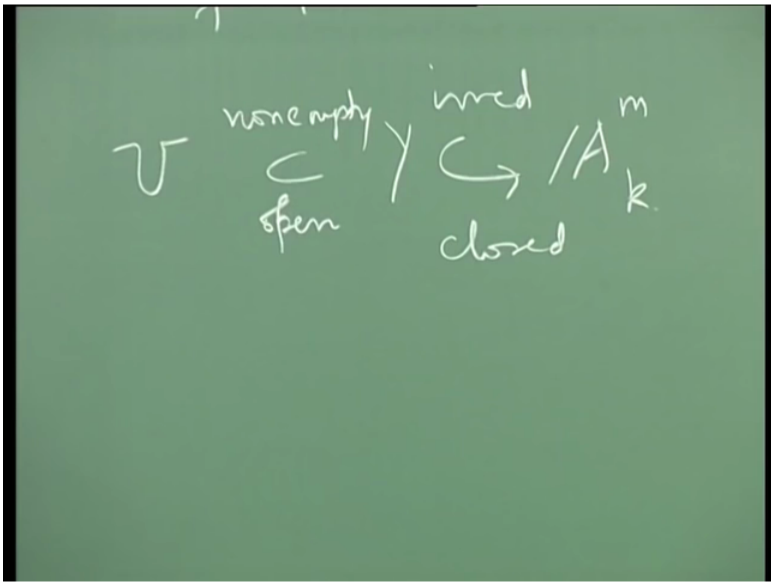

And then in that you take U non-empty open then this U will be a quasi-affine variety by definition of quasi-affine variety is a non-empty subset of affine variety.

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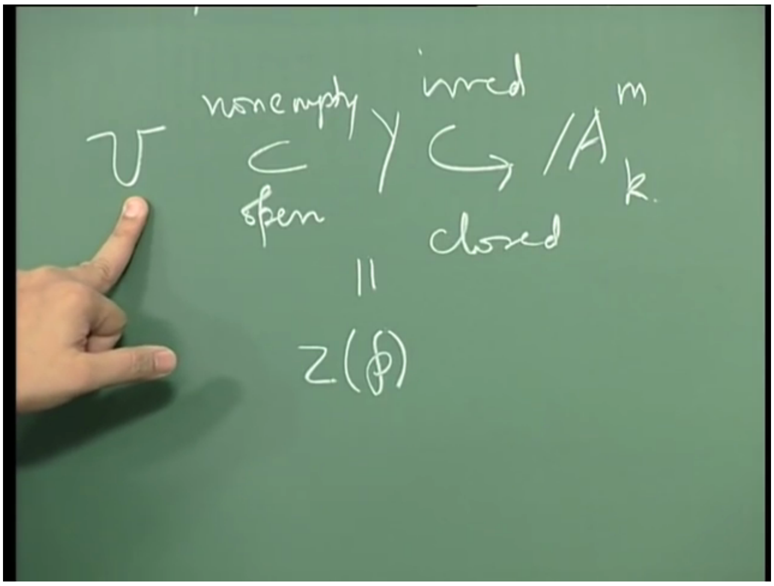

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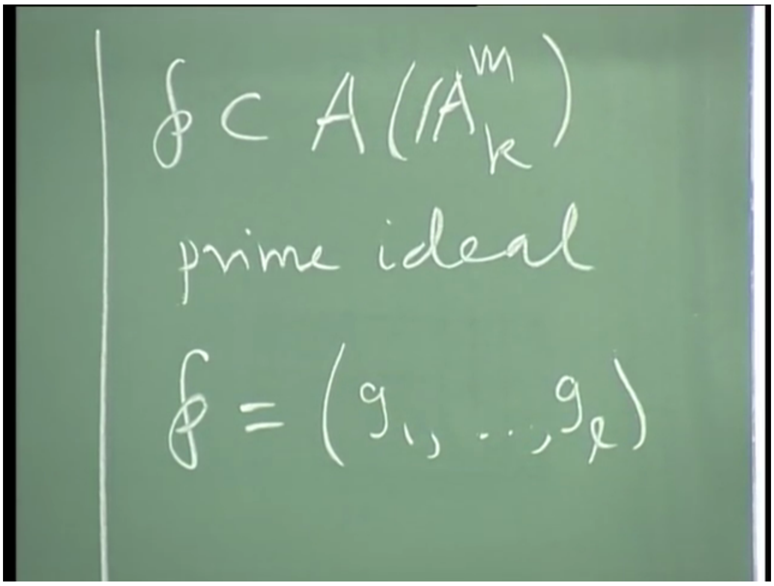

And now Y is E Z of P where P is a prime ideal in the P inside the affine coordinate ring of this affine variety A m prime ideal. So it is a zero set of a prime ideal because it is a variety ok. It is an irreducible closed subset so it correspond to a prime ideal, the zero set of that prime ideal and this offcourse anyway this ideal is in this which is a polynomial ring in N variables and it is by Hilbert's Basis theorem it is noetherian so the prime ideal P is also finitely generated any ideal is finitely generated so you can write if you write P as the ideal generated by G1 through Gl by the Hilbert's Basis theorem then you know that no, this is not what I want sorry.

It is no-no this is not what I want, so I have so U is an open subset here alright, what I want is a following I want to show that this U which is a quasi-affine variety is a finite union of affine variety I want to show that and the way to do that is to realize that the this the fact that (Y) U is a, so the compliment of U inside Y is a closed subset of Y alright and it therefore corresponds to an ideal ok.

