**Basic Algebraic Geometry**
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**Module 10**
**Lecture 24**
**Translating Projective Geometry into Graded Rings and Homogeneous Ideals**

Affine space automatically gives you a geometry on the projective space. So what this tells you is that philosophically the geometry of the projective space is controlled by the geometry of their affine space and that is going back to (())(2:10) ok philosophical yeah.

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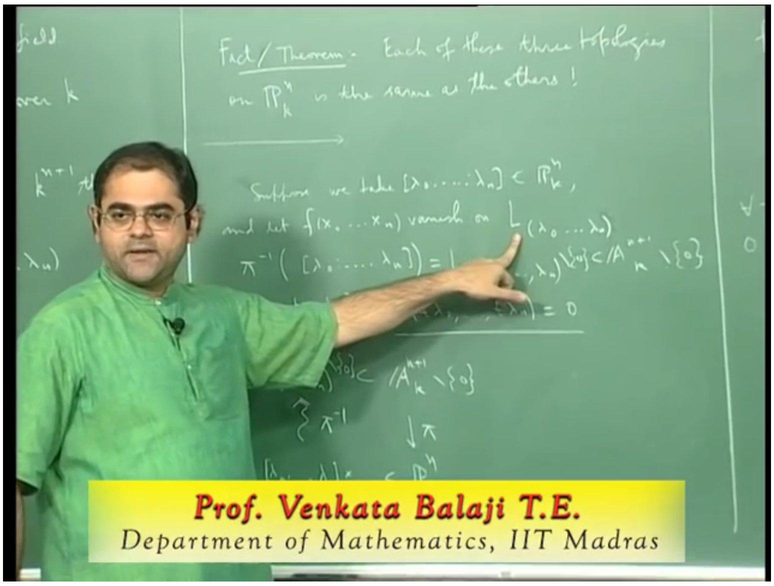


Ok so, so you know lets recall that if you take a point in N dimensional projective space and you take a polynomial vanishing on the line above that point ok in the affine space N plus 1 dimensional affine space then we saw that every homogeneous component of that polynomial will also vanish on that line and in particular that polynomial will not have any constant term ok.

So it will be a some of homogeneous components and there won't be any homogeneous component of degree zero alright. Now this leads into the study of what I called as homogeneous ideals and the uhh when you try to translate from the algebraic geometry of projective space to commutative algebra you end up studying properties of homogeneous ideals ok and the key to defining homogeneous ideal is actually it comes actually from this observation. So let me explain that in more detail.

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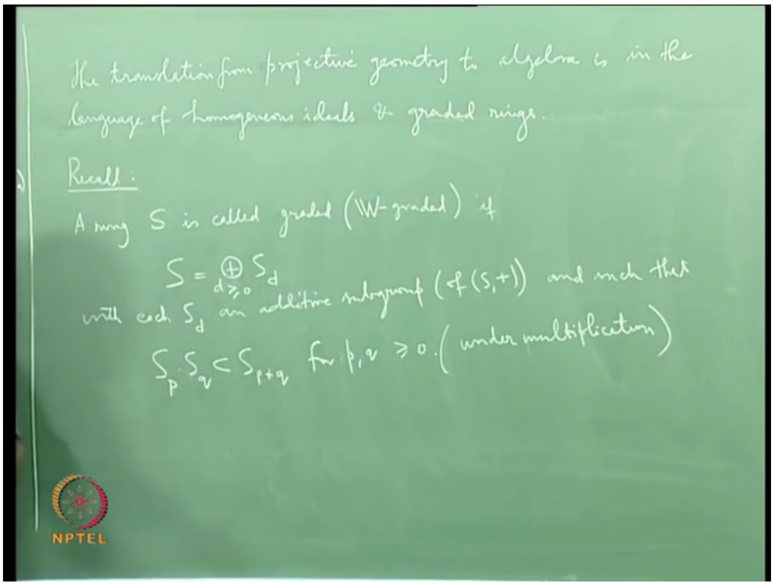


So the first thing is the translation from projective geometry to commutative algebra is in the language of homogeneous ideals and graded rings. So this is a little bit of algebra that one needs to recall alright. So recall the following things, the notion of a graded ring first, a ring S is called graded which is suppose to mean N graded or rather whole numbers graded ok, if a S is the direct sum of S d, d greater than equal to 0 ok, with each S of d and a abilian fact offcourse a abilian, a sub-group an additive sub-group of S and such that S p into S q lands inside S p plus Q for p q greater than or equal to 0 ok.

So under multiplication, so you see what is a here the W corresponds to whole numbers which means that you include zero and all the natural numbers which star from 1 ok and this indexing is on the whole numbers alright and the ring should break into a direct some of pieces each piece is additive sub-group of the, the additive group underline ring ok and there is a multiplication in the ring, the multiplication should the multiplication of the Pth piece, the Qth piece should land you inside the P plus Qth (P) piece ok.

And offcourse you know well if my ring could my ring need not have 1 if you want when I want to make a general definition it need not even have 1 and did not even be commutative and in that case I will also it also follows that S q into S p will also land into (S) P plus Q by this ok and therefore the Pth piece if you take an element in the Pth piece and the element in the Qth piece and you multiply them, they land in the P plus Q piece ok.

We say that this is meant to, this is also sometimes refer to us the multiplication this was the gradation ok it respects the gradation. It takes a, so what you do is you think of elements of S d as elements as homogeneous elements of degree D ok. So the elements each S d are called homogeneous of degree D ok and what you are saying is that every element of the ring can be decomposed into homogeneous elements of certain degrees and the decomposition is unique. The uniqueness of the decomposition is because the direct sum ok.

The direct sum tells you ok that every element of this can be broken down can be written as a sum of finitely many elements which have, which are homogeneous, which belong to certain homogeneous pieces and for each (homo) the each component the each sumant in that sum is unique, for that homogeneous. So the if you give me an element here the its for any D its Dth homogeneous pieces uniquely determined.

That is what the direct sum suppose to mean alright and the multiplication preserves the uhh it respects the homogeneity in the sense that homogeneous element of degree p multiplied by homogeneous element of degree q leads to a homogeneous element of degree p plus q ok. So this is the definition of what graded ring is offcourse.

