Basic Algebraic Geometry
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Module 10
Lecture 26
Translating Homogenisation and Dehomogenisation into Geometry and Back

So this is the continuation of the previous lecture where we were discussing projective and quasi-projective varieties so the last thing that we saw was that if you have a projective variety then there are no non-constant global regular functions and as corollaries we saw that the only morphisms from projective variety to affine variety are the constant maps and putting the condition of projectiveness and affiness on a variety reduces it to a point ok.

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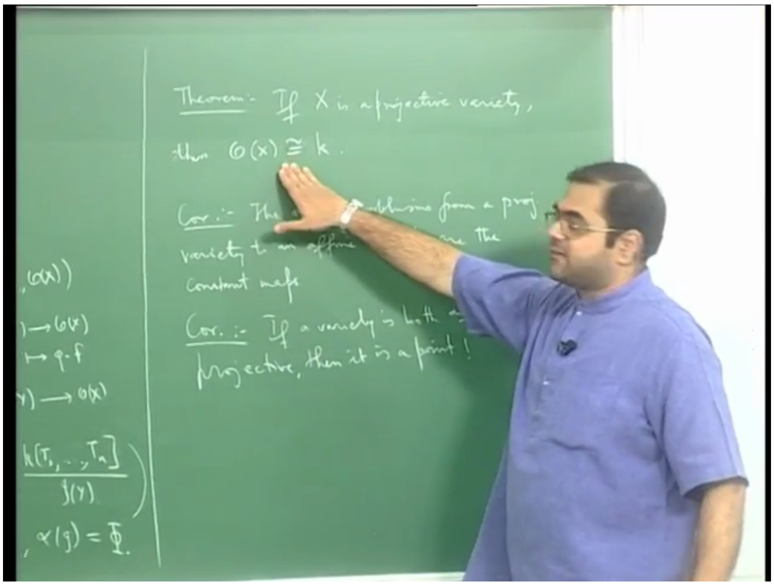

These are all just reflections of the fact that the no non-constant global regular functions in a projective variety ok and therefore you know one thought of one line of thought from this is that if you still belief once still goes by the philosophy of Felix Klein that the geometry of space is controlled by the functions on it.

Then it is very clear that the geometry of a projective variety cannot be controlled by just looking at the global regular functions because there aren't any non-constant global regular functions and therefore you will have to concentrate on regular functions on open subsets or the projective variety and this leads to what is called as bi-rational geometry and that is studying the geometry of open subsets ok and therefore you the clue is that if you cannot keep track of the geometry of the projective variety by looking at its global regular functions which are only constants.

You can still keep track of the geometry of the projective variety by looking at regular function on various open sets ok and this is covered in by studying sections of line bundles on a projective variety and this is something that you would see in a second cosine algebraic geometry ok. So I am just trying to say that the statement of Felix Klein the philosophy of Felix Klein that the geometry is controlled by the functions still applies in the also in the case of projective variety only thing is that you is not enough to just consider global functions because they only constants.

You have to consider functions on open sets ok and the device that attaches which keeps track attaches to every open set the regular function on that open set is what is called a sheave of regular functions ok. So one need to keep track of this using sheave theory alright. Which most probably you would see in a second course in algebraic geometry. So now what I want to do, I want to, I just want to continue saying the following that, so basically you know I want to go back to the tour earlier argument where I think a couple of lectures ago where I showed that the affine (space) the projective N space was union of N plus 1 copies of affine N space.

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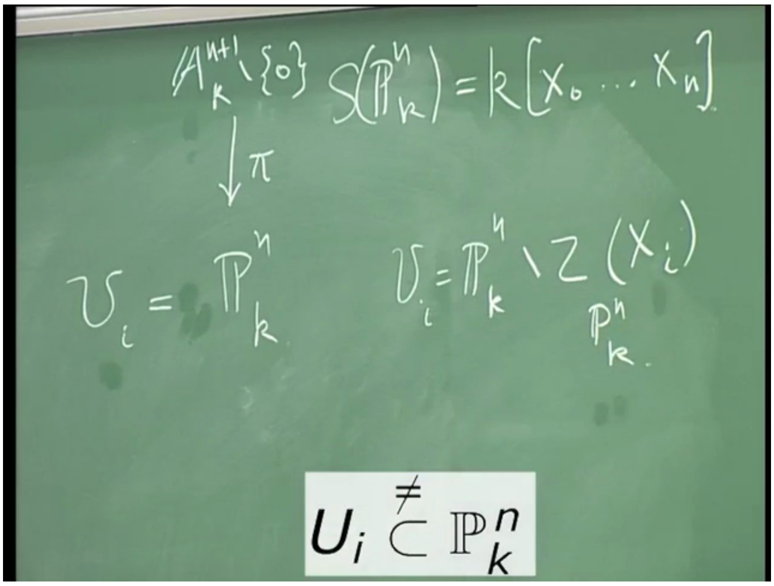

So you know if, so let's recall that, so you see we have projective N space and we offcourse take the homogeneous coordinate ring for the projective N space as K X knot to X n and these Xi's are they are the coordinates on the affine space above to projective space. So we must remember that there is an A n plus 1 minus the origin of which the projective space is a quotient, why this quotient map which is quotient by the equivalence relation that identifies all points on a line passing through the origin in the affine space above.

And the coordinates here are there Xi’s and the corresponding coordinates here are called homogeneous coordinates because here only the ratios are the coordinates are ratios and now you look at the set U i, where U i is a projective space minus the zero set of Xi in the projective space. Notice that each Xi is homogeneous of degree one therefore its zero set is closed subset of projective space it is called hyper plane because it corresponds to the ith coordinate being zero and its compliment is open set U i. This consists of all points with homogeneous coordinates with ith with the subscript I coordinate from zero ok.