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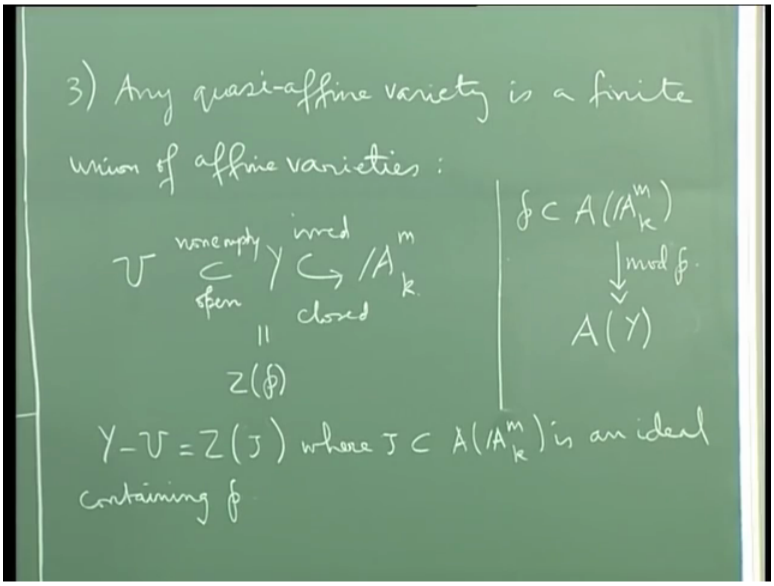

So Y minus U is E Z of J bar, where J in A-A ok A n is an ideal containing P, ok this is what will happen.

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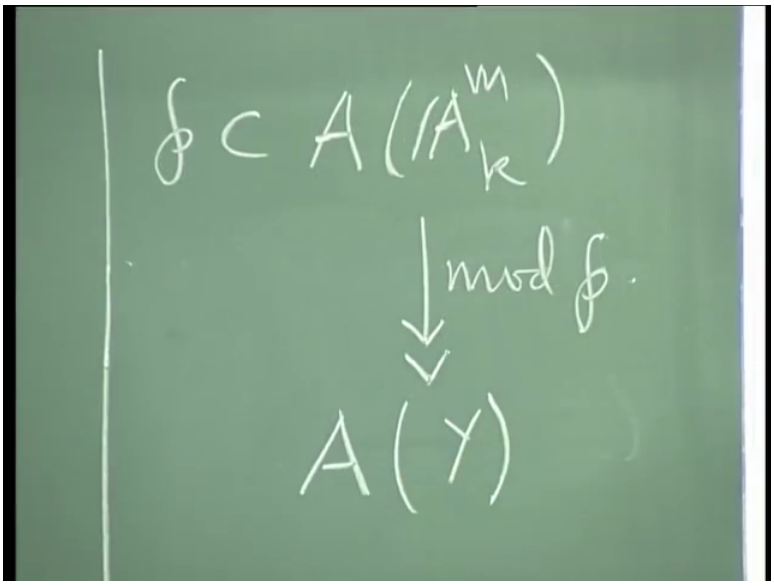

So you see I have this quotient map quotient homo-morphism from this to the affine coordinate ring of affine space to the affine coordinate ring of the fine variety Y and this is just given by Mod P ok and what is a Y minus U? it is a closed subset of Y ok and Y is already a closed subset of A m so it is a closed subset of A m also, a closed subset of a closed subset is also a closed subset, therefore it is given by an ideal in the affine coordinate ring of a big affine space and infact if you want to look at it, it will given by zero set so infact maybe I should not put J bar I should simply put J ok.

Just put J, it will be the zero set of J and if you consider it as a zero set in the full affine space ok and offcourse J will be an ideal which will contain P because Z J is contain inside Z p ok

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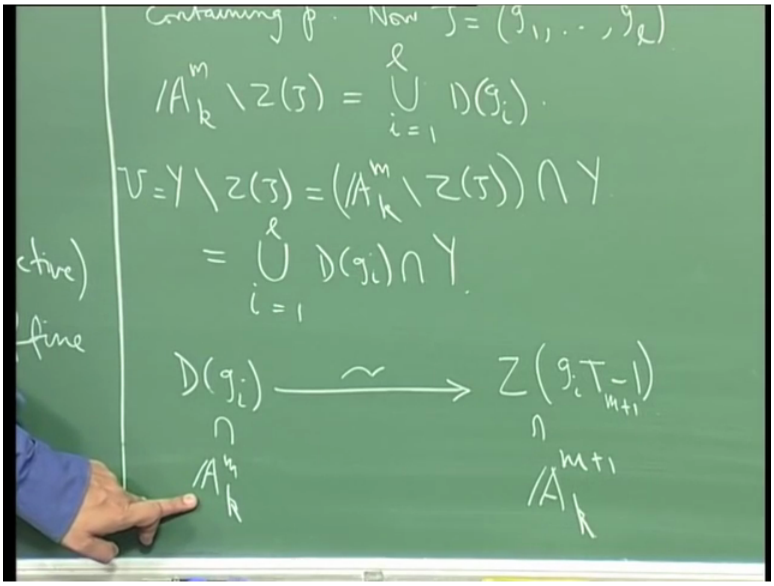

And now J is generated by finitely many elements, so G1 through Gl ok and what will happen is and this is offcourse because any ideal is finitely generated because this is a noetherian ring this is a polynomial ring in M variables that is a noetherian ring by Hilbert's Basis theorem or Emmy Noether theorem. So this is finitely generated, so if you take this as generators then offcourse if you take the compliment of J in the full affine space that will be a union of the basic open sets given by all this Gi's ok.