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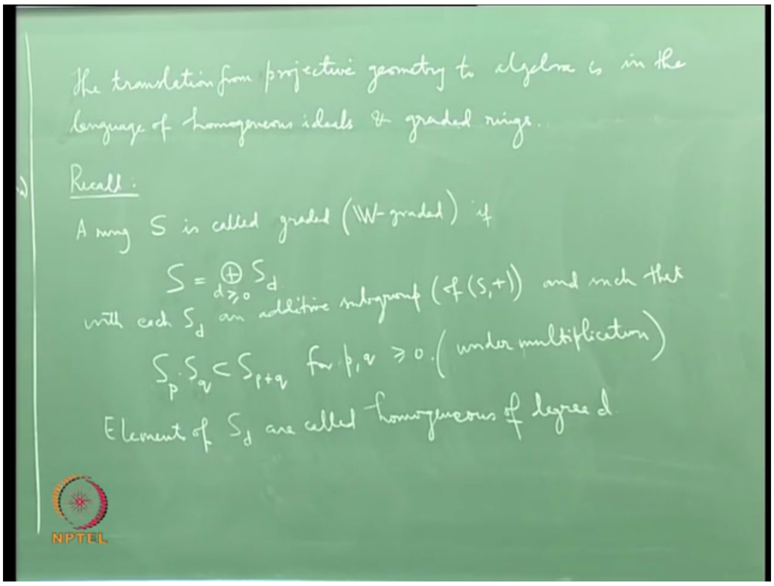


So let me write this elements of S d are called homogeneous of degree D alright and offcourse the particular case at we are interested in is polynomial rings and their quotients by prime ideals, their quotients by ideals which are homogeneous ok.

So what is the basic example, the basic example is offcourse the polynomial ring in finitely many variables I will take the variables to be N plus 1 variables because I am always thinking of projective space.

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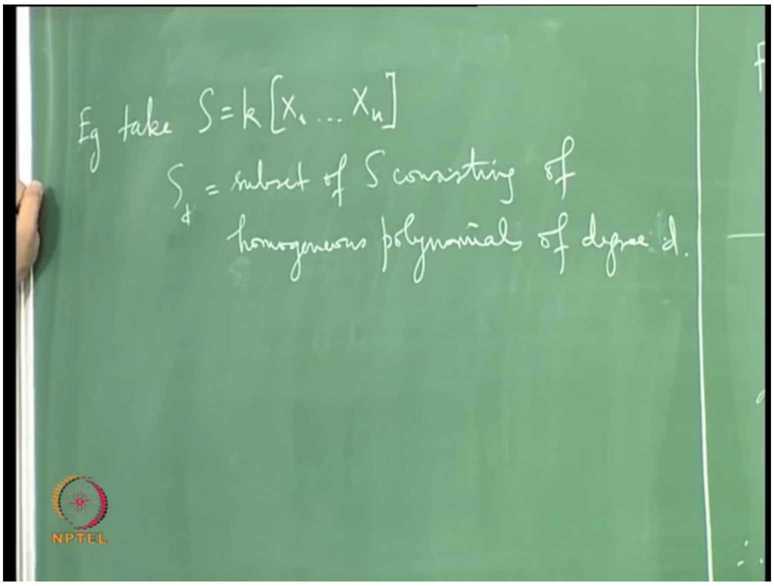


So example take S equal to K X knot etc X n S d is equal to subset of S consisting of homogeneous polynomials of degree D ok and so you know that the whole polynomial ring is a direct sum of homogeneous polynomials of various degrees that is just reflection of the fact that you take any polynomial you can break it down uniquely into homogeneous components each component a homogeneous polynomial of certain fixed degree ok.

And this is the example that we keep in mind ok and offcourse it is not just to what its just not enough to work with this but we need to also work with graded quotients of this ok. So for that so it is a you know the intuitive  area is very clear if you to get a quotient you have to go (())(11:56) ideal ok but to get a graded quotient ok you have to go modulo what is called a homogeneous ideal ok. So what is a homogeneous ideal?

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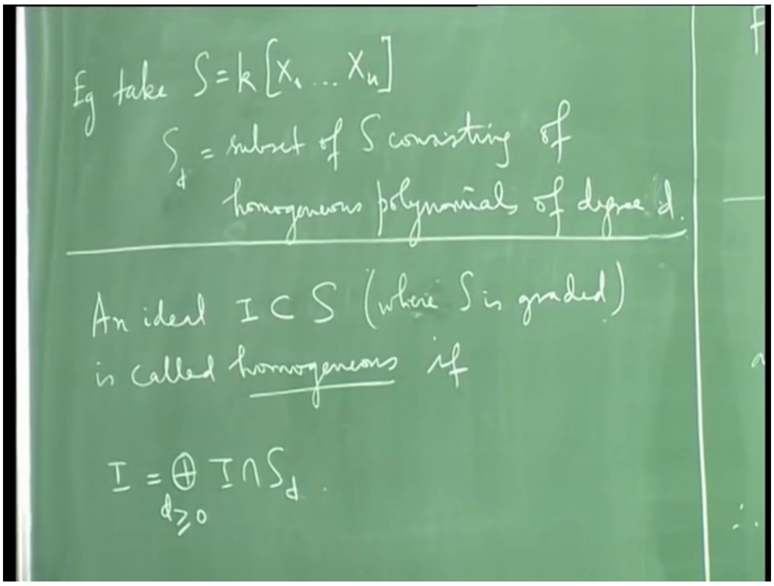


An ideal I in S which is graded where S is graded so I will draw a line here ok will draw a line here and I am again going back to the old situation where I take a graded ring which is direct sum of homogeneous pieces ok.

I should say it is a direct sum of pieces which corresponds to homogeneous elements of certain fixed homogeneous degree ok. So take an ideal S an ideal I C S it is called homogeneous if I is I intersection S d direct sum D greater than equal to 0. So look at this definition of what a homogeneous ideal is. So definition is you take the ideal alright you intersect it with S of d, when you intersect the ideal wit S of d what you get is mind you not an ideal.

Because you are only intersecting with a additive sub-group and you know the ideal is also an additive sub-group therefore the intersection is again additive sub-group therefore each of these is an additive sub-group of the additive group underline their ring S ok, and now you take the direct sum ok offcourse it is a direct sum because all the S d's themselves are pieces of direct sum and you take the direct sum and that you should it is obvious that this will be contained inside I ok.