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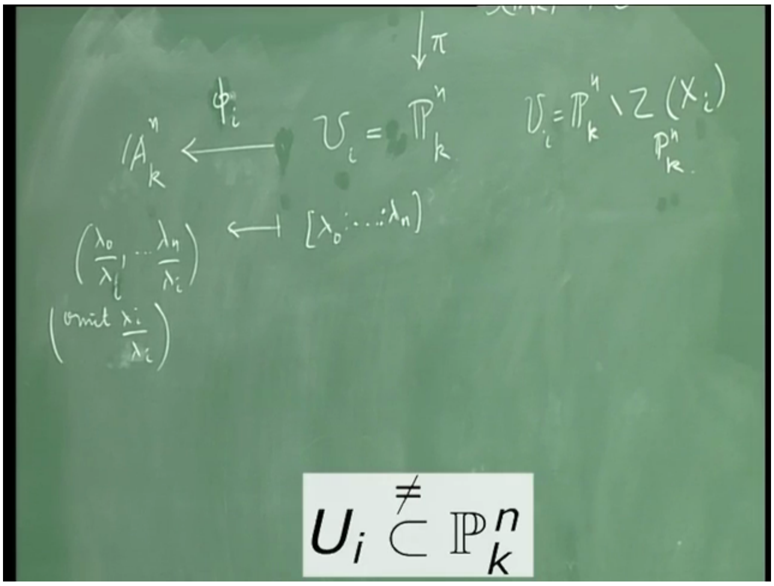

And then I told you that there is a map Phi I from A n to U i and I would ask one of you to check whether Phi I was defined this direction in this direction or the other, this the map Phi I or was it the other way around ok, it was the other way around. So let me so that I stick to the notation I started with earlier and this map was very simple, what you did was you took a point with coordinates lambda knot etc and N and simply send it to the point lambda knot by lambda 1 lambda I dot-dot-dot lambda n by lambda I.

Where offcourse omit lambda I by lambda I ok. So this is the map we defined and we set it is easy to check that this map is a bijective map ok and I asked you to check that this map is actually a homo-morphisms of topological spaces and so you know, so what I want to say now is that suppose I start with a topology on the projective space given by the quotient topology for this map with the topology above being is one induce by the Zariski topology on affine space. Then this topology with P n with that topology will induce a topology on the U i and for that topology Phi is the homo-morphisms with A n.

With A n having the (())(08:20) Zariski topology. So the way this is interpreted is that if you want to give the topology on P n there are three ways of doing it, one is you define the closed sets zero sets of bunch of homogeneous polynomials then the other way is by giving the P and the quotient topology via this quotient map.

The third way is that you make each of the Phi I's homo-morphisms namely you transport on the you give the topology on U i that makes this a homo-morphisms namely a set here is open or closed if and only if its image here is open or closed and then these topologies in the individual U i's will agree well on the intersections and therefore they will to define global topology on the projective space and that will be the same as the quotient topology or the topology for which closed sets are given by zero sets of bunch of homogeneous polynomials ok.

So all these are the same. So to you know now to make things more clearer infact what I want to say is that these Phi I are not they are not just homo-morphisms these Phi I's are actually isomorphism of varieties ok.

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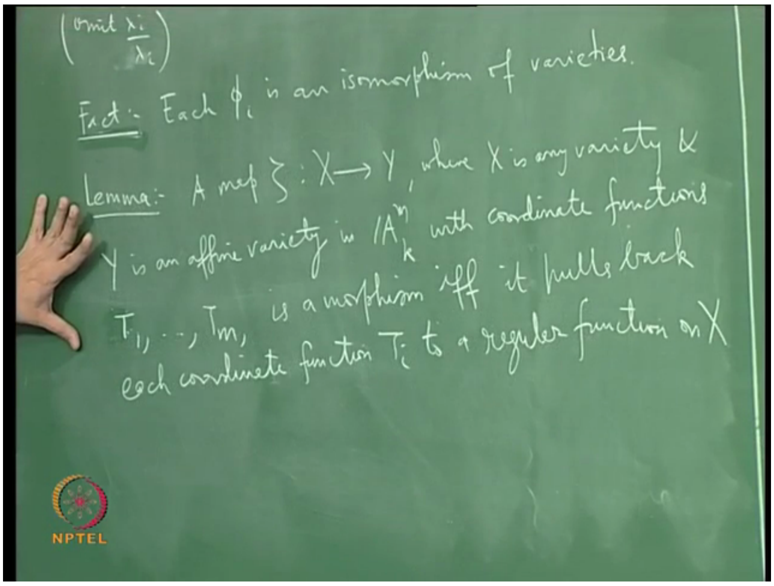

So fact each Phi I is an isomorphism of varieties ok and yeah ok, this is not equal I am sorry this should be a subset of P n ok.

Student: is that the topology changes, each U i if have the topology now because of these maps. Now to bring a topology on P n using these, that is the topology generated by A n.

Yeah you can so it is a fact, you can take the topology generated by this or so ok so thank you for pointing out this should have been subset not equal to offcourse. So when I said that there is a topology on each U i that makes Phi I a homo-morphisms and then you all this topologies put together give a topology on P n I mean the following thing. A subset of P n is defined to be closed or open if its intersection with each U i is respectively closed or open that is the definition and for this definition to work a subset of U i intersection U j is closed in U i only if and only if it is closed in U j and is open in U i if and only if it is open in U j.

That is the compatibility that you will have to check. So the topology on P n that I want to give by gluing the topologies on the U i is a following, you define a subset of P n to be closed respectively open if its intersection with this cover is closed respectively open. In particular these U i's themselves will become open because if you take the set U i you will have to check that U i intersection U j is open in U j ok every j different from I and U i intersection U i is U i and that is offcourse open in U i.

So each of the U i's by this definition will automatically become open subsets and then you are only requiring that subset of the ambient space is closed respectively open if and only if that if its intersection with respect to this cover is relatively closed or relatively open. So that is the topology that you can get by gluing. The topologies on each U i ok. Now the fact I want to make is that each Phi i is an isomorphism of varieties alright and how does one do this? You will have to show that Phi I is a morphisms you have to show that Phi I inverse is a morphisms.

