Basic Algebraic Geometry
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Module 10
Lecture 27
Adding a Variable is Undone by Homogeneous Localization-What is the Geometric Significance of this Algebraic Fact?

Ok so we continue with our discussion about general varieties namely projective spaces and projective varieties.

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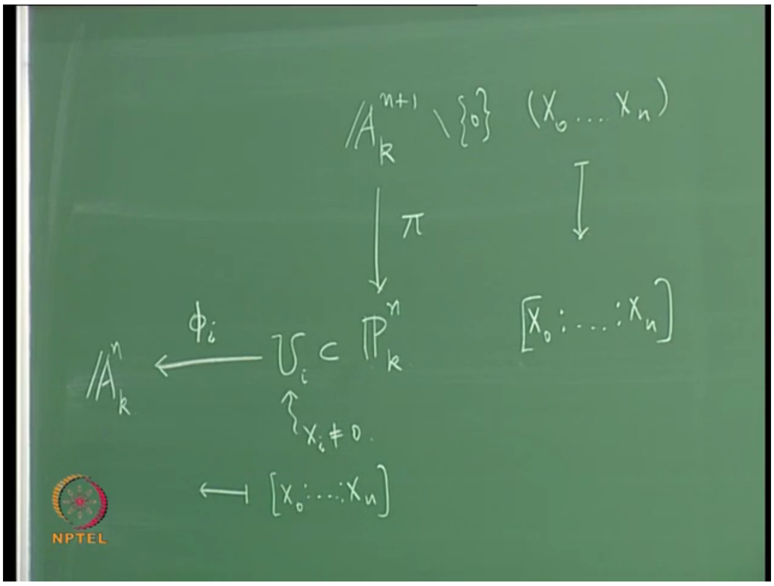

So if you recall from the last lecture what I told you was that if you take projective space which is you take a affine space of dimension N plus 1 over an algebraically closed field K, so this as the set just Cartesian fir the K N plus 1 times but given as Zariski topology and then you take away the origin and look at the space of lines then you get the projective space over K and there is this map which is the natural projection map.

You can think of it is a quotient map and you can also think of it as associating every point in projective space to the line through the origin that it corresponds to ok and what we saw last time is that this projective space is it can be given as a Zariski topology in three ways, one is to take the quotient topology given by this map I. which makes use of the topology above this Zariski topology above. The second way of giving Zariski topology on projective space is to declare closed sets to be of the form given by zero, common zero loci of a bunch of homogeneous polynomials in a N plus 1 variables ok.

And the third way is to consider an open cover of projective space by open sets which are homorphic of affine space and then or which are bijective to affine space and then transport variety structure. So it is a third aspect namely the affine cover of projective space by N plus 1 copies of affine space which is what I explain last time. So you know you define the subset U i, which is the locus where Xi does not vanish.

So basically you have, so the coordinates here are X knot through X n that is how a point here looks like and well it goes down to a point here with which will now have homogeneous coordinates X knot semi-colon to X n and I am using the same Xi to denote both the coordinate function as well as the coordinates, general varying coordinates and U i corresponds to Xi not equal to 0, this is an open subset because its compliment is the zero set of Xi which is the zero set of a homogeneous polynomial Xi is a homogeneous polynomial equal to E1. So it is a compliment of a closed set it is an open set.

But the point is that you have this map Phi I from U i to A n, N dimensional affine space which is gotten by sending a point with coordinates X knot etc X n to what you do is you divide every coordinate by Xi and then at the entry Xi you will get Xi by Xi which is one an you forget it.

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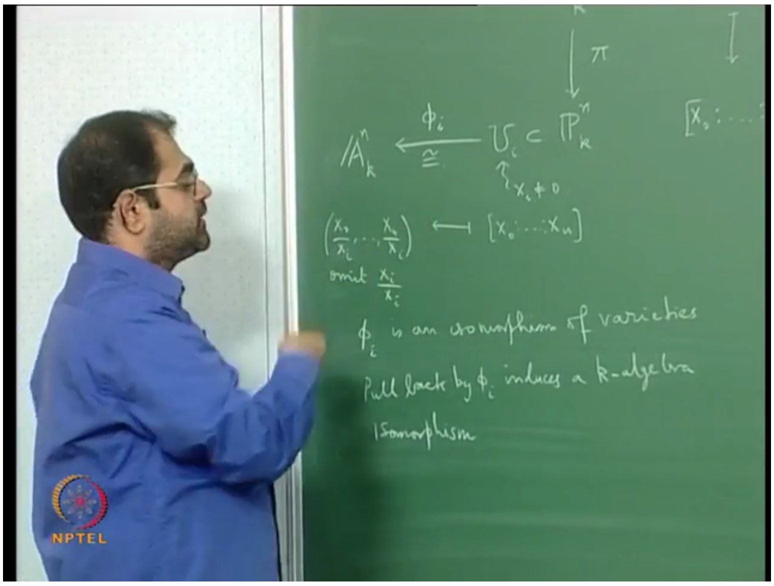

So just send it to the point X knot by Xi comma X n by Xi and here you omit Xi by Xi. So what we saw last time was that, this phi I is actually an isomorphism of varieties ok. So phi I is an isomorphism of varieties and what it will, what you will induce is pullback by phi I induces a K algebra isomorphism.