The right side every piece ok I intersection S d refers to all the elements of I which are homogeneous of degree D. what is an element I intersection S d? It is an element of I which is in S d but elements of S d are called homogeneous elements of degree D. So I intersection S d is, those elements of I which are homogeneous of degree D ok and offcourse if you take a direct sum of this mind you the direct sum an element in a direct sum only consist of a finite sum, even though that direct sum is over a collection of infinite, infinitely many sub-scripts ok.

So an element here is certainly here by definition but the requirement is every element here comes from here, that is the homogeneity definition ok. So you know what it means?

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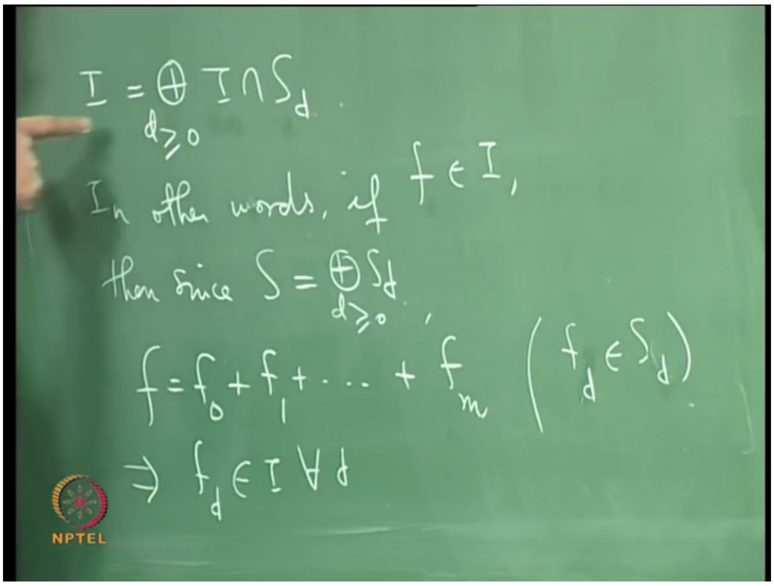


It means that so, see in other words if F is in I ok, if you take F is in, F is an element of I then since S is a direct sum of all the S d's D greater than or equal to 0, what you will have is F will be F knot plus F1 plus etc upto F m ok you will get this alright, where so you will get a finite expression like this ok.

You get a finite expression like this because it is an expression in direct sum it will leave only upto a finite index ok, beyond this all the F G S will be zero right. So here F I or in S I or F D is in S D you get a breakup like this ok and so if you take an F and you break it into homogeneous pieces then the condition is that this condition will tell you that each F I is also an F ok. So you see the, so implies that F d belongs to I for every d for every index d ok. See why is that true? That is because you see take an F here, F is because F is in I and I is in this graded ring and this graded ring has its graded decomposition, F has a decomposition alright.

Where each of this pieces come are homogeneous of the corresponding degrees alright. On the other hand since you have written like this F also has a decomposition here because of the equality F belongs here. So it corresponds to an element here. So it also is a, it also has a decomposition in terms of homogeneous elements ok but both decompositions are to be valid in S but is S there is only one decomposition, the decomposition S is unique, therefore what it forces is that each F d is already in I.

So the moral of the story is an ideal is (homogeneous) saying that an ideal is homogeneous is the same as saying that every element of that ideal you take any element in that ideal, every homogeneous piece of that element is also in that ideal. One way of saying that ideal is homogeneous is saying that you take any element in that ideal ok then every homogeneous piece of that element is also back in that ideal ok and you see that is exactly geometric, that is exactly algebraic reflection of this geometric fact.

If you're polynomial vanishes on a line then every homogeneous component of that polynomial vanishes on that line. So what you are saying is that if the line is in the zero set of an ideal suppose you take the line to be in the zero set of an ideal ok and you take an element of that ideal that means that is a polynomial which vanishes on that line then what you are saying is that every homogeneous component of that polynomial is also vanishing on that line so if the zero set of an ideal contains a line what you are saying is that every element in that ideal ok, every polynomial in that ideal is such that each of its homogeneous components is also again in the ideal of that line ok.

So this is just a geometric reflection of this algebraic fact. So this is the key to defining a what a homogeneous ideal is and the advantage of having a homogeneous ideal is that once you have a graded ring and you have a homogeneous ideal the quotient ring S mod I automatically gets a graded structure, it becomes a graded ring ok. So the key to translating project from projective geometry to commutative projective algebraic geometry to commutative algebra is that you have to change uhh from ordinary ideals to projective ideals from ordinary ideals to homogeneous ideals and you have to change from ordinary rings to graded rings.

So the language instead of just looking at rings and ideals commutative rings and ideals in them, the language becomes the language of homogeneous ideals and graded rings ok. That is the language that you should use that is the algebra that you should use for projective algebraic geometry ok and well now the fact is, so let me tell you what happens there are a few nice facts.

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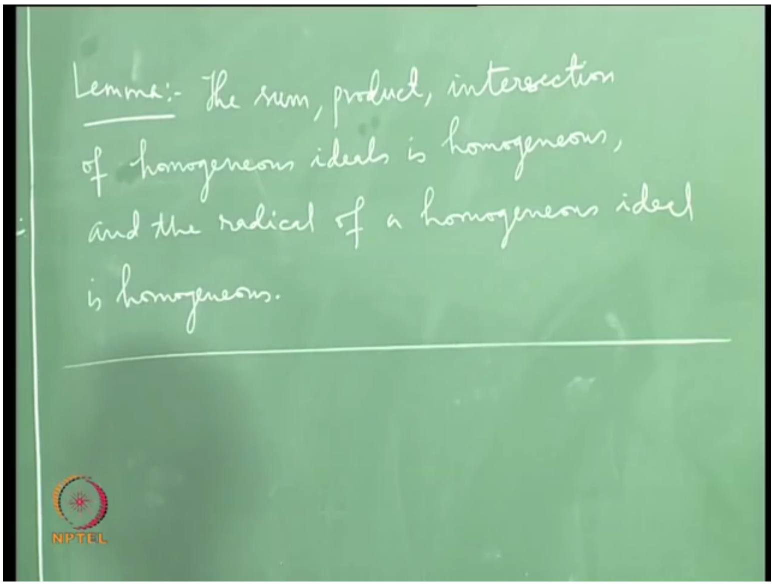


So lemma there is a lemma that you can easily check the sum product intersection and radical, so the sum, product, intersection of homogeneous ideals is homogeneous and the radical of a homogeneous ideal is also homogeneous ok.