So let me do that properly so what will do is. So on the so will have to you know what I am trying to say is that the Phi I's are isomorphism so I am trying to say that each U i is isomorphic to A n but offcourse every each of these U i's is a quasi-affine variety because it is an open subset of a projective variety ok. So sorry quasi-projective variety because it is an open subset of a projective variety ok and I am just saying that these quasi-projective varieties are actually affine because they are isomorphic to affine these affine varieties.

Namely I am just saying that this natural open cover of quasi-projective varieties is actually isomorphic to so many copies of affine space ok and then so you know so the way to check it that you do the following things you use this lemma which I have stated earlier the lemma is a map zeta from X to Y where X is any variety and Y is affine variety in say A n in A n with coordinates functions X1 or let me put T1 etc T m is a morphisms if and only if it pulls back each coordinate function T i to your regular function on X ok.

So this is the fact that we have already seen. I mean we saw this in the context of affine variety or quasi-affine varieties but it, you can go and check that the proof has got nothing to do with the source variety being affine a quasi-affine could have very well been projective or quasi-projective so the idea is if you are just given a set theoretic map ok when do you check it some morphisms? I mean how do you check it some morphisms? When it is a morphisms?

Is a very-very important, it is a very powerful lemma but is very easy ok and it is very easy to use. The proof is also easy but it is a very powerful lemma because if you want to check something as a morphism you have to check two things, you will have to check its continuous number 1 then you have to check that it pulls back regular functions to regular functions, so the checking involves two steps. But this lemma tells you that you can you know do away with that will do it one go by just simply saying that if your target is an affine variety ok and your map pulls back coordinate functions to regular functions then it is a morphisms ok.

So if you use this lemma it is very clear that it is very easy to see that Phi I and Phi I inverse of morphisms and since they are inverses set theoretic inverses of each other it will follow that Phi I is an isomorphism ok. So you know for example if you try to apply it so try to apply this lemma Phi I my source (varies) variety is U i, which is a quasi-projective variety and the target variety is an affine variety is just affine space ok and you know if I take so if I take a coordinate function on this and compose with this, what I will just get so for example suppose I take the first coordinate function.

The first coordinate function will be if I take the coordinates here as T1 through T m ok, then if I composed lets say T1 with Phi I, I will simply get this X knot through X n homogeneous coordinates going to X knot by Xi ok and if I compose it with J it will be just X knot through X n going to X j by X i and X j by X i is offcourse a regular function on the source projective space. Because it is a quotient of two homogeneous polynomials of the same degree, degree 1 and it is defined on the set I am dividing by Xi is correct because I am on U i where Xi is not zero.

Therefore it is very trivial to see that the pullback of the coordinate functions or regular functions. So that makes Phi I a morphisms alright and similarly if you go in this direction also ok. So you can go in one direction because this is affine and this is any variety ok. So for showing that the map is a morphisms in the other direction I will need to do something more. So let me write this down, let me write that down.

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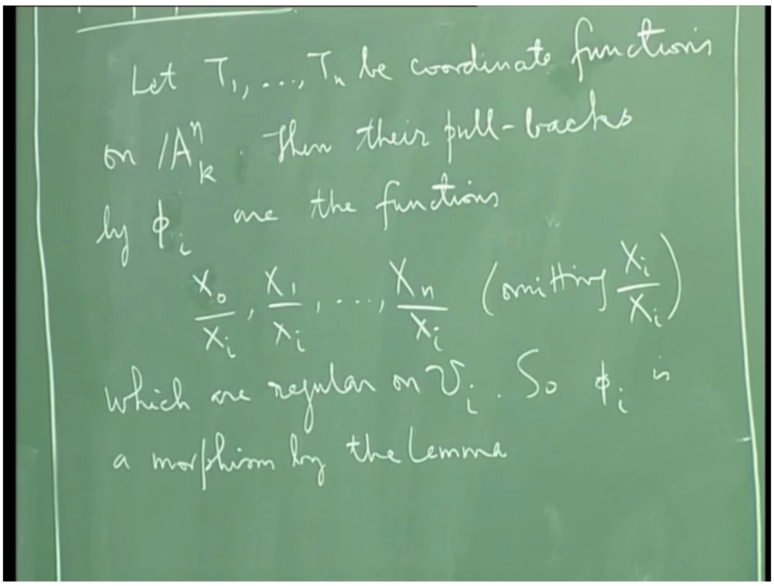

So proof of fact, let T1 etc T n be coordinate on coordinate functions on the target A n then if I look at the pullback then their pullbacks by Phi I are the functions, well T knot by I mean X knot by Xi, X1 by Xi and X n by Xi omitting Xi by Xi ok, which are regular on U i. Therefore by the lemma each Phi I is a morphisms. So phi I is a morphisms by the lemma ok.

Now so what this will tell you is that Phi I is a bijective bi-continuous morphism but even that is not enough to unsure that Phi I is an isomorphism because in the category of varieties the problem is that you can have a bijective map which is a morphisms in one direction.

It could even be continuous in the other direction but it may fail to be a morphisms in the other direction. So you will have to do something do say that Phi I inverse in also a morphisms ok and for that offcourse I have to check that Phi inverse is a morphisms I cannot apply this lemma because the target now is U i and I don't know for sure that U i is an affine variety.

To start with I don't know that U i is an affine variety it is only a quasi-projective variety I cannot apply the lemma to show that Phi I inverse is a morphisms. So I will have to prove Phi I inverse a morphisms I will have to do it the hard way namely I have to check its continuous and then I have to check that it is a it pulls back regular functions to regular functions. Now continuity is something that I have already asked you to check but it is probably a easy to write down.