So the K algebra isomorphism will be say a pulling back regular functions. So given a regular function here the pull it back to a regular function here, pulling back means you given a functions here you compose it with this to get a function here.

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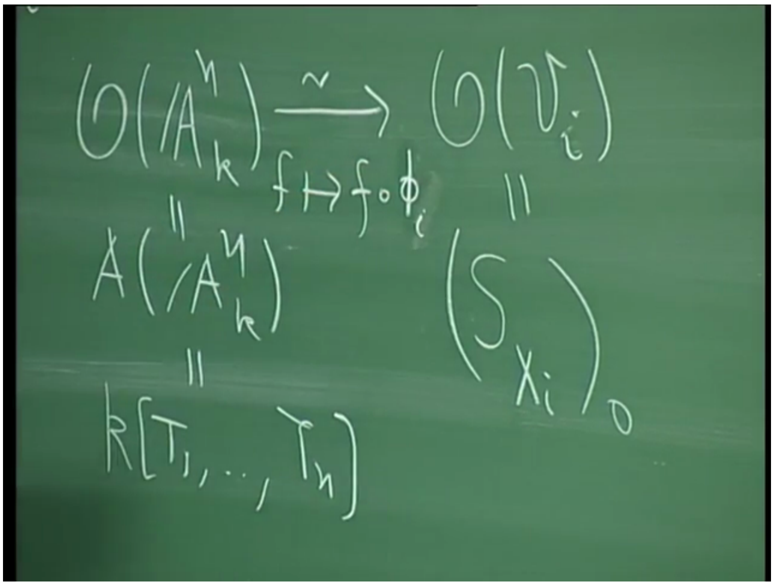

So you get a K algebra isomorphism from the regular functions on A n to the regular functions on U i and the fact is that this is a same as the coordinate affine coordinate ring of affine space and this by here is can be identified with the homogeneous localization or the polynomial ring corresponding to this affine space at Xi.

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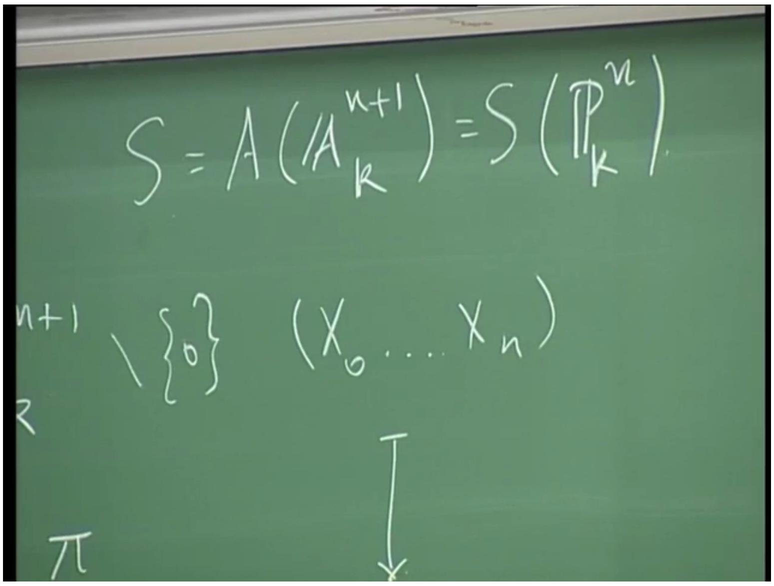

Which means that you take you call S as the affine coordinate ring of affine N plus 1 space and it also called as the homogeneous coordinate ring of the N dimensional projective space and what you do is, you take that is just the polynomial ring in the variables X knot to X n, you localize with respect to Xi ok and S has grading that is a natural grading on (X) S which is given by degrees of polynomials homogeneous polynomials every polynomial in S breaks up into uniquely into its homogeneous components and each homogeneous component has a fixed degree that is the grading on S.

And this that grading will induce a grading on this localization by simply by very simple and obvious manner namely an element here is going to be just of the form some element of S divided by a power of Xi ok and then what you do is you define the degree of a homogeneous element here is of the form homogeneous element of S divided by some power of Xi and its degree will be the degree of the numerator minus the degree of the denominator ok. That makes this into graded ring and then you take the degree zero point ok.

So this is the ring of regular functions on this open subset of projective space. Mind you on projective space or regular function is defined as (local) something that is locally a quotient of homogeneous polynomials of the same degree ok. Whereas on affine in affine space it is simply a regular functions is simply a quotient of two polynomials. Offcourse it is suppose to be define where the denominator polynomials does not vanish right. So this is, this is isomorphism and infact so you know if I call the coordinates on this A n as T1, T2, etc upto Tn then this isomorphism which is given by pullback of maps.