So you know A m K minus Z J will be union D Gi, I equal to 1 to L ok and therefore if you take Y minus Z J, so Y minus Z J which is U ok this will be A m minus Z J intersected with Y ok and so it will be union I equal to 1 to L D Gi intersection Y alright and therefore but the point I wanted to make is that each D Gi inside A m you know is an affine variety itself though it is an open subset of A m it is isomorphic to an irreducible close subset of affine space of what dimension one more by the Rabinowitsch trick so this is so you know from A m so you have A m plus 1 K and you take the zero set of Gi into some extra variable T minus 1 ok.

So T m plus 1 if you want and you know that there is this there is an isomorphism of varieties like this ok. Each D Gi basic open set in affine space is itself an affine variety ok which means it is isomorphic to an affine variety what is affine variety? It is the affine variety given by g times Gi times the new variable an extra variable minus 1, zero set of that, in an affine variety of dimension one more, we have seen this earlier.

So now what I wanted to understand is that you know Y is a closed subset of A m, therefore Y is also close subset, Y intersection D Gi is closed subset of D Gi ok, Y is a closed subset of A m of the affine space so Y intersection D Gi is a closed subset of D Gi for the induced topology, therefore it will be and Y intersection D Gi is also irreducible that is because of the following fact. D Gi intersection Y is a non-empty subset of Y and Y is irreducible therefore D Gi intersection Y is also irreducible. So the moral of the story is that D Gi intersection Y is an irreducible close subset of D Gi and via this isomorphism it will give you an irreducible close subset of this and therefore it will became affine variety.

So I am just trying to say why each D Gi intersection Y is itself isomorphic to an affine variety ok and the finite union of all such is your U. So what this argument shows is that any quasi-affine variety is a finite union of affine varieties and if you combine that with 1 and 2 you will get that any even a quasi-projective variety will also be a finite union of affine varieties. So if you put everything together you will get that any varieties is a finite union of affine varieties ok, any variety has a finite cover by open subsets each of which is isomorphic to an affine variety and thus we say that affine varieties are the building blocks of varieties .ok

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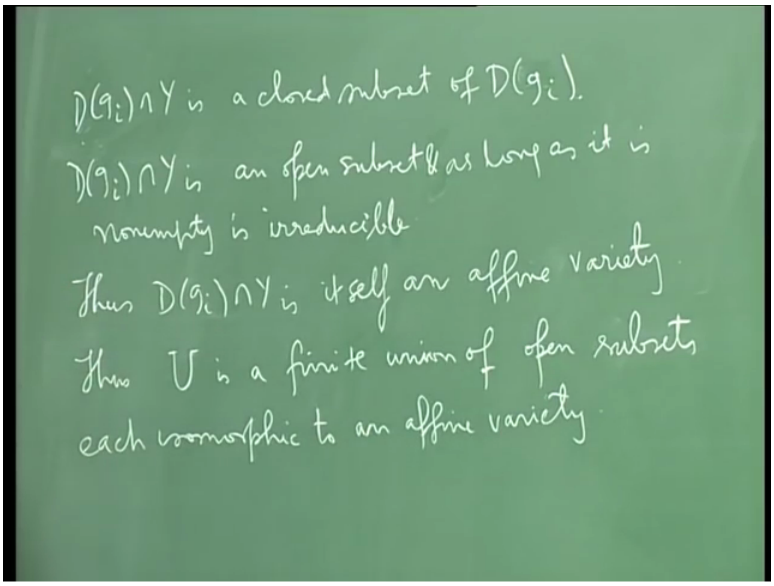

So let me write that down here, that is very important. So thus D Gi intersection Y is a closed subset of D Gi, D Gi intersection Y is an open subset as long as it is non-empty and it is an open subset and as long as it is non-empty is irreducible. Thus D Gi intersection Y is itself an affine variety ok because it is irreducible close subset of affine variety or something that is isomorphic to affine variety. Thus U is a finite union of open (affine) finite union of open subsets each isomorphic to an affine variety. So this is point number, this is observation number 3, that any quasi-affine variety is a finite union of affine varieties.

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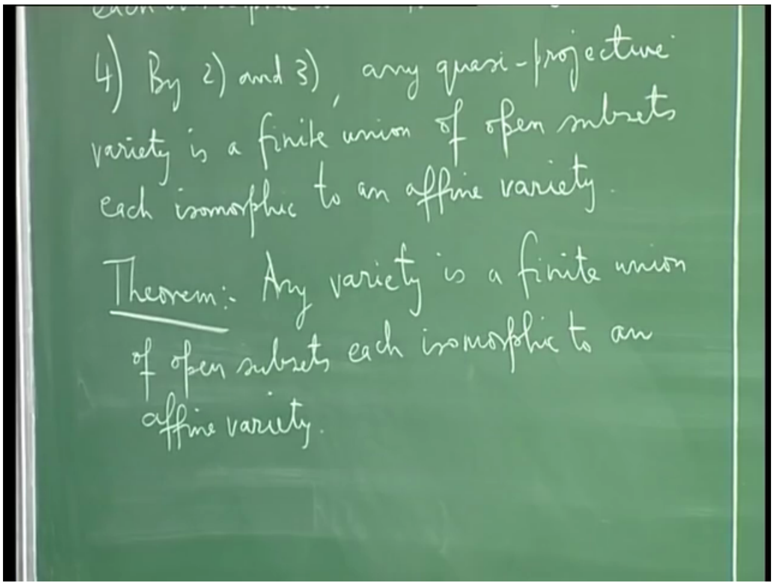