So this collection of homogeneous ideals in a graded ring is a well it is well behaved under the operation of taking sum product intersection and radical ok. This is a I mean this is a very straightforward verification algebraic verification which I leave you to do ok and therefore you know the, so you know the with armed with this we can now translate from projective geometry projective algebraic geometry to commutative algebra.

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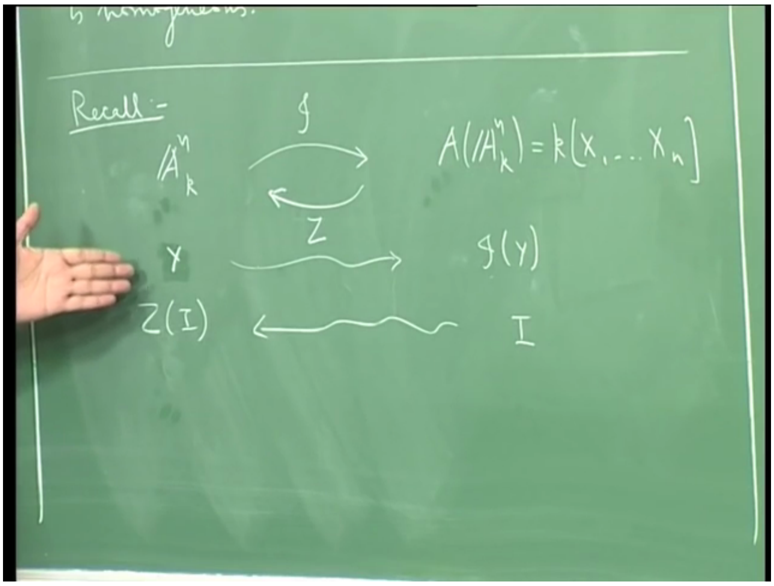


So you know let me recall that as far as affine geometry was concerned what we did was if you recall we had affine space ok and we have this is the geometric picture the algebraic picture is the coordinate ring of affine space the algebraic picture is the coordinate ring of affine space is polynomial ring in N variables ok and you know well you had a map like this which is called as I and you had a map like this which is called as Z and what did these maps do?

Well if you give me a subset T or if you give me a subset Y of affine space then I get I of Y the ideal of functions polynomials that vanish on Y and conversely if you give me an ideal I here in the polynomial ring in N variables I get the close subset Z of I and every closed subset if of this form ok and you know that so you get a correspondence between closed subsets here and on that side you have to take radical ideals ok and we had things like, so on this side if you take sub-varieties affine sub-varieties which are reducible algebraic sets they corresponded on that side to prime ideals which were offcourse radical ideals and points here will correspond to maximal ideals there ok.

So we had this nice translation from a algebraic geometry to commutative algebra alright, this is for the affine space. Now we go to, we can do that now for the projective space as well right. So in the same way what we do so I will have to make a statement here, so here is one more lemma which I forgot to mention probably.

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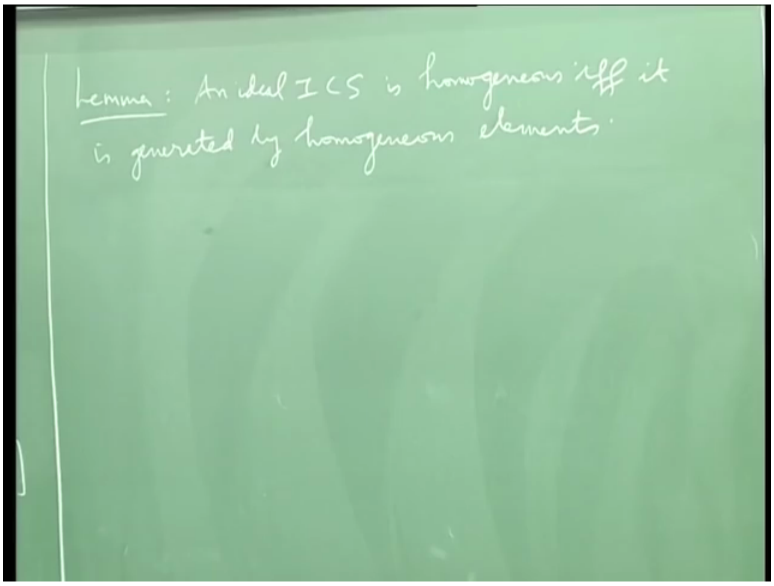


Let me mention it here, an ideal I C S is homogeneous if and only if it is generated by homogeneous elements. So this is another definition of when an ideal is homogeneous. This definition of an ideal being homogeneous requires its generate has to be homogeneous ok. So earlier definition of homogeneity is that you take the ideal is a sum of its components homogeneous components and which translates to saying that given any element in the ideal each of its homogeneous components is again back in that ideal ok. So this is again a simple algebraic fact that you can verify as an exercise ok and the reason I need it is the following.

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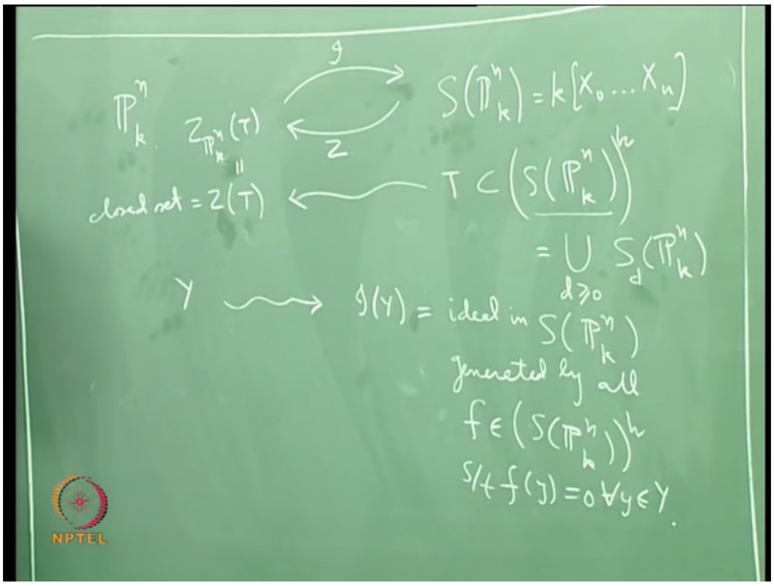

Recall that we have defined in on the projective space ok we have defined algebraic sets closed sets as zero sets of a bunch of homogeneous polynomials ok and again instead of just taking the zero set of a bunch of homogeneous polynomials you can take the zero set of the ideal generated by this homogeneous polynomials and by this lemma that ideal will be homogeneous ideal ok. So what you will do is so will do the following thing, will use S will use this rotation, S of P n this is the commutative algebraic picture this is K X knot to X n ok.