So the map is very simple you give me a regular function here its pullback is just composition with Phi I so it is just F going to first apply Phi I then apply F. So this is the pullback map and this is an isomorphism ok. So moral of the story is that this each of these U I's there are N plus 1 of them ok. Each of these U I's is just an affine space affine N dimensional space and this cover this N plus 1 affine space is the cover projective space and the fact that this an isomorphism actually tells you that the projective variety structure on projective space is given by gluing the affine variety structures on each of this affine spaces via these isomorphism of varieties ok.

So yeah so there are I had ask you to check that this equality holds probably it is not so difficult to check that so let me check this and let me also tell you that in terms of rings this isomorphism is given in a very nice way. This way it is given by homogenization map and this way it is given by a de-homogenization map ok so let me just explain that. So uhh first thing that I want to tell you is that as to Y the regular functions on U I are given by this ring ok. So you know the situation like this, I have so let me draw a diagram.

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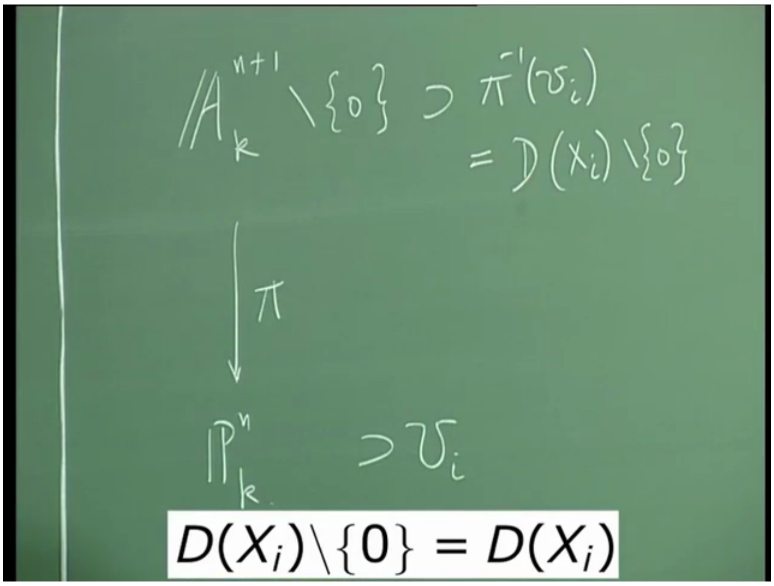

So I have affine space, so here is A n plus 1 minus the origin, this is the punctured N plus 1 dimensional affine space and this is the projection onto the projective space and here I have U I ok and well the if you take Pie inverse of U I, if you take its inverse image under this map then you get Pie inverse U I which is a actually, which actually is if you will if you think about it for moment it is the D of a Xi minus the origin ok.

So U I in the projective space corresponds to the ith homogeneous coordinate Xi not vanishing and its inverse image above will correspond to ith coordinate not vanishing that means the equations so you are looking at Xi not equal to zero and Xi not equal to zero is the basic open set the of Xi you know D of F always denotes the locus where F doesn't vanish is the compliment of Z of F which is the locus where F vanishes and we have already seen that D of F is itself an affine variety isomorphic to affine variety because of the Rabinowitsch trick ok.

It can be (D) so this D of Xi can be embedded in an affine space of one dimension or more namely it can embedded in A n plus 2 as a closed sub-variety ok. So this of Xi in the whole affine space are certainly it is an affine variety and its and since I am looking since I am looking at its intersection with this I have to throw away the origin ok and that is the inverse image of this U I ok and the point is that this map Pie, so first observation is that this map Pie itself is a morphism of varieties ok.

See this map is natural projection from the punctured affine N plus 1 dimensional space to the projective space this itself is a morphism of varieties and the reason is I mean the, the reason is obvious because it is continuous and regular function here if you give me a by definition a regular function here is a quotient of two homogeneous polynomials. So if you compose it with Pie I will simply get a regular function here. So not every regular function here on an open set goes down to a regular function below ok.

But regular function this map Pie offcourse pulls back regular functions in open sets to regular functions in open sets. So it is obvious that Pie is a morphism ok. So the continuity is obvious and in one of the definitions you give this the quotient topology from that ok. So this is certainly continuous map and if there is an open set here what is a regular function on open set here, locally it is of the form F by G where F and G are homogeneous polynomials of the same degree but if I take that function and compose it with Pie, you know if you pull it back it will correspond to a function there ok and that function is offcourse it is going to be a function that is constant on lines.

It is going to be a function that is constant on lines, lines passing through the origin because it is that is what will happen any function on this set when you compose it the projection map will give you a function above which is constant on lines through the origin and conversely any function above which is constant on lines towards the origin will go down to define a function below ok. So if you take a function of the form F by G F and G are homogeneous polynomials offcourse defined on the open set contained in the locus where G does not vanish then its inverse image will be simply the same function F by G which will which is again a quotient of polynomials and that certainly quotient of polynomials a quotient of homogeneous polynomials is also a quotient of polynomials and any quotient of defines a regular function locally on affine space. So it very clear that Pie pulls back regular functions to regular functions. So Pie is a morphism ok.