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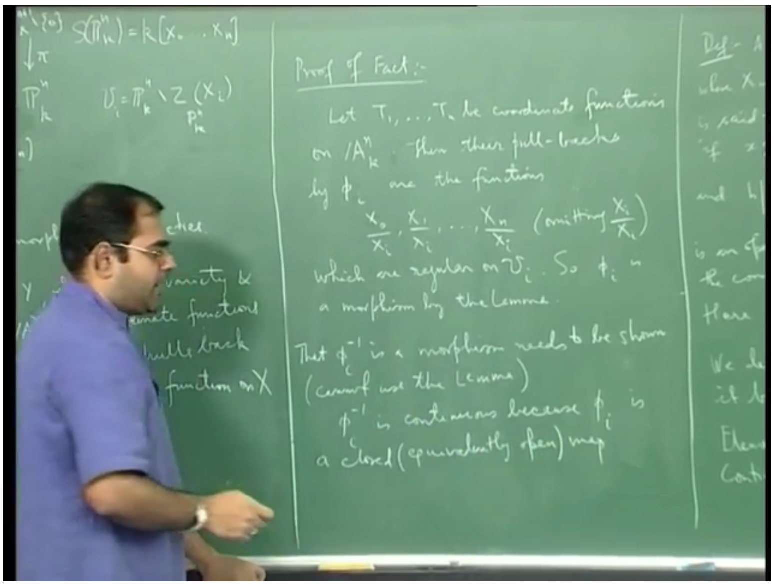

So phi I inverse is a morphisms that Phi I inverse is a morphisms needs to be done, needs to be shown cannot use the lemma because the target is U i and I don't know U i is affine. But offcourse once I prove this fact it will follow that U i is isomorphic to affine variety and therefore U i will become affine. But I cannot assume it when I try to prove it ok. So phi I inverse is continuous because you know how will I show that it is continuous, so you know I start with to show it is continuous I have to show that if I take a close set here, its inverse image under Phi I inverse is the same as its image under phi I, it is a closed set in A n.

So will just have to show that phi I is a close map or an open map ok. So because phi I is a closed map equivalently open map and why is phi I a closed map? The idea is very simple, if you take a closed subset here ok that closed subset is, the closed subset in the projective space intersect with U i because that is the induced topology, now since it is a close, if you take it closure if you take a closed subset here and take its closure in the projective space you will get a closure, you will get a closed subset of projective space.

So it is the zero set of a homogeneous ideal ok. Now what you do is you take all those elements in the homogeneous ideal and then just de-homogenize them in terms of these coordinates ok, you will get an ideal here and it is a zero set of these ideal which is the image of that close set ok. So let me write these down. So what you must understand is ok, so that gives me an opportunity to say something, what is the algebraic translation of this morphism? The algebraic translation is offcourse the algebraic part here is the ring of, it is a polynomial ring.

It is the ring of regular function on A n. Is the polynomials in T1 etc Tn ok and what is happening here the fact is that the regular function here are just the resists this polynomial ring localized at Xi ok, that means you invert Xi but then after invert Xi it is still a gradient ok because you have inverted a homogeneous element and then you take this degree zero part of that, that will give you a sub-ring and that is the ring of regular function in U I ok. So the fact is that this map it is an isomorphism of these rings and that is the geometric translation, that is the algebraic translation of the fact that this map is a isomorphism of varieties ok.

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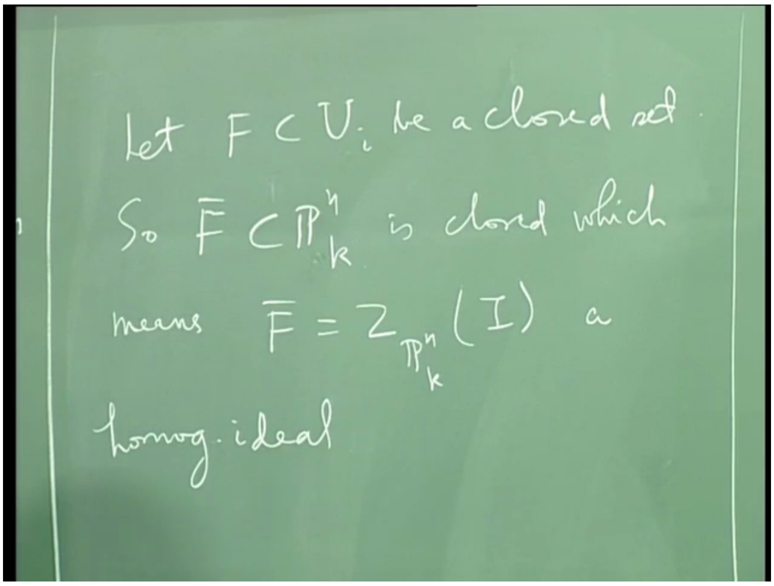

So let me explain that, so what you do is let F in U, U y be closed be a close set so F bar in P n it is the , this is the ambient projective space in which U i is sitting is closed which means F bar is the zero set in P n of I here homogeneous ideal ok and now what you do is that.

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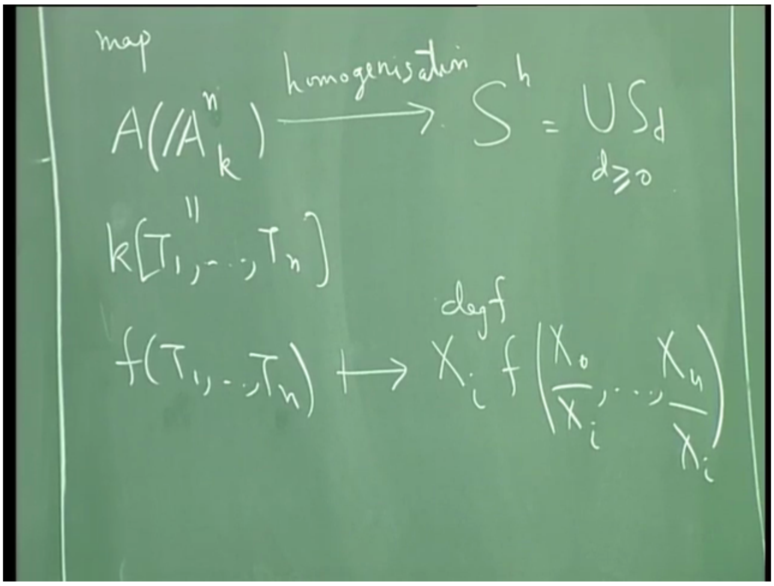