And then 4th by 2 and 3 any quasi-projective variety is a finite union of open subsets. Each isomorphic to an affine variety and the raptured of all this is the theorem that any variety whether it is projective or quasi-projective or affine or quasi-affine is a finite union of open subsets each isomorphic to an affine variety. So sometime we say this also in following way, we say that any variety admits a finite cover by open affine varieties ok that is one way of saying it, the other way of saying it is that affine varieties are the building blocks of all varieties.

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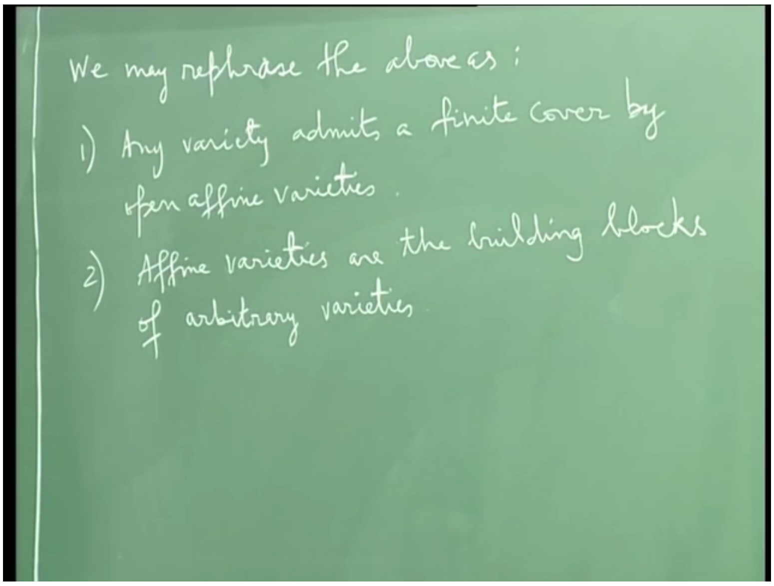

So let me write that down , we may rephrase the above as number 1 any variety admits a finite cover by open affine varieties, the other way of saying it is that, affine varieties are the building blocks of arbitrary varieties ok.

So I mean the importance of this is you will see the importance of this in the following sense so the first immediate thing is if you are going to study if you are going to focus attention in a neighborhood of a point of a variety then you could as well look at an affine neighborhood of the point and deduce your question to studying a point on affine variety that is an advantage ok. So let me repeat that if you have a variety and you want to study in a neighborhood of the point the variety then what you do is you choose a neighborhood which is an open set which itself is an affine variety ok.

The variety you started with may not be affine, it could be projective, could be quasi-projective ok it could even be quasi-affine but then if you wanted to restrict attention to a point you can find an open neighborhood which is itself isomorphic to an affine variety and then thereafter it reduces to studying a point on an affine variety ok. So the advantage is that if you are trying to restrict attention to a point on a neighborhood of a point on a variety is just enough to study points on affine varieties ok, that is one advantage.

The other advantage is rather philosophical, it is that when you it is it concerns the more general definition of what is called as Kim which you will see in a second cosine algebraic geometry. So the roughly Kim is something that locally looks that is locally isomorphic to something which is affine ok. So the affine’s they build up what is called as Kim alright. So you can so roughly speaking here you know if take an affine variety you have its coordinate ring ok and then from the coordinate ring, how can you get back the affine variety by applying max-spec ok.

So you can say any variety is got by topologically atleast as topological space it is got by gluing max-specs ok alright. If you take a variety which is cover by finitely many affine varieties ok, then it is the same as saying that you are gluing all these affine varieties together to get that big variety ok. But then each affine variety is a max-spec of its coordinate ring. So you can say that this big variety is gotten by gluing the (maximum) maximal spectra of these various affine coordinate rings ok.

And more generally the more general thing is what is called as Kim and that is an object that you get by gluing just not maximal spectra but just prime spectra of rings ok. So this is helpful later when you take a second cosine algebraic geometry but I am saying that the philosophy is already here, that affine's are the building blocks ok alright. So that is a very important thing. Now you know there is one fact that I need to prove, I need to tell you that the only regular function on a projective variety are the constants ok.

But for that I need to uhh understand what is happening at a point ok in a neighborhood of a point and I also need to keep track of information of what is happening on an open set ok. So I have to study regular functions in the neighborhood of the point and also regular functions on open sets and the devices that handle these things are respectively called the local ring at a point and the function field or field of narrow-morphic or rational functions ok, which is what I am going to deduce next ok.