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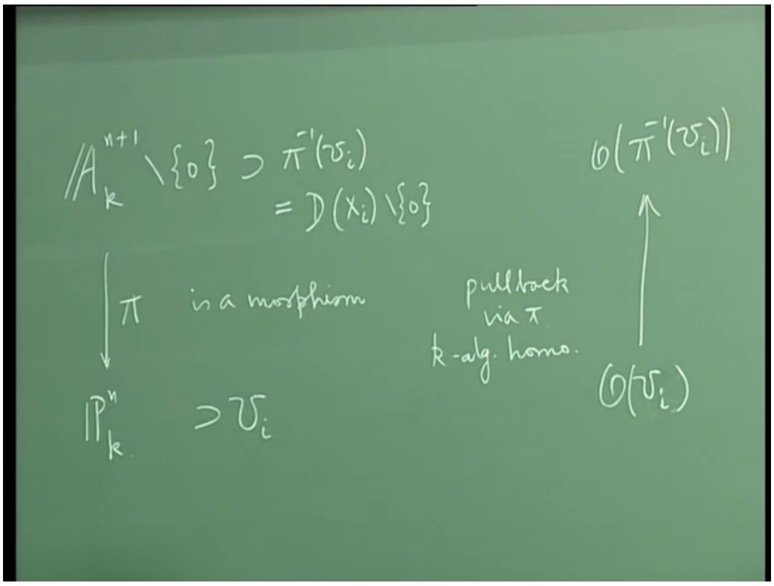

So Pie is a morphism of varieties and you know therefore pullback of Pie pullback via Pie will give me a map from you know regular functions here to regular functions there. So I am looking at Pie restricted to this. So I am going to get O U I, O of U I to O of Pie inverse U I, this is what I am going to get and so namely you give me a regular function on U I, you compose it with Pie I will get a regular function on Pie inverse U I, this is the pullback map alright.

So this is and offcourse this is a K algebra homo-morphism any morphism of varieties will induce a K algebra homo-morphism which corresponds to the pullback of regular functions ok and offcourse mind you U I is a variety now as far as our definition concerned because our definition allows varieties to be either affine or quasi-affine or projective or quasi-projective, U I is the quasi-projective variety. It is an open subset of a projective variety in this case projective space ok and the point is that Pie inverse U I is also a variety, Pie inverse U I is actually you know it is the uhh it is just the this variety D X I it is a basic open set.

First thing is certainly if you take an element here and element here is of the form F it is of the form F by Xi to the power of degree F where F is homogeneous of certain degree alright.

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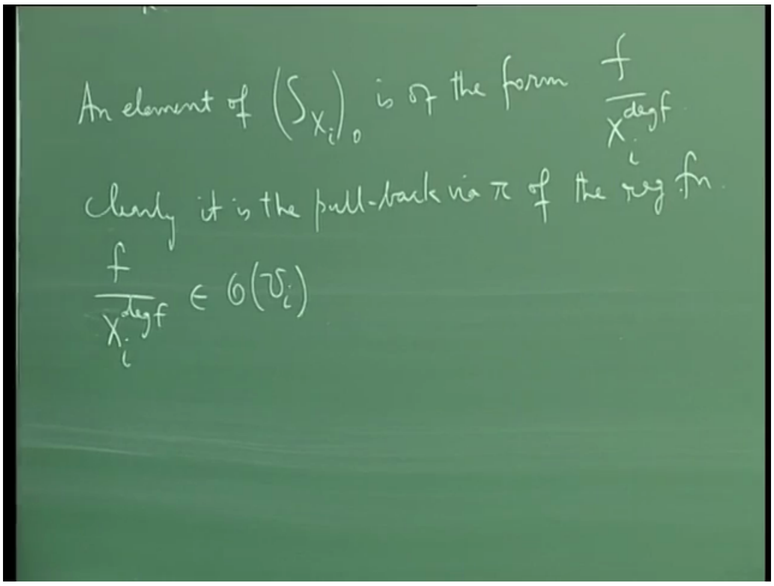

So an element of a S Xi 0 is of the form F by Xi to the degree F ok. So S Xi S localize of Xi is just inverting Xi. So elements there will be of the form simply F by some power of Xi. But then if I want degree 0 part then I want the degree of this element to be zero and the degree of this, degree of a quotient is defined as degree of the numerator minus the degree of the denominator.

So if you want that to be zero then the power of the Xi in the denominator should be equal to the degree of the numerator polynomial which is suppose to be homogeneous ok. So this is how an element looks like and this element is if you think of this element this element is offcourse a function which is constant on lines passing through the origin, because it is both the numerator and denominator of homogeneous of the same degree. In other words it goes down to define a function on the projective space and where it will define a function, it will define into a precisely function on U I because that is where Xi is not zero.