So you know take, define a map from the fine coordinate ring of A n which is K T1 etc T n to the set of homogeneous elements in H. this is just union of S d D greater than equal to 0, and what is this map. So this map here is homogenization, so this is the homogenization map and what is the homogenization map. If you give me a polynomial in T1 etc T n then this polynomial in T1 etc T n has a certainly it has a maximal degree ok. Now what you do but this has only N variables, so what you do is that you homogenize it namely you make every monomial equal, you make the degree of monomial appearing in this polynomial to be equal to this highest degree by adding as many you know the required power of a new variable ok and that is the new homogenizing variable that you introduce ok.

And so the homogenization is I put Xi power degree F, F of X knot by Xi and so on X n by Xi ok. This is the homogenization process. So what you do is that if you do it like this you must realize that when F of, when I plug in for T1 through T n the X knot by Xi, X1 by Xi and so on upto X n by Xi offcourse omitting Xi by Xi ok, then I will get a polynomial will have the Xi's in the denominator ok. I get a polynomial in (X) Xi by X j I mean rather polynomial in the X j by Xi where J varies and I is fixed.

But then my multiplying by Xi to the degree F I clear all the denominators ok. So what you must understand is that is polynomial is an homogeneous polynomial of degree equal to degree of F ok. So this method, this is called homogenization alright and there is a map like this direction which is called de-homogenization, so there is a map called de-homogenization and that is very simple.

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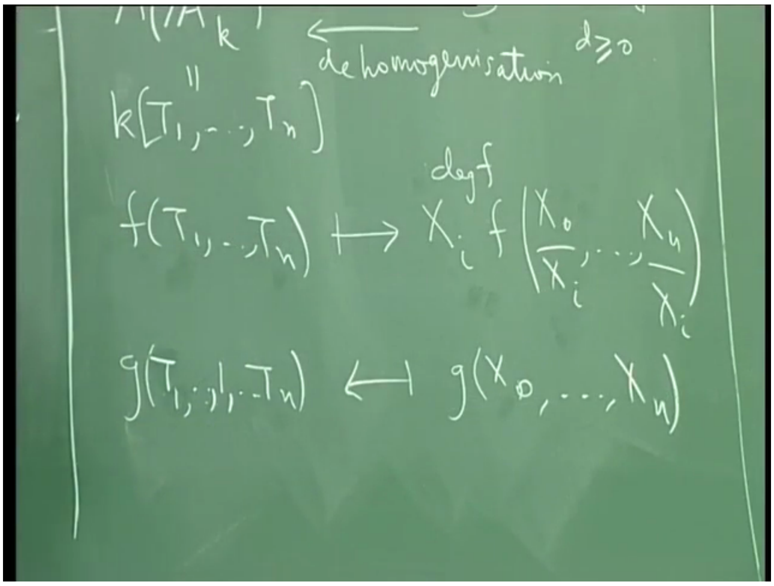

You give me a polynomial G X knot etc Xn and I do the obvious thing I simply send it to G X knot etc, G T knot I mean G T1 etc.

I put one where I have Ti then I put T n ok. So this is called de-homogenization. So what you do is there are these variables X knot to X n ok now for Xi you put 1, ok if you put 1 for the Xi then you get the remaining polynomial is only a polynomial in X knot to X n without Xi so it is only N variables and these for these N variables you put (three) T1 through T n in that order ok. This is called de-homogenization and the fact is that let me call this as F sub-H the subscript H mean being homogenization and let me call this as G sub-D H which means the de-homogenization ok, instead of giving names to these maps.

So any F which is the polynomial in the Ti's is homogenize to give an F H which is a homogeneous polynomial in the Xi's conversely you thought of a homogeneous polynomial in the Xi's you can de-homogenize it to give G sub-D H is inhomogeneous polynomial not necessarily homogeneous polynomial in Ti's ok. So you have this map and the reason I want, the reason I have tis map because you can now check it is a very simply set theoretic exercise to check that you know if you take, so I have now see I am trying to check that this map Phi I is closed ok.

So I started with an F which is closed here and I took its closure in the full space and this F closure is the zero set of an ideal it is a homogeneous ideal so now what I am going to do is I am simply going to take, since it is a ideal it is generated by homogeneous elements alright. So what I am going to do is I am going to take those homogeneous elements and I am going to just de-homogenize it ok. So essentially what I am doing is, see the map is only from the homogeneous elements to the homogeneous elements, so what you do is this ideal breaks up into a direct sum of its homogeneous pieces and so the ideal is equal to I intersection S d, direct sum of I intersection S d's and each I intersection S d is a subset here and it takes its image there ok.

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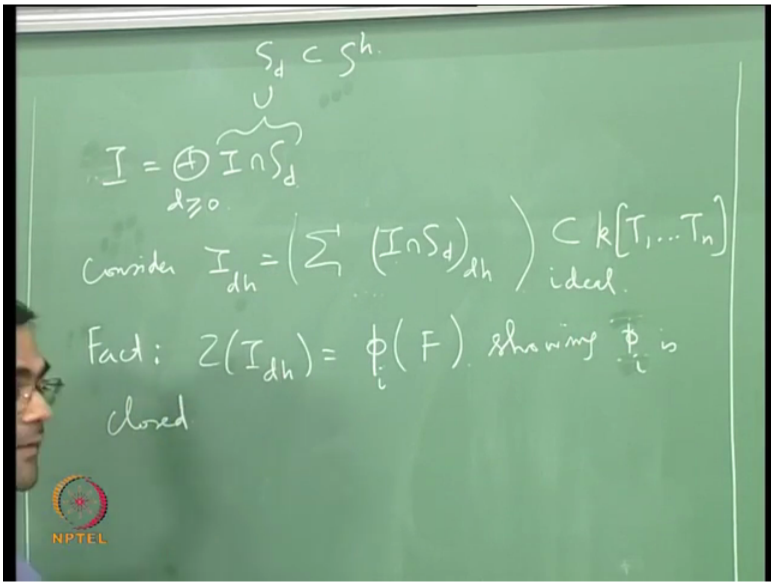