So this thought of this is called the homogeneous coordinate ring of projective space ok, projective N space. See in the affine situation ok, A we use the word A to give you the affine coordinate ring ok, which is a number of (pol) which is a polynomial ring in as many variables as the dimension of the affine space. Now what you do is in the projective case the analogue is the so called homogeneous coordinate ring ok and you know why they are called homogeneous coordinates because when you write a point in projective space you these coordinates are only when you put them together they are only common ratio ok.

I mean they are given by set of ratios, that is the reason we put a colon, a point in projective space has coordinates X knot colon X1 colon etc X n and the colon means that there is a ratio involved ok and therefore it is a that's why it is called homogeneous and that is why this is called the homogeneous coordinate ring ok and for that matter each Xi is a homogeneous polynomial of degree 1 right and what we do is well, how do we start we say close set is of the form E Z of T where T is a in S it is in this I will put a I will put this H ok, which means the union of the various degree D pieces.

Namely all the possible homogeneous elements ok. See this, this is homogeneous coordinate ring is direct sum of its degree D pieces ok. Which is just trying to say that polynomial at degree n is uniquely expresses expressible as a sum of its homogeneous components but what you do instead of taking if you take a direct sum you will get the homogeneous coordinate ring. Instead of taking the direct sum if you take union will get all the homogeneous elements.

Because by definition a homogeneous element is suppose to be an element in one of this pieces ok. So where offcourse S d of P n is homogeneous polynomials of degree D in this variables that is what it means ok. So you take what I am doing is why I am writing it like this is, I am taking a subset of homogeneous elements I am taking, my T is a is not just any bunch of polynomials in this polynomial ring, it is homogeneous elements, that is the reason I have put the sub-script the superscript H ok.

And that is just gotten by taking this union ok and what you do is that for this T you take the zero set of T but now you see you are taking the zero set in projective space ok. Mind you sometimes if you are working with both the affine space and the projective space at the same time you will have to worry about where you are taking the zero sets you need better notations. So sometimes it is if you don't want any confusion you put Z sub P n of T which means you are looking at the zero sets, the zero set of T in P n ok.

And this is how the close sets in projective space are defined, this is how the Zariski topology is defined ok. This was the second definition ok. We had three definitions of Zariski topology. The first one was a as quotient topology of the punctured N plus 1 dimensional affine space above, the second one is this where the closed sets are given by zero sets of bunch of homogeneous polynomials and the third is offcourse the topology that is gotten by gluing the N plus 1 pieces though we each of which is are which is isomorphic to an affine space ok of dimension n.

So well so this is how we have defined it ok and now what we can do is well, so you have this just as in this case you have this map Z ok and there is also this map in this direction, what is this map in this direction? In the affine case you give me any set Y then you look at all those polynomials which vanish on Y ok and as you go like this and then this is automatically an ideal here. So I land on the collection of ideals on this side ok. So you know I also need to put an I here right and you have to be careful that you should simply not say all the polynomials here which vanish on a given subset here.

Mind you if a polynomial vanishes on a subset then it has to be homogeneous I mean each homogeneous piece of that polynomial has to vanish on that subset. You see what we just saw sometime ago was that you know if a polynomial vanishes on a line passing through the origin ok then each piece of that polynomial each homogeneous piece of that polynomial will also vanish on the line through the origin. So it means that if you're so you must think of the polynomial vanishing on a line on the origin on a line through the origin as you must think of it like this, take the point in projective space corresponding to that line and that point is a zero of that polynomial in the projective space.

So what you are saying is if your polynomial vanish at a point in projective space then each of its homogeneous components will also vanish at that point in projective space and offcourse the constant term will not be there right. So if I want to make sense of a polynomial vanishing on subset of projective space I need to make sure that every , that first of all that I has no constant term and I also need to make sure that every homogeneous piece of that polynomial also vanishes on that subset of projective space alright.

So what you do is you see find the everything reduces to vanishing of homogeneous polynomials. So when you define this I you define it very carefully you, keeping this in mind you define I F Y to be the ideal in the homogeneous coordinate ring generated by all F in the all homogeneous F namely all homogeneous polynomials such that F of Y is zero for every Y in Y ok. So this is how you define, when you define the ideal of Y you define it as the ideal generated by all those homogeneous polynomials which vanish on Y alright.

So you see therefore this is an ideal which is generated by homogeneous elements therefore it is a homogeneous ideal because that is what the lemma above (())(34:30) says ok. So this is actually this is a homogeneous ideal. So what has happened is, if you start with a set of homogeneous elements you get the zero set of that which is a closed subset of projective space and if you start with any subset of projective space you get the ideal of that subset and that will be homogeneous ideal by definition and whatever happened here more or less will happen there except for one or two certainties.

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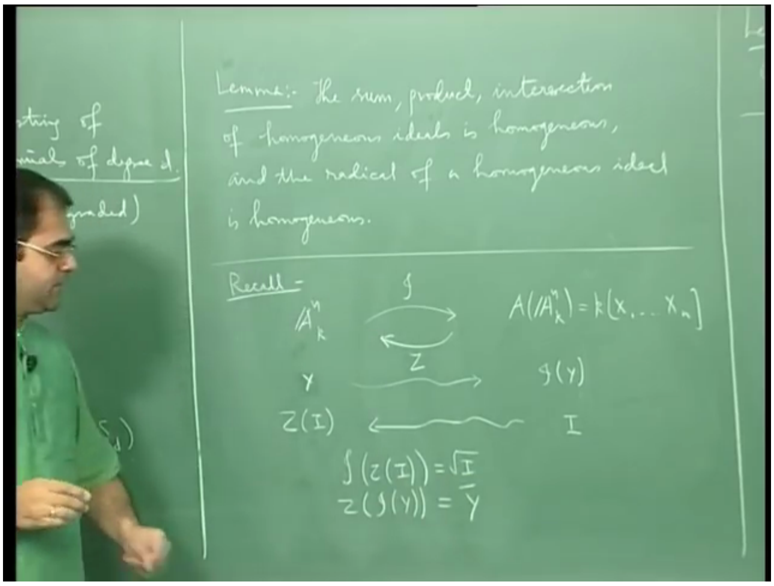


So let me tell you what are the things that would happen you know a few things in the affine situation what do you know? You know that you know if I take a uhh so the Nullstellensatz says that I of Z of I is rad I that is one fact ok then if I take E Z of script I of Y I will get Y bar the Zariski closure of Y so E Z of script I of Y is Y bar ok and the fact is this, is that the same thing will hold here except with one certainties for the Nullstellensatz  so let me state it here.