So it is very clear that clearly it is the pullback via Pie of the regular function F by Xi to the degree F which is in O of U I, this is clearly a regular function on U I because a regular function on projective space is suppose to be on an open set is suppose to be quotient of polynomials homogeneous polynomials of the same degree. So in this case it is certainly a quotient of homogeneous polynomials of the same degree. That is because you have taken it in the degree zero in the graded localization and therefore it is a regular function ok. So what I am trying to say is that you know for the moment.

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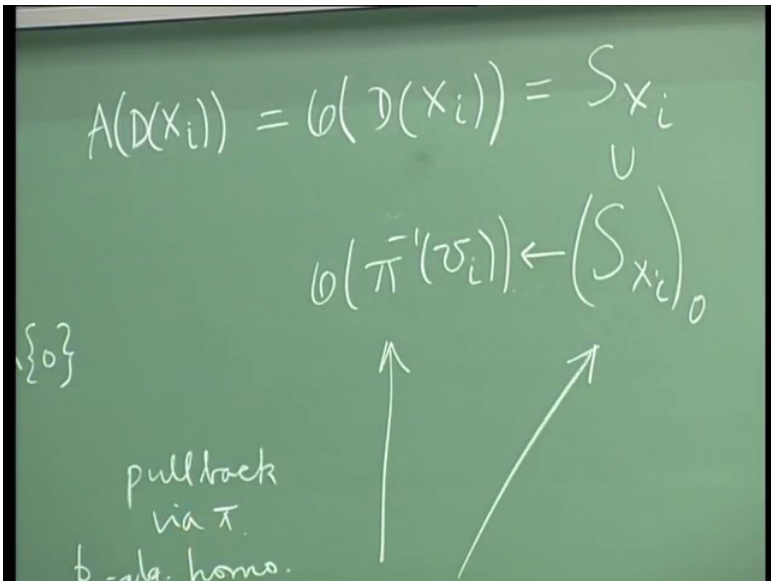

So you know I have if I want to draw a more particular diagram I have O, so I have O of D Xi in the fine space above this an affine variety and you know that this can be identified with S sub Xi ok. If you take T Xi in the affine space that is the locus where Xi does not vanish and it is an affine variety. It is affine coordinate ring is S Xi.

Infact this is same as A of , you can write A of D Xi reason is because D of Xi the basic open set defined by Xi is of the form D of F and every element of the form D of F is an affine variety alright and its coordinate ring is just the given by localization at F. So in this case instead of F I have got Xi ok, but don't confuse with the F here ok. So and you know you can see that certainly this is a sub of this ok and what is happening is it is very clear that if I take an element here, if I take an element of this form it certainly going to be a regular function on Pie inverse U I ok so going infact be going constant on lines.

So it is very clear that this is going to sit inside this alright I want to say that you know if you go there is a map like this ok. You start with an element here is certainly regular function here ok and it covers from a regular function here ok. So what I want to say is that you know so there is a map like this ok. So let me explain so you know you start with, you start with an element here, this element is of this form. Now this element certainly defines it defines a regular function here ok which goes down to a regular function there and the pullback of that regular function is this regular function ok.

So you have map like this, mind you this is also an element O of Pie inverse U I, this is also an element it is also a regular function there and the same thing goes down to a regular function below and if you pullback this regular function you will get this regular function above. So when I write like this here I am thinking of it as regular function above on Pie inverse U I. Infcat this makes sense as regular function on D Xi this certainly makes sense a regular function on D Xi ok.

So when I write like this this can be thought of as a regular function above ok and it is since it is constant on lines passing through the origin it also goes down to define a function and that regular function is also given by the same expression. It is of the form one homogeneous polynomial by other of the same degree ok.

So it is the same regular function then it defines below also ok and the and if you pullback this function below via Pie you will get this regular function above alright. Here D Xi minus 0 is the same as D Xi because I don't have to remove already origin is not there certainly and the other thing is offcourse

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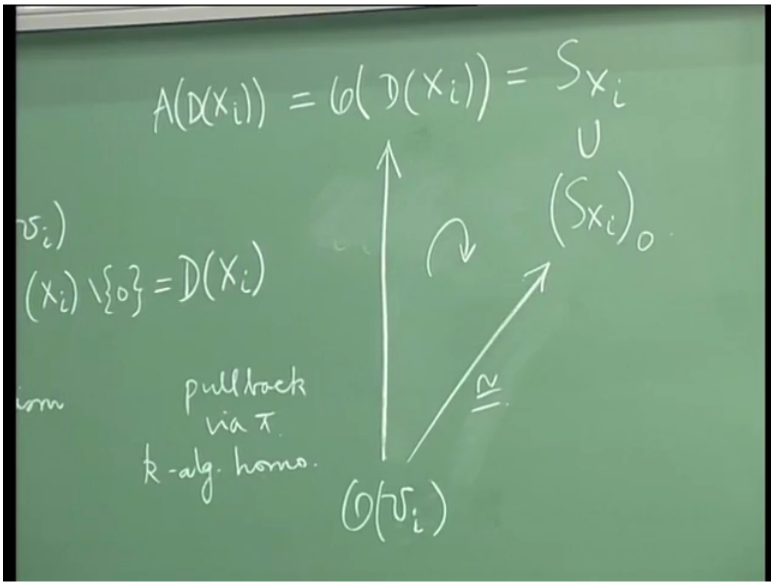

So Pie inverse U I is just D Xi and what I will have here is and this D Xi will offcourse have affine coordinate ring S Xi of which if you consider it as a graded ring the degree zero part is S Xi localize at 0 and the claim is that this map from, so this pullback map which goes from (U) regular functions on U I to regular functions on D Xi, this map the claim is that it factors through this ok and it factors through this and it is an isomorphism that is the claim alright. So this is the claim, this is what one has to prove. So the argument is as follows.