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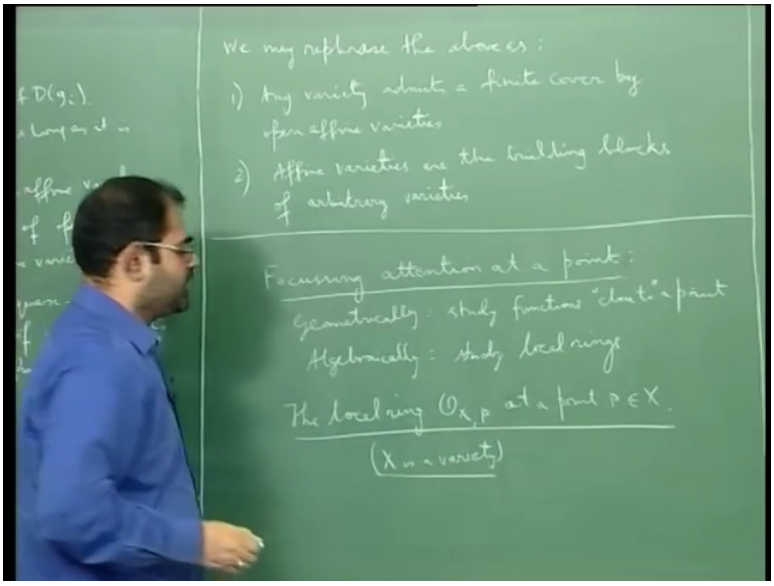

So what I am going to do next is focusing attention at a point, this focusing attention at a point, so well geometrically study functions closed to a point and algebraically commutative algebraically this is the same as studying local rings ok. So you know one important distinction between algebra and analysis is that, in analysis you have the limit concept ok so you have a limit and therefore it helps you for example to take small neighborhood about a point and you can take smaller and smaller neighborhoods and using that you can define various things existence of a limit and then you can define continuity, differentiability and you can do analysis.

But in algebra handicapped is that you don't have any such notion of limit a priory so it appears that you cannot do for example study smaller and smaller neighborhoods of a point. So it appears like that because to study smaller and smaller neighborhoods of a point I need small and small neighborhoods first but then the neighborhoods if they are open neighborhoods there is Zariski neighborhoods and Zariski open sets are you know irreducible and dense so the huge sets ok.

So you cannot think of epsilon neighborhoods as you would think of in usual analysis. So it appears that it is not possible to do that kind of analytic study at a point but the answer is wrong that is not true, the truth is that you can even though you don't have small enough neighborhoods, the information is still available ok and that is that comes by studying so called local rings ok.

So if you recall a local ring in commutative algebra is ring which has a unique maximal ideal and so in other words if you take its maximal spectrum it is just a single term ok and every element outside that maximal ideal will be a unit ok and you know conversely any ring with an ideal with the property that very element outside that ideal is a unit has to be a local ring with that ideal as unique maximal ideal ok.

So these local rings are what we need to study infinite assemble neighborhoods around a point ok and we do it even though we don't have small neighborhoods Zariski neighborhoods even though we don't have the notion of limit ok. So let me explain so what will do is, so the idea is that in algebraic geometry whenever you want to study something closed to a point you have to always think of think and terms of local rings ok. So what I am going to do now is I am going to define the local ring at a point of a variety ok.

So the local ring O x p is the notation of a point at a point so maybe I will use ok at a point P ok. So here offcourse X is a variety. So we going to define this and how does one define it? Defined in a it is defined in way in which extends to defining local ring of functions of any type you can use this to define local rings of continuous functions local rings of continuous differentiable functions, local ring of if you want holo-morphic functions and so on and so forth. So the philosophy is very-very general.

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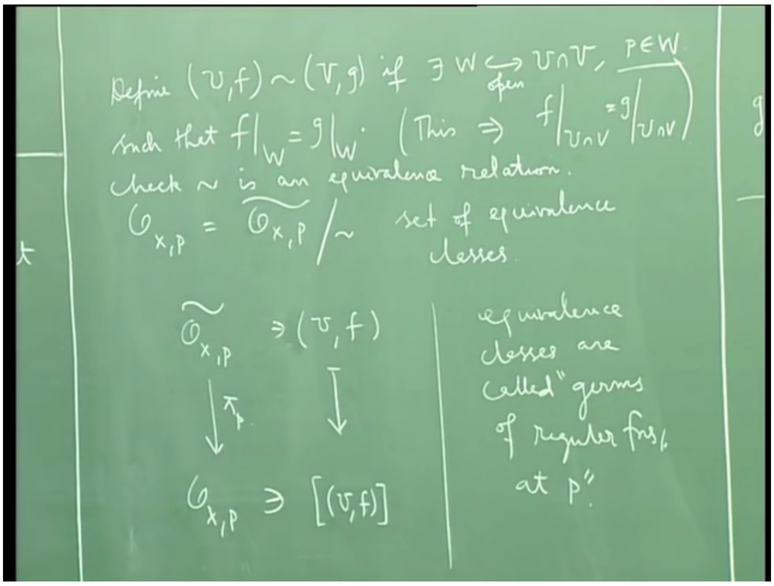

So what you do is that so you do the following thing define O x p tilda to be the set of all pairs U, F such that U inside X is open X is a point of U and F is a regular function on U ok.

So you look at pairs like this ok, so what I am doing is I am just looking at regular functions define on open neighborhoods containing oh yeah ok, my point was not small x it was capital P, so I will change it thank you for pointing that out. I have taken my point as capital P right. So you are just looking at pairs consisting of regular functions defined on an open neighborhood of the point P alright and what will do with this is that will introduce an equivalence relation ok. So define U f equivalent to V g if there exist W open in U intersection V such that F restricted to W is equal to G restricted to W ok.

