Basic Algebraic Geometry
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Lecture 23
Gluing (n+1) copies of Affine n-Space to Produce Projective n-space in Topology, Manifold Theory and Algebraic Geometry; The Key to the Definition of a Homogeneous Ideal

Ok, so let's continue with (your) with our discussion of projective varieties alright.

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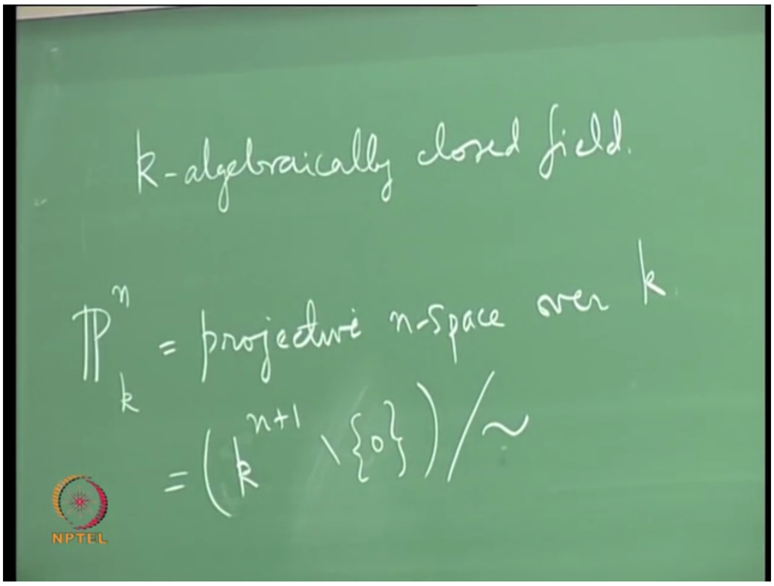

So what we will do is recall from previous lecture that we are taking K to be algebraic closed field. So we always atleast when I define the projective space I don't need this and in principle according to general the most general version of algebraic geometry which is Kim theory infact for defining projective space K may not even be a feel it can even be a ring .

But then lets not go we are not working in that generality but we assume that atleast we are working with fields and since we are going to do geometry we are working with algebraic closed fields ok. So atleast for the initial part K just need be a field it need not even be algebraically closed so you know we have projective N space this is projective N space over K ok and what is this? This is a it is a quotient of the affine space the affine N plus 1 space modulo and equivalence and what is the equivalence? The equivalence is you identify points on a line.

Offcourse I don't want to I want pearly non-zero points so I throw out the origin and then I go modulo and equivalence relation ok and what is this equivalence relation? This is the equivalence relation that identifies two points if they lie on the same line passing through the origin.

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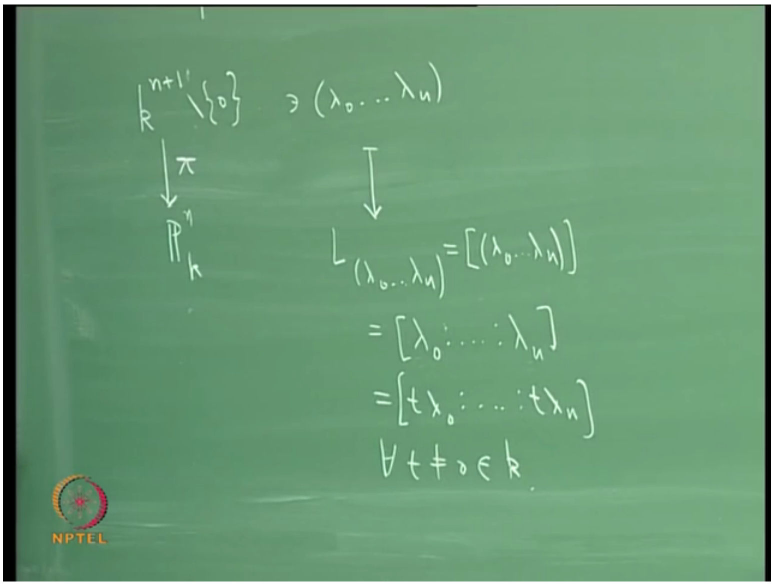

In other words this is just the space of lines in K n plus 1 ok and is so you have a natural map K n plus 1 to P N, K n plus 1 minus 0, this is the punctured affine space to projective space there is a natural map which is the quotient map ok.