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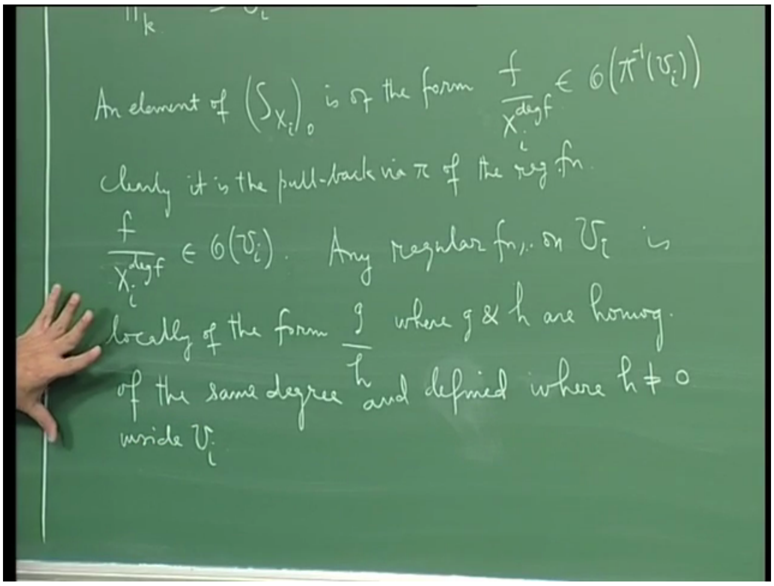

So any regular function on U I is locally of the form G by H where G and H are homogeneous of same degree and with a and defined in where H is not zero ok inside U I. This is how, see any regular function on U I looks like this, locally it is of the form quotient of two homogeneous polynomials of the same degree and it is defined where H is not zero and it is inside U I ok.

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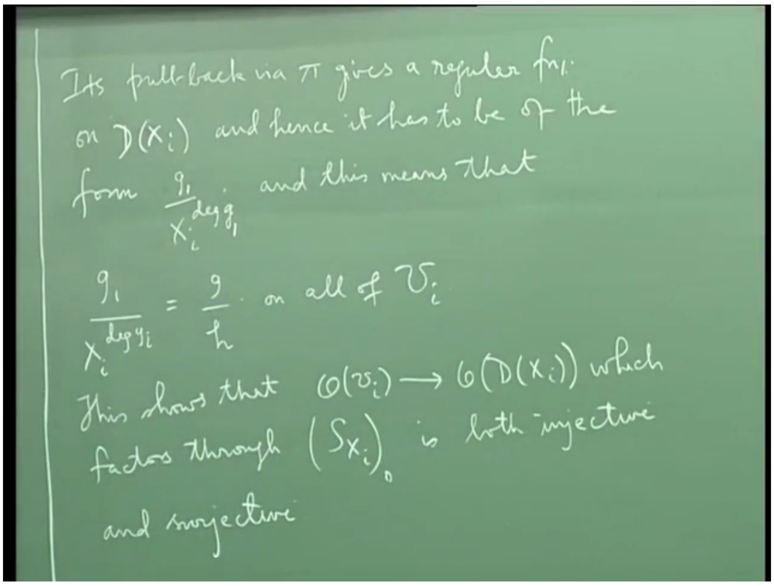

See it is pullback via Pie gives a regular function on U I on D Xi and hence it has to be of the form G by Xi to the power of degree G ok. Because regular functions the regular functions on D Xi they have to be of the form G by Xi some power of Xi and if they go down if this power if they go down to a function on projective space this the power of Xi below should be equal to the homogeneous degree of the polynomial above. It has to be the first of all, any function any regular function on D Xi will look like some F by some power of Xi ok.

But if it has to go down to a function on projective space ok then the numerator polynomial F has to be homogeneous and the denominator polynomial will be a power of Xi the power being equal to the degree of the numerator polynomial. Therefore if since I am pulling back if regular function below I will get a regular function above the fact that it has come below will tell you that when I pull it back it has to look like this ok and therefore this will also be the expression for the regular function below ok.

Because this below G by H is on some open subset ok and this coincides with this on some open subset alright and that will tell you that it has to coincide on the largest possible open set ok. So it has to be of this form so maybe it may happen that there are some common factors but so if you want I can put G1 here, I can put G1 by degree G1 ok and this means that G1 by Xi to the power of degree G1 is the same as G by H on all of U I ok. So I am using the fact that you know on a variety if two regular functions define the on a variety you know if they coincide on a open non-empty subset then they coincide everywhere ok.