So you put this condition, you define two such pairs to be equivalent if they define the same function in smaller neighborhood of P ok. Offcourse you know this implies that this is true this implies that F and F restricted to U intersection V itself will be equal to G restricted to U intersection ok, this will happen because you know F and G will now be regular (functions) F restricted to U intersection V and G restricted to U intersection V will be two regular functions on U intersection V which is a variety ok and if they are equal on non-empty open set then they have to be globally equal.

I only have to ensure that U intersection V itself is irreducible ok and in think that is alright because U intersection V is an open subset of U and U is irreducible and therefore U intersection V is also irreducible ok.

So these are two regular functions on U intersection V which coincide on an open subset of U intersection V therefore they coincide everywhere. So well though the equivalence only requires F and G to coincide on a small neighborhood containing offcourse I want P to be point of W that is very important ok. Probably I don't even need that I mean if, it shouldn't create any problems yeah probably I don't need this but anyway I will add it ok. Yeah so this is an equivalence relation alright and now you define O x p to be so check this is an equivalence relation that is very trivial because obviously it is, it is obviously reflexive symmetric and for transitivity as far as transitivity that is also quite obvious ok.

So this certainly an equivalence relation, maybe I have to think from moment about if yeah so the point that anyone thinks with respect to the usual topology one can be misled one should remember that in this Zariski topology any two non-empty sets will intersect that is something that one should not forget. So you cannot have imagine it picture where you have three successive open disks capacitance second intersecting the second and third intersecting but the first and third not intersecting that kind of a picture will never happen.

Any two open sets will intersect ant two non-empty open sets will intersect in Zariski topology ok. So that is something that you have to remember. So we take O x p to be the set of equivalences classes which is just O x p tilda modulo is equivalence set of equivalences classes ok and you have this, so you know you have this natural quotient map O x p tilda to O x p and this is Pie P if you want ok and this is this sense an element U, F here to its equivalence class which I will put which I signify by putting a square bracket, this is the map ok.

Now what I want to tell you is that, I want to tell you that the two things I want to tell you, first thing is that there is a name for these equivalence classes they are called germs of regular functions ok. So equivalence classes are called germs of regular functions at P ok. So a germ of a regular function is represented by on some neighborhood of the point is represented by a regular function ok and it is also represented by another regular function on another neighborhood of the point if it is also represented by another regular function on another neighborhood then these two functions coincide on the intersection ok.

So you know in some sense you must think of these equivalence class as defining a particular class of regular functions at that point. So this makes sense as thinking of a function class of functions at a point right and two distinct classes means that they are they corresponds to regular functions at that point which will not coincide on any neighborhood of the point ok because if two (regu) if you have two regular functions locally define at the point and if they coincide in some neighborhood of the point then they will give rise to the same germ.

So if you take two distinct germs they will corresponds to regular functions which do not coincide in any neighborhood of that point that is what it means ok. So the moral of the story is that if you want to think of regular functions at a point and you want to distinguish them this is the way to do it ok, so that is the first point.

The second point is that this is the local ring ok that is the point and more importantly you can do this whole construction not only in algebraic geometry you can do this in with respect to any thing. So for example you could have started I mean you could be working on a topological space and you could just take you could fix a point P on the topological space and you could simply construct pairs where you have the neighborhood of the point and you have a function which is continuous real valued or complex valued function ok and then you could add more conditions like you could make that function not just continuous you can make it differentiable.

If you want you can make it, if it is complex valued you can make it holo-morphic or analytic and you can put the same definition ok and you will see that again you will always land in a local ring. So this definition, this whole process of identifying equivalence classes of functions which agree in a neighborhood of a point automatically produces a local ring.

So in a nice way you know this is how a local ring is completely an algebraic concept ok. It comes out completely out of geometry ok. If you restrict the, if you want to you know restrict the study of functions good functions at a point you automatically end up with a local ring ok.

So if I take my space to be an interval on the real line close interval on the real line and if I take my functions to be real valued contained functions then I will get a local ring which corresponds to germs of real valued continuous functions ok. Or I could have taken the space to be the plane or some N dimensional ingredients space and I could have taken functions F which are defined on neighborhood of a point and which are C infinity which means which infinitely differentiable ok and then I will get the local ring of germs of differentiable functions C infinity functions and that is what you get when you look at a manifold ok.

On the other hand I could also taken X to be uhh domain in the complex plane or domain in C n, N dimensional complex space ok and I could have taken F to be I could take the functions to be holo-morphic functions holo-morphic in several variables ok which means which is equivalent to saying holo-morphic in each variables separately ok. Then I will end with again a local ring namely I will get the local ring of followed germs of holo-morphic functions at that point. So this method of producing local rings is completely geometric ok.

So this algebraic concept of local ring comes out through geometry in this way by restricting by trying to look at functions at a point ok. Now the fact I want to say is that this is actually a ring, why this is a ring, because you can define addition and multiplication infact it is not just a ring in our case it is actually a K algebra it is a K algebra and.