So I is direct sum I intersection S d, d greater than equal to zero, this is because I is homogeneous ok and consider I D H which is definition is summation of I intersection S d, see (my) mind you I intersection S d is this I intersection S d is sitting inside S d, which is sub of F H is a homogeneous elements and on S H I have this de-homogenization map ok. So what I do is I simply take I intersection S d and I de-homogenize it ok and then I take the sum ok.

Let me for safety take the ideal generated by that alright and the fact is so this bracket is suppose to mean ideal generated by this alright and the fact now is that the image of the F under this Phi I is I have to verify it is a closed side, and what is the closed side? It is just the closed set of this ideal mind you this ideal is an ideal in on the left side, it is an ideal in K T1 etc T n, this is an ideal ok, this is an ideal there and so it defines a closed set and the fact is, what is that closed set? That close set is actually the image under F, the image under phi of F.

So fact is E Z of I D H is equal to phi of F, which shows that phi is closed ok, phi I of F showing phi I, phi I is closed ok and since I have written down all these, it is also easy to say why phi I inverse is closed and phi I inverse is (also) closed is also easy to check in the same way. What I will have to do is that (which) I will have to start at the closed set here and show its image under phi I inverse is a closed set there and what will I do it is very simple. A closed set here is given by an ideal, it is given by an ideal in T1 etc T n and then I take this ideal in T1 etc T n and simply homogenize it.

I will get an ideal, I will get an homogeneous ideal and the zero set of that homogeneous is going to give me a close subset of T n if I intersect it with U i that is going to be the image of this closed set under phi I inverse and that is how you show check that phi I inverse is close ok. So let me write that also. So this is something that you can check ok, so instead of saying fact I should say check.

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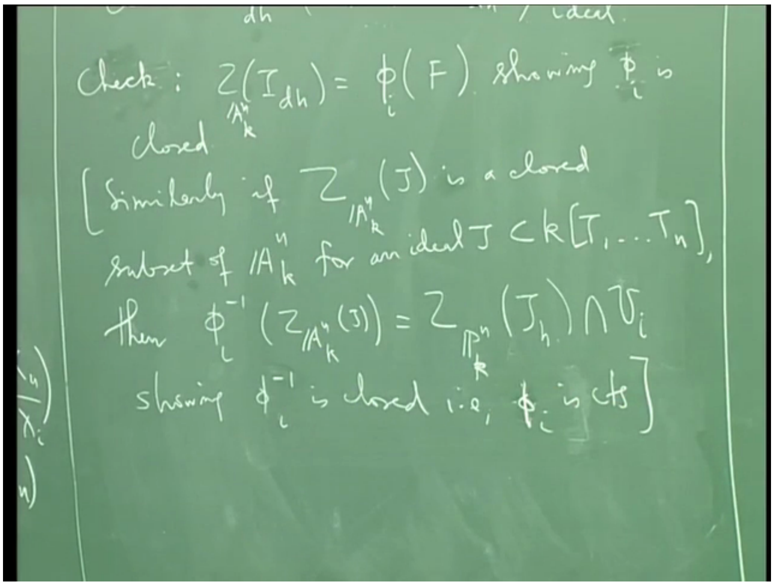

Similarly if E Z in, so mind you this set is Z this Z I an A n, this Z is the zero set in A n. So similarly if Z in A n of J is a closed subset of A n for an ideal J in K T1 etc T n then phi I inverse of Z in K j is simply Z of, this is Z in P n of the homogenization of J which is J h ok, this is the homogenization of this ideal and it is that intersection with U i showing Phi I inversed closed that is phi I is continuous ok.

So once you understand this homogenization de-homogenization, you see clearly that is what is going on at the you know algebraic level in fact to be more precise what is happening is that you know once this fact has been proved, what it is going to say is that U i isomorphic to each U i isomorphic to A n as affine variety therefore you expect that the ring of regular function on U i is the same as polynomial ring in N variables ok.

So that should give you some (isomoring) isomorphism and you should have a nice ring description here ok. That also can be checked, so but before I do that let me complete the proof of this fact I needed to check that phi is a inverse is a morphism I have already checked that phi I inverse is continuous I will have to only check that phi I inverse pull back regular functions to regular functions.

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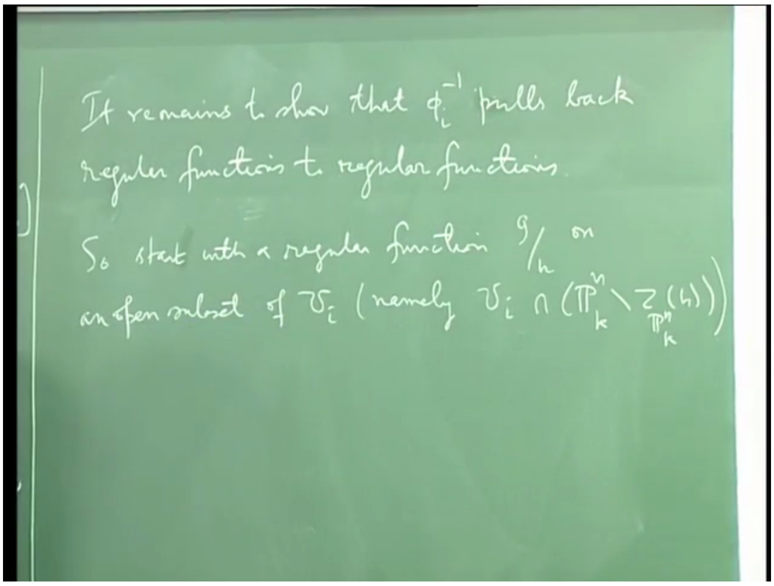

So it remains to show that phi I inverse pull back regular functions to regular functions and that would make phi I inverse a morphism and that together with the fact that phi I is a morphism will tell you that phi I is an isomorphism.