You can think of so you can think of every point being send to its equivalence class and the equivalence class can be taught of as a line ok so you send a point it coordinates lambda knot N to the line passing through its equivalence class which is the line passing through lambda knot etc lambda N and the origin ok. This is the same as the equivalence class of that point lambda knot etc lambda N. So at the square bracket denotes equivalence class and this is also the same it is also written in this form.

It is written as lambda knot (semico) colon dot-dot-dot lambda N but this colon is to be thought of as a ratio and we put square brackets to say that this is a set of N plus 1 scalars which have this given ratio ok. So in other words what it means that you know if I multiply this by any non-zero element of K ok which means I multiply each entry by a non-zero the same non-zero element of K then the line will not change alright and the equivalence class will not change ok and this also will not change namely that will be the same scalar multiplied through each of this entries and the rule is that the idea is that whenever you have this colon, you can cancel of the common factor.

So this is the same as T lambda zero T lambda N for every T not equal to zero in the field ok. So that is because if you take T lambda zero to three (to) through T lambda N that also lies on the same line joining lambda zero lambda N through the origin. It defines the same line. So it is the same equivalence class and it is the same point but the idea here is that whenever you have a ratio whenever you write a ratio as A is to B you also write as T A is to T B so the ratio A is to B is the same as the ratio T A is to T B ok and you can cancel the T off from the T from the ratio T A is to T B to think of it as a ratio A is to B(that was).

This is for the case of two ratio of two numbers but then this is a ratio of N plus 1 numbers ok and that is the meaning I mean that is the reason for this notation this are called homogeneous coordinates ok, they are called homogeneous coordinates because you can cancel out a common divisor alright. So well I have now, now the point is that we would like to do geometry on this space, so first of all you know the starting point is always to first make it into topological space and there are two ways to do it ok.

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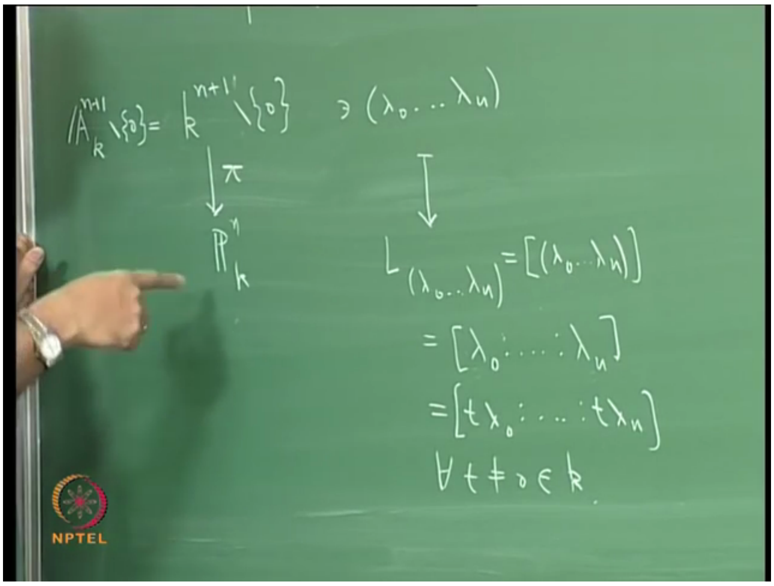

So I will tell you one way the one way is offcourse use the notion of quotient topology which comes from the from topology itself that is because the space above is actually the affine space affine N plus 1 space which is punctured at the origin ok and this as Zariski topology ok. So this are topological space and this is a surjective map from a topological space onto another set ok and then you can make this into a topological space by giving this what is called the quotient topology for this map ok, what is that quotient topology?

It is the open sets here are precisely those for which they inverse image under