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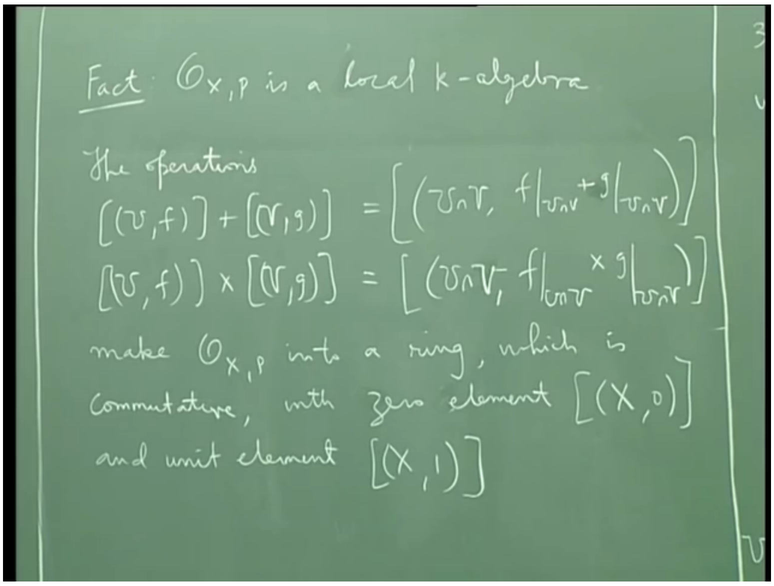

So let me write out fact O x p is a local K algebra that is the fact. So first of all how do you make it into a ring? How do you define addition multiplication and so on?

So you define so the operations, so you know if you give me U, F and if I want to add it to V, G this is germ of U, F, this is germ of V, G ok. Mind you U and V are neighborhoods of the point P therefore U intersection V makes sense, that is also neighborhood of the point P. So I will simply define this to be U intersection V and F restricted to U intersection V which is the same as plus G restricted to U intersection. So this is how I define addition.

Then I can define multiplication in the same way, I simply restrict and multiply on the intersection ok. So you can check that since I am defining these operations on equivalence classes you can check that they are well defined ok. So this is my addition this is my multiplication and offcourse the make this into a ring which is commutative, it is offcourse a commutative ring because the way the product of functions is commutative ok. The function define by F times G is the same as function define the G times F ok, because functions are defined product functions and some functions are defined point wise and therefore everything is are point wise being done in the base field and the base field is offcourse commutative and so this makes it into a ring which is a commutative ring with unit element with zero element.

The zero element is very simple it is just the class of X, 0 so you will have the constant functions are always there. So you take the function zero on the whole of X, that is a regular function and you take its germ, that is the zero element and unit element is given by X,1 you take the constant function 1, constant functions are offcourse regular functions always, so that is the reason why the constants K the field base field seen as constant functions in the ring of regular functions on any open set. So every if take U any open set then O U is offcourse a K algebra, because K sits inside O U as a constant functions ok.

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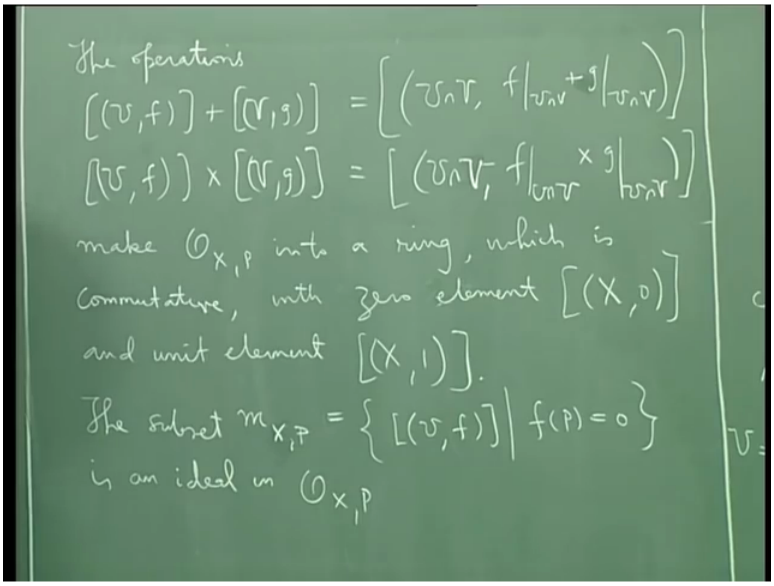

So this makes it into a commutative ring with this a zero element and this as unit element and further it is a local ring why because the subset M p, M of P so let me write M x p to be the set of all germs U, F such that F vanishes at P ok. So you whenever you take a germ it is defined is represented by a regular function on neighborhood and you take all those take the subset which corresponds to function vanish at that point ok.

Then this is a maximal (ideal) this is an ideal in a O x p that is very clear because if a function vanish at a point then a function multiplied by any other function will also vanish at that point and sum of two functions vanishing at a point will also vanish at a point. So it is an ideal alright and every element which is outside that you take a function you take a germ of germ which corresponds to a function that does not vanishes at a point it will have an inverse. Because the moment F does not vanish at P then by continuity F will not vanish in a neighborhood of P ok.

On that neighborhood 1 by F will be a regular function and F into 1 by F will be 1 on that neighborhood. Therefore every element outside this ideal is going to be unit and that will tell you that this has to be the unique maximal ideal for that local ring and that ring will be local with this unique maximal ideal.