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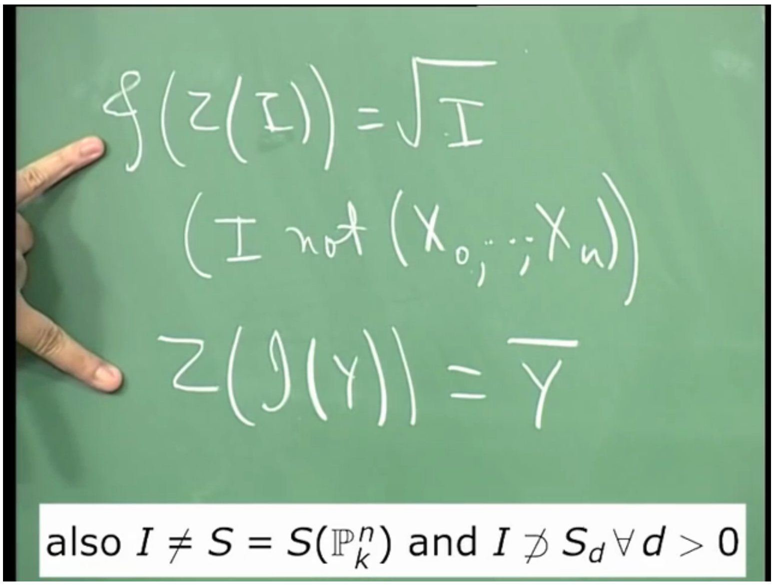


So what is going to happen here also I am going to get I of E Z of I is rad I ok and for offcourse I knot X knot etc X n so and well the other thing is E Z of I of Y will be Y bar ok. So both facts will be true here also except that here the ideal I start with should not be the maximal ideal corresponding to the zero in the affine space above because you know I have thrown it out, when I got the projective space below I have thrown out, I have taken the punctured affine space and then I have gone modulo and equivalence relation.

Namely I have taken the lines in the punctured affine space passing through the origin ok. I have thrown out the origin but the origin corresponds to this ideal in the affine space above the point 0, 0, 0, 0, 0 N plus 1 coordinates that corresponds to the maximal ideal candidate by the variables and this is the only ideal that you have to leave out, it is a maximal ideal but you have to forget and it will not, so that maximal ideal will not it is also a homogeneous ideal, because it is generated by the coordinates which are homogeneous functions.

They are all homogeneous of degree 1 ok. So it is a certainly a homogeneous ideal but the point is that it is a maximal ideal it is homogeneous ideal but it is not going to come into the picture ok. So on this side you are only going to consider homogeneous ideals which are different from this particular maximal ideal that is the ideal generated by all the variables and therefore this particular ideal generated by all these variables is given a very special name it is called irrelevant maximal ideal ok.

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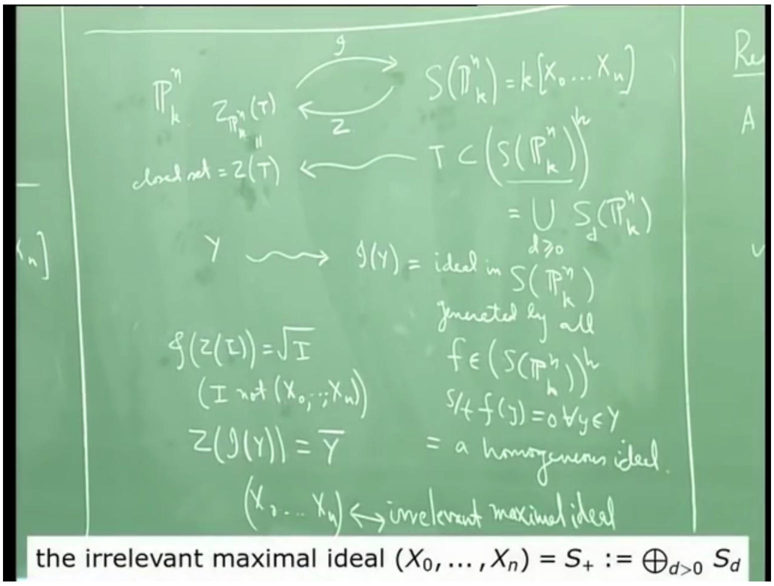


So there is a name for this, X knot through X n is called the irrelevant maximal ideal, it is the irrelevant maximal ideal ok and it is a homogeneous ideal but then the homogeneous ideals we are interested in are everything except that and that is why that is called irrelevant alright. It is irrelevant with respect to the projective geometry right and uhh so what I want to tell you is that you can prove the statements from the corresponding statements for affine space.

If you just remember that the quotient on the projective space is this is the quotient topology given Zariski topology on the punctured affine space above ok. So all the statements can be proved by translating everything to the affine space above ok and by using the corresponding results in the affine case. So what you must understand is you must understand the following, I mean this is a picture that should help you to think of what is going on.

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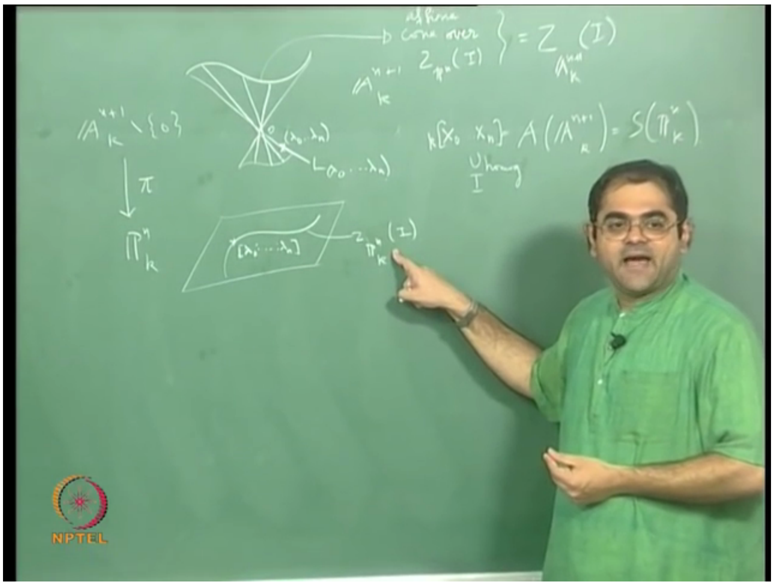


So you see you must always think of this is the affine space this is the punctured affine space and there is this projection onto the projective space ok and how you should think of it is that if you take if I draw a picture like this on the projective space and well I take the zero set of a homogeneous ideal here ok and this is a zero set in P n of this ideal ok.