So what should I do? So I have to show that this map pulls back regular functions to regular functions so I should take a regular function here, I should compose with this map and show that the resulting function here is regular. So that is also easily return in terms of the de-homogenization map ok. So start with regular function let me make sure that I am not messing up some notation. Start with a regular function G by H on a open subset of U i namely U i intersection projective space minus zero set in projective space of H ok.

So you know regular function on a quasi-projective variety or projective variety is just quotient of homogeneous polynomials of the same degree. So I start with a regular function G by H on a n open subset of this, ok.

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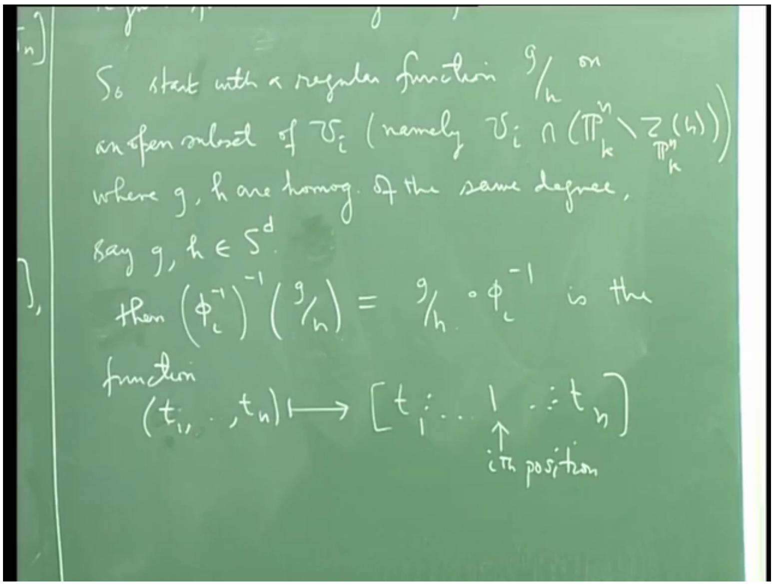

Where G H are homogeneous of the same degree say G , H are in some degree D part ok, D offcourse I will assume now what you have is, you look at, so look at then phi I inverse of G by H ok, so this rather confusing notations so you know I wanted to go back and look at this diagram, phi I inverse is in this direction ok, my regular function G by H is here, it is define on this I compose it with phi I inverse ok then I get the pullback.

So this means this is simply, it simply means that you first apply Phi I inverse and then apply G by H and if you write it out is the function, if you give me a point with coordinates T1 etc T n what I am supposed to do is, I am suppose to apply phi I inverse to it. So the point to which it will go is, is going to be T1 dot-dot 1 T n, (this is the), these are the homogeneous coordinates, where this one is in the ith position, this is the map which is Phi I inverse ok.

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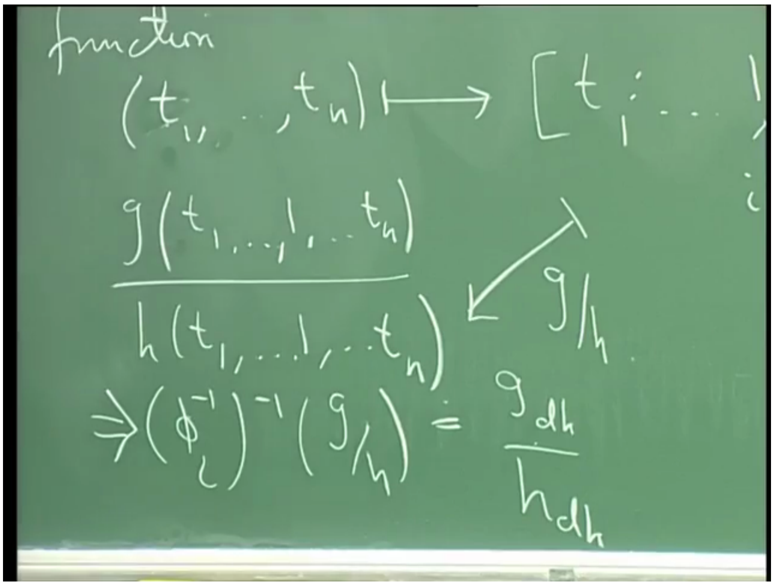

And then what I am going to do is I am going to apply G by H to this ok and so what I will get is I will get G of T1 etc 1 and T n by H of T1, 1, T n ok, this is what I will get.

But then, what is G1, G of T1? If I put G of T1 etc T n with one in the (ith) T ith position then this is the de-homogenization of G I of G. So this is actually so what this tells you is that phi I inverse of G mod H is nothing but the (G) de-homogenization of G by the (D) de-homogenization of H, which is offcourse regular, which is regular on D of H D H, so I am done. So I am just saying that I mean this is if you write it down it seems a little complicated but it is not, it is quite straightforward.

I am just saying that if you give me a regular function here which is the quotient of polynomials ok. If I pull it back the function I get here is simply the same quotient of not the same polynomials but the corresponding de-homogenized polynomials. But since it is again a quotient of polynomials it is a regular function on affine space because regular function on affine space is supposed to be define by locally by a quotient of polynomials.

So it is pulling back regular functions to phi I inverse is certainly pulling back regular functions to regular functions and what is helping is, this language of homogenization and de-homogenization ok. So that completes the proof or the fact of the fact that Phi is an isomorphism ok. So that completes the proof of this fact and then let me also tell you once this is done.