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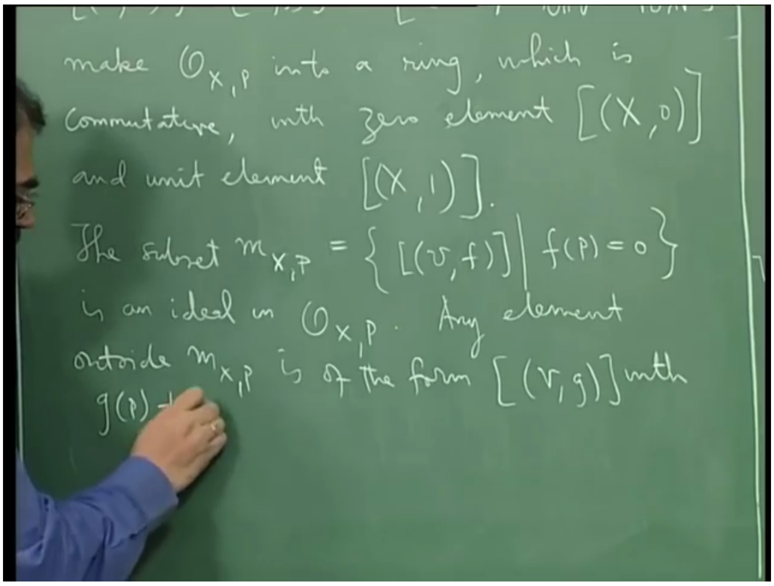

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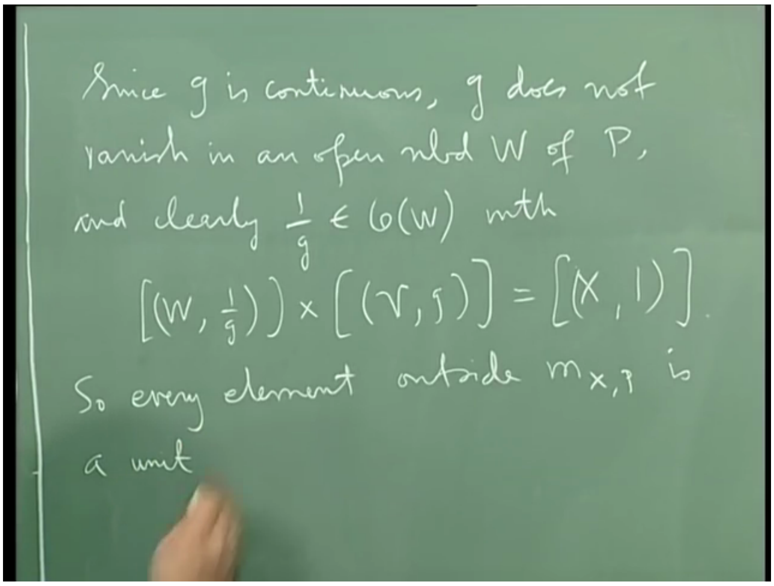

So let me write that, any element outside M X P, P is of the form V, G with G of P not equal to 0, since G is continuous G does not vanish in an open neighborhood W of P and clearly 1 by G belongs to O W with W 1 by G germ of this into V, G is equal to X, Y.

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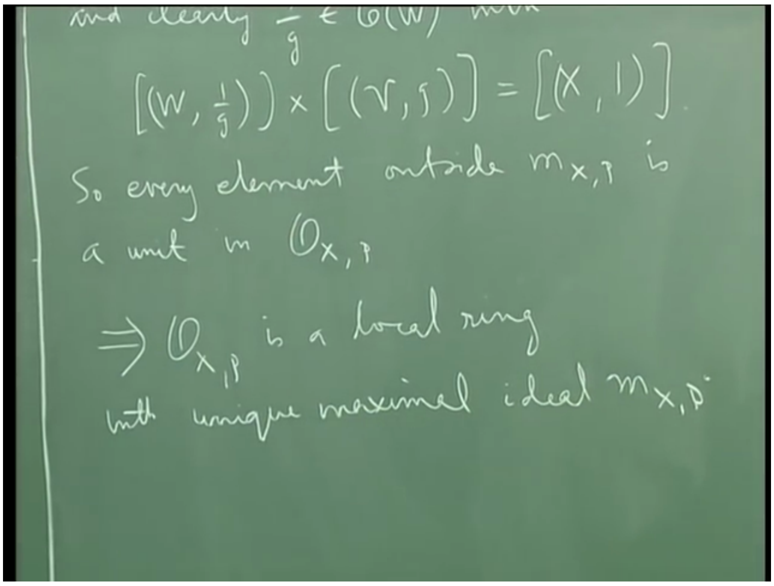

So every element outside M X P is a unit in that is an invertible element in this ring which implies that O x p is a local ring with unique maximal ideal. M X P. So this is the local ring at a point of a variety ok and the so there are so immediately there are couple of facts about this, so the question is that if you will ask is, we have defined this local ring just using the regular functions in a very abstract way, can you really write it down commutative algebraically and the answer is yes you can write it down. If you are looking at a point on an affine variety ok you can write it down as the localization of the affine coordinate ring at the maximal ideal that corresponds to that point ok and that is one fact the other fact is that the local ring will not change if instead of X you take an open subset which contains a point P inside X.

So instead of computing the local ring of X at P if I compute the local ring at of U at p where U is an open subset of X which contains the point P I will still get the same local ring which means that you know you the local ring depends only on the neighborhood of P it doesn't depend on the ambient variety ok. So the moral of the story is that if you want to really work with the local ring at a point of a variety you know already that, that point can be surrounded by an affine open because I have told you that any variety can be covered by finitely many affine opens and therefore the local ring of the variety at that point is the same as a local ring of the open affine sub variety at that point.

But then when you look at the local ring of an affine variety at a point there is a precise commutative algebraic expression namely it is the localization of the affine coordinate ring at the maximal ideal corresponding to that point. So this all this helps us to reduce study of local properties to a study of local rings of affine coordinate rings ok so that is the importance. So with that I will stop now.