Therefore I start with a regular function here I start with an element here. It is locally of the form G by H ok it may have local many representations but I am just taking one such representation G by H ok, where G and H have to be homogeneous of the same degree. Now if I pull this function there locally to the affine space above punctured affine space above, offcourse I will land in D Xi ok and locally on D Xi will get a regular function that I get by pulling back this regular function U I will be a regular function on D Xi, which has to be this ok.

Therefore there a function I pulled from below has to be also of the same form, it is given globally in this form ok and so what I, so I am just I mean this is just the statement that you know this map this mapping is surjective ok. This shows that O U I to O D Xi factors through which factors through S Xi 0 is both injective and surjective. I mean it is surjective first of all the map from here to here actually lands in here ok.

And the second thing is that it is surjective because if you take anything here it is certainly defines regular function below, that is the first line that I wrote ok and it is injective for obvious reasons because if a function above goes down to a function below and if the function is the function below is zero then the function above you stated must be zero and conversely if the function below is zero, if the function above is zero offcourse the function it goes down to is zero ok.

So it is very clear that this both the that this map is injective and this map is surjective is very-very clear ok. So here what I am doing is I am using the fact that this is the morphism and this is and I am pulling back the functions ok. But you can also think of it in another way by taking any element here which is an element of this form and that defines a regular function below. So you have also a map like that ok and then you can check that if you do it like that then you are just defining the inverse map ok.

You start with an element here, it defines a regular function below the you will get a map like this ok you can check the that map is also an isomorphism for the same reasons ok. The only thing the way I have done it I have used, I have just used pullback I am just saying that this morphism it corresponds to the natural quotient of punctured affine space to projective space if you take the pullback the pullback induces a K algebra homo-morphism but that K algebra homo-morphism from the source to the target it is injective its image is precisely the degree zero apart.

In other words it is an isomorphism onto the degree zero part just proving that the regular functions on U I is exactly this the degree zero part of the localization at Xi is called the, which is called the homogeneous localization at Xi ok. So that is the proof of this statement that these two are one and the same and you know and to show that this pullback via Phi I this map this is an isomorphism in term of rings you can actually give the ring isomorphism from k T1 Tn to this homogeneous localization. So what is that? So what is the other way, what is the other statement?

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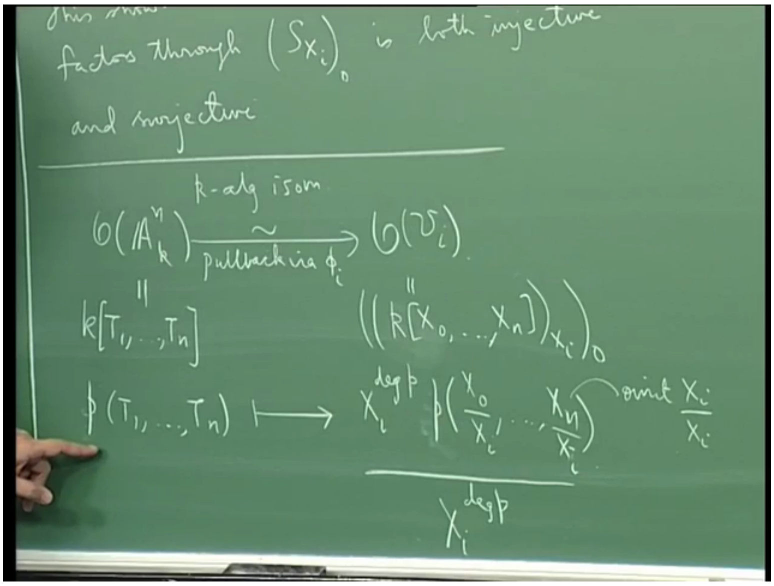

So you see I have O of A n and I have this isomorphism K algebra isomorphism which is given by pullback via Phi I pullback of regular functions via Phi I, this is suppose to give as we proved last time an isomorphism with O U I but now you know this is K of T1 etc Tn and this is as we have seen just above it is just S localize at Xi. So let me use let me expand what (S) is X is K X knot etc up to Xn localize at Xi and then degree 0 part ok, this is S localize at Xi degree zero part.

And what is this? What is tis isomorphism? What is a K algebra map in this direction? What is a K algebra map in this direction? This map is given the map in this direction is given by homogenization, the map in this direction is given by de-homogenization, which what we use last in the last lecture to prove that this is an isomorphism ok. So what is a map in this direction? So you take F of or let me take P of T1 etc Tn what I am going to send it to? I am going to send it to the following thing.

I homogenize it ok so which means I take Xi to the degree P of times P of X knot by Xi and so on Xn by Xi and offcourse when I write this I omit Xi by Xi alright and I take this and you know and then I divide by Xi to the degree P ok. See if I take P which is a polynomial in N variables ok and if I just take the numerator this is the homogenization of P. it is you add the new variable Xi ok and then you get the homogenized polynomial.