That is how close subsets in projective space look like then how you should think of it is, if you take the affine space above so in the affine space above the diagram will be something like a cone so it will be, so this is the diagram in the affine space above ok and what has happened is that for each of these lines they go down to a particular point. So this line each of these generating lines L throughout certain point lambda knot through lambda N goes to the corresponding point in projective space (lambda) with homogeneous coordinates lambda knot through lambda N and this is to be thought of us simply the line above.

This is just L of lambda knot etc lambda N. so you think of this point as a line above ok. So what you will get is if you give me any projective any close subset of projective space take its inverse image here and then the only thing that will be missing is zero you which is what you will get when you take it closure you will get a closed subset there ok, zero is the only thing that will be missed. So if you add it you get this picture which is which you can easily think of as a cone over this closed subset in projective space.

So you see this thing is the cone it is called the affine cone over E Z, A n I, this is called the affine cone alright. So and you know so the picture is something like this, so if you give me any closed subset of projective space then you take the inverse image in the affine space above and close it up so that you add the origin, what you get is a cone above and what is this cone? What is it? This is actually this is none other than this is just the Z zeros of I in the affine space, it is the same I, take the same ideal, the same ideal mind you the ideal is an ideal in the affine coordinate ring of A n plus 1.

Which is thought of a projective coordinate ring, homogeneous coordinate ring of P n. note that A of A n plus 1 is s of P n and this is offcourse polynomial ring in these N plus 1 variables ok and I is sitting here ok. So if you start with I homogeneous here, the zero set is a close set in projective space if you take its inverse image and add the point zero you will get projective cone, it is called the affine cone. It is the cone in the affine space above and what is the affine cone? It is just the zero set of the same ideal considered as a zero set in the affine space above ok.

So any questions about Z I in P n can be translated to questions about Z I in A n plus 1 ok and then in affine space offcourse I know I have a good dictionary, I have the Nullstellensatz I have all that I need. So I use that to prove things in gets same as the projective case ok. So all so the point is somehow already they geometry that you know the affine geometry that you know that kind of helps you to get the projective geometry ok.

It controls the projective junction. So now you see and well so what you will get so there are two facts that I want to say, here you get a bijective correspondence between closed subsets and radical ideals. So if you look at this situation the projective space and the homogeneous coordinate ring you will get a bijective correspondence between closed subsets and homogeneous radical ideals ok and in that collection you will have to get rid of this particular homogeneous radical ideal which is his maximal ideal corresponding to the origin above which you have thrown out ok. So this is the irrelevant maximal ideal.

So what you get in the projective space is a bijective correspondence between closed subset of projective space on one side on the other side you will have to take homogeneous ideals, homogeneous radical ideals which are different from the irrelevant maximal ideal. You take the collection of all homogeneous radical ideals which are different from the irrelevant maximal ideal that is in bijective correspondence with the closed subsets of projective space ok.

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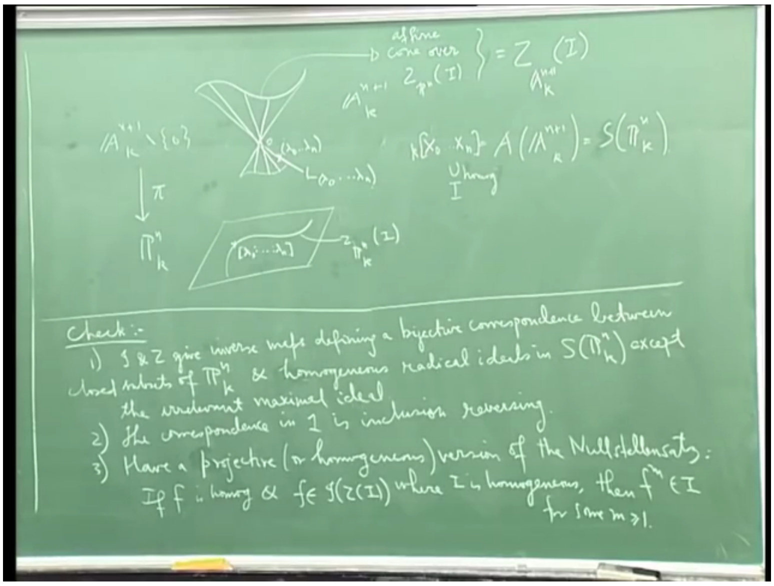


So let me write that, check number 1, I and Z give inverse maps defining a bijective correspondence between closed subsets of projective space and the set of and homogeneous maximal, homogeneous radical ideals in S of P n except the irrelevant maximal ideal. You will get this bijective correspondence ok. So the difference from the affine case is that there you simply consider I radical ideals here you consider homogeneous radical ideals and there you consider offcourse all ideals but here you consider you leave out that particular irrelevant maximal ideal ok and that is one thing.

Then the second thing is offcourse that in this case the correspondence in both directions is inclusion reversing it is an inclusion reversing correspondence because as the ideal grows bigger the zero set becomes smaller ok and conversely so the same thing happens here as well ok. So this is an inclusion reversing correspondence, the correspondence in one is inclusion reversing that is also true then offcourse whatever versions of the Nullstellensatz that you had for the affine case also have corresponding version of Nullstellensatz for the projective case.

What is the Nullstellensatz for the affine case? If a polynomial vanishes at every point of a variety then some power, if a polynomial vanishes on the zero set of an ideal then some power of the polynomial is in the (radical) some power of the polynomial is in the ideal that is the Nullstellensatz ok and the same statement will work for the projective case if you take homogeneous, but the only thing now you have to use homogeneous polynomials and you have to use homogeneous ideals. So if you have a homogeneous polynomial which is positive degree and if it vanishes on the zero set of a homogeneous ideal then some power of that polynomial is certainly in that ideal, that is the homogeneous version, projective version of the Nullstellensatz.