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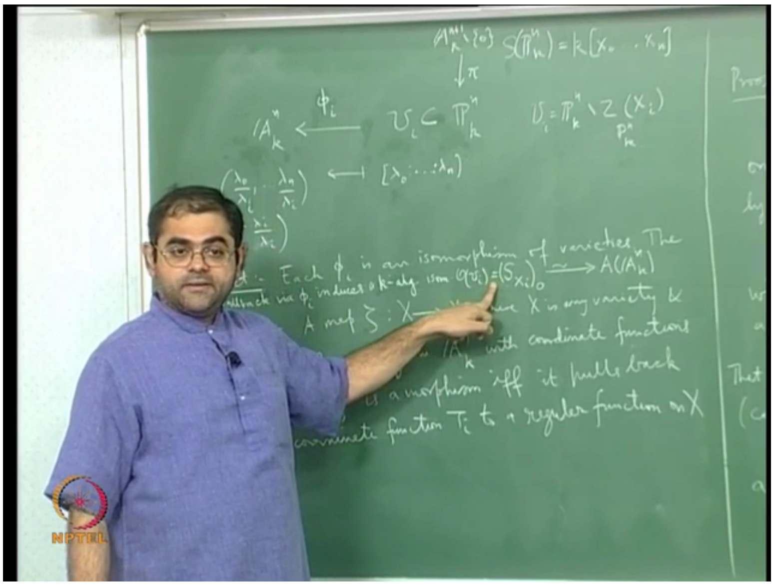

So let me add one line to this ok the pullback via phi I induces an isomorphism, induces a K algebra isomorphism from the regular functions on U i which is the following you take these this polynomial ring in N plus variables the homogeneous coordinate ring of projective space you localize it at Xi, S sub Xi is the localization of Xi ok and then you take its degree 0 part ok.

See localizing Xi means you are inverting Xi, so any element in the localization will look like your polynomial by a power of Xi. Now foe such a polynomial for such a quotient name, your polynomial by a power of Xi you can define degree to be the degree of the numerator polynomial minus the degree of the (power) minus the power of the Xi that is occurring in the denominator. With this degree definition you can check that the localization S sub Xi is also a graded ring and you take its degree 0 part ok.

That will be a K algebra, that K algebra is exactly this that is exactly the ring of regular function on U i ok, and so in other words I am just saying the regular functions on this U i are simply quotients of they are globally given by quotients of polynomials with the numerator polynomial being a homogeneous polynomial, the denominator polynomial being a power of Xi whose power is equal to the degree homogeneous degree of the numerator polynomial.

So the only regular functions on this U I, global regular functions are the form G by Xi power N, G by Xi power T ok, where T is the degree of G, they are the only (regular) global regular functions on this and that is what you get when you take the this homogeneous coordinate ring localize at Xi namely invert Xi and then take the degree zero part for the induced gradation ok and this you will get isomorphism of this K algebra with the affine coordinate ring of a fine space. This is what pullback of regular functions will induce an isomorphism from the ring of regular functions here to the ring of regular functions here ok.

And that is described like this, ok and you can check this (statement) this equality ok it involves a little bit of region checking that the global regular functions on U i is given by this ring and you can also check that this map is given by the pulling back of pullback of regular functions from the ok I think my directions are wrong. If I want to use phi I then I must pullback regular functions from here to here, so my directions are wrong.

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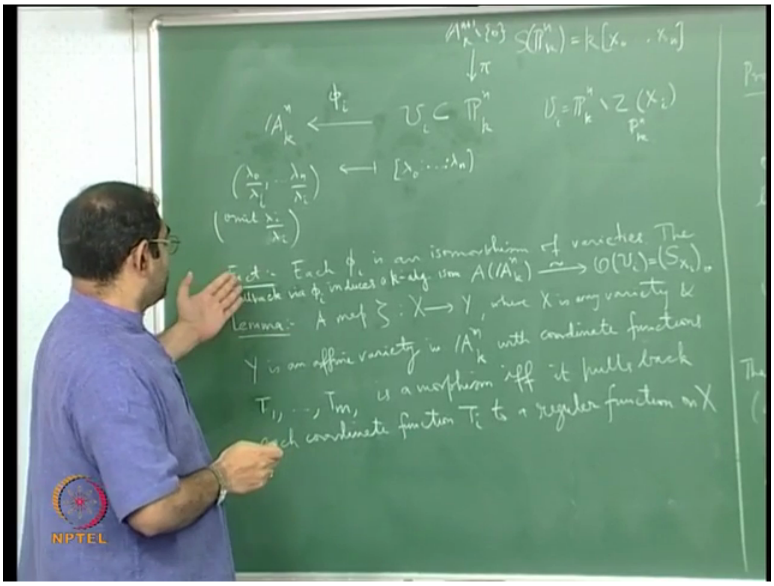

So maybe what should I say is I should rather change this to Phi I inverse or if I want to use phi I, I should change this, so let me change it and phi I induces a K algebra isomorphism from A of A n to the regular functions on U i which is S localized at Xi and you take degree zero part. So this is the, so the fact is so let me repeat it not only is each phi I an isomorphism but once you know phi is an isomorphism I am just saying that the variety U i, which originally was a quasi-projective variety is actually isomorphic to affine variety.

Because it is isomorphic to this A n it is infact in affine space and you know that whenever you have an isomorphism of affine varieties it has to induce an isomorphism of the corresponding regular functions and therefore this isomorphism by pullback should give me an isomorphism regular functions here to the regular functions here.

The regular functions here are just the polynomial in N variables and the regular functions on U i are you can check exactly the homogeneous localization of this graded ring with respect

Xi which means you localize with respect to Xi and take the degree zero part. Which means you are taking quotients of the form homogeneous polynomial of certain degree by Xi to that degree as raised as a power ok and you will get an isomorphism like this right and this map is also described in terms of (homoge) this map and is inverse as you can check is also described in terms of homogenization and de-homogenization you can write it down right. So with that I will stop.