Now that's will continue to have degree P that will be homogenize of degree P ok. Here degree by degree P I mean the degree of the highest monomial here ok, the degree of the polynomial like this will consist of monomials product of powers of Ti ok and multiplied with some coefficients and sum of such finitely many such and among them you take the monomials of the highest degree and take the highest degree and call that as a degree ok.

That could be offcourse several monomials of the same highest degree ok. But you take the highest degree monomial ok and call that as a degree of P and if you take this expression on the numerator what I will get is a homogenization of P. So this homogenization is achieved by adding a new variable Xi ok and raising it to the degree P and multiplying with this ok. Now once I do this the numerator becomes the homogeneous polynomial in X knot through Xn of degree P alright and if I divide it by Xi to the degree P what I get is a regular function on U I, because regular function on U I is of this form.

It is some homogeneous polynomial in the X knot through Xn divided by a power of Xi which is equal to the degree of that homogeneous polynomial. So this is certainly an element of O U I. So this is how the map goes in this direction, you can explicitly write this map ok and what is the map in this direction? The map in the other direction is pretty simple. That is also very easy to write.

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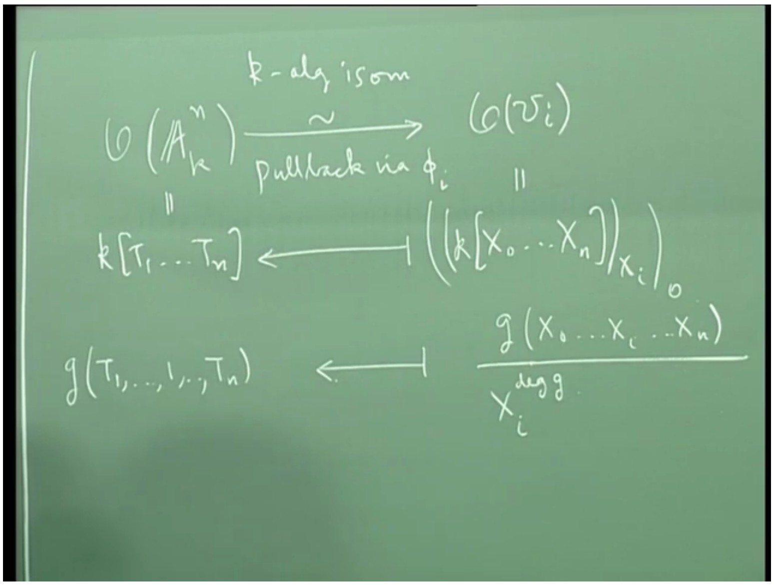

So lets write that down, let's just de-homogenization. For what is the map in this direction? So this is pullback via Phi I, so I have here let me write that K X knot etc Xn localize at Xi till the degree 0 part and here this is K T1 etc Tn you have this, so what is the map in this direction?

The inverse map is you start with any G by Xi to the degree G ok and what you will (do it) do is simply send it to you know G of T1 dot-dot-dot 1 Tn this is what you will do. So what you are doing is you are simply sending, so this is de-homogenization, this is just de-homogenization alright. So this G is a polynomial in X knot etc (Xi) Xn. G is a homogeneous polynomial in these N plus 1 variables alright and now what you do is in wherever Xi comes you put 1 ok.

Wherever Xi comes you put 1 ok and for the remaining X knot through Xn with Xi left out which are remaining N you simply substitute T1 through Tn that is what this means. So this is de-homogenization. So this map is this is homogenization divided by the right power of Xi. So this is this map in this direction is given by homogenization and this map is given by de-homogenization and therefore. So the moral of the story is that and you can see if I now start with this G and homogenize it and divide by (degree D) degree G I simply get back this.

So (it will) you can very easily see that this is the inverse of this map ok. So it is so what I am trying to say is that you see that this isomorphism of this U I with A n geometric isomorphism in if you translate it to commutative algebra it is just this isomorphism between a polynomial ring in N variables ok and polynomial ring in N plus 1 variables localize at one of those variables and taking the degree 0 part, the significance of this isomorphism which you can write down just from commutative algebra you don't need any geometry for that ok.

You can write this maps down just using commutative algebra right. The fact that this isomorphism is an algebraic fact and the geometric manifestation of that is that it is giving you an isomorphism of a corresponding open subset of projective space with an affine space. That is what it means, so this is the commutative algebraic translation of , this isomorphism commutative algebraic translation of this geometric isomorphism ok and this a very-very important thing in algebraic geometry whatever you see in terms of geometry you should translate it commutative algebra and whatever you see in terms of commutative algebra you should try to translate it into geometry ok.

And I am saying that in this case it is completely you know you can really write it down there is nothing complicated about it. One is able to write down all the maps ok. So what I need to say is next which I will do in the next lecture is to tell you that I will use this to tell you that you know affine varieties are the building blocks of all varieties. So actually that any variety can be covered by finitely many open sets which are each themselves isomorphic to you know affine varieties ok. So affine varieties are the building blocks of all varieties ok. So I will do that in the next lecture.