Again the projective version of the Nullstellensatz can be you know derived from the affine version by going to affine space above ok. So this is the whatever you wanted to do here you go above and do it ok. Because there you already have a clear picture you have affine geometry already there so use that alright. So let me write that have a homogeneous projective version have a projective version or homogeneous version of the Nullstellensatz and that is just if F is homogeneous and F belongs to I F E Z of I where I is homogeneous then F for M is an I for sum M greater than or equal to 1. So this is the homogeneous version of the Nullstellensatz.

And so the only thing that has not been said in all this is, what happens to this irrelevant maximal ideal? So that is the only thing that I have to tell you and that is pretty easy to state. So here is a fact that it is a fact about irrelevant maximal ideal but certainly it is not irrelevant to our discussion.

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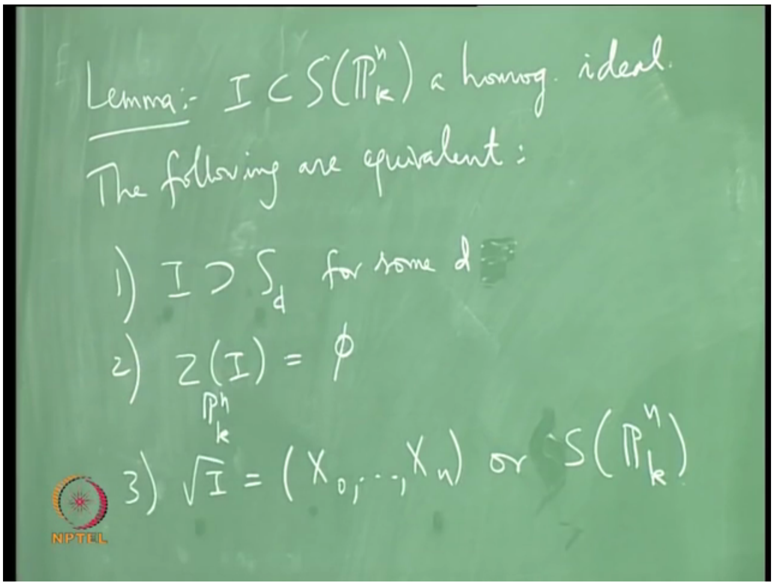


So you know, so here is a lemma which you can check I in S of P n a homogeneous ideal. The following are equivalent, number 1 I is I contains S d for some d greater than or equal to for some d ok.

Then 2, Z of I in P n is empty ok, number 3 rad I is irrelevant maximal ideal ok. So these are, this tells you why you throw out the irrelevant maximal ideal ok. So these are all, this three are equivalent conditions and you know if I contains S d then it means that I will contain Xi power d for every I therefore rad I will contain Xi, therefore rad I will contain the ideal generated by the Xi's ok and offcourse there is another there is one more possibility it is either this or it could be the whole ring.

So I should also write or of S of P n itself right, so it can happen that see the ideal may contain S, S knot, S knot is homogeneous polynomial of degree zero they are the constants. So if the ideal contains constants it will contain non-zero elements of the field so it is the ideal will be a unit ideal and therefore the radical of ideal will also be the whole ring (it will be the even at ideal). So these three are equivalent conditions and this is the exact reason why you throw out the irrelevant maximal ideal to get a bijective correspondence ok (and) ok.

So with that we have now a nice dictionary between projective algebraic geometry and on the one side on the geometric side and on the algebraic side we have the homogeneous coordinate ring and homogeneous ideals there. Now let me tell you a point of surprise. We have seen for affine variety that offcourse we, so that reminds me we define an affine variety to be an irreducible closed subset of affine space ok.

In the same way we define a projective variety to be an irreducible closed subset of projective space it will follow that by the same argument we follow that you know if you know a closed subset here is in affine space is irreducible if and only if the corresponding its ideal is prime. The same thing will hold also in projective space a closed subset of projective space going to be irreducible if and only if the ideal its ideal is a homogeneous prime ideal ok and the fact that will have to remember when you go here is that under continuous map the image of an irreducible set is irreducible ok.

So that is a fact that is a topological fact that you have to remember and use ok. So to get proof of the fact that a closed subset of projective space is irreducible if and only if it is ideal is a homogeneous prime ideal ok and there are two big differences if you take an affine variety you know that there are (globe) the ring of regular functions is a same as its coordinate ring ok and the coordinate ring is just polynomials ok and there are lot of polynomials ok there are lot of polynomial functions at the worst if it's even a single point you have, I mean you have constant functions but if it is not a point then you have many functions many non-trivial polynomial functions on your affine variety ok.

However if you to the projective space there also you can define regular functions and the amazing thing will that will happen is, on a projective variety namely an irreducible closed subset of projective space, the only regular functions are constants ok. So that is a major point of difference between affine geometry and projective geometry. The other major point of difference is the following. We saw the two affine varieties or isomorphic if and only if they are affine coordinate rings are isomorphic as K algebras ok.

But here the projective or homogeneous coordinate ring is not such an invariant, so what will happen is you can have two projective varieties which are isomorphic as projective varieties but their homogeneous coordinate rings are not isomorphic which means that the way they are homogeneous coordinate rings will depend on the way in which they are embedded in the ambient projective space ok. So offcourse here the definition of homogeneous coordinate ring of projective variety is similar to the affine case.

Namely in the affine case you take the all the polynomials on the ambient affine space in go modulo the ideal of the variety ok. Here also you do the same thing, you take the homogeneous coordinate ring of the ambient projective space and go modulo the ideal of the projective variety and you get what is called the homogeneous coordinate ring of the projective variety. But the fact is that this is not an invariant of the projective variety. It will depend on which projective space into which you are putting the projective variety.

So you see the geometry of projective varieties is far more complicated than the geometry of affine varieties, this is what the complications due to one is because there are no global regular functions which are different from constants, there are no non-constant global regular functions that's is one point of difficulty. The second point of difficulty is that the homogeneous coordinate ring of projective variety is not its not an invariant ok.

So this adds lot of richness to an variety to the geometry of projective varieties ok. So and that is what more serious algebraic geometry is about studying projective varieties ok. So will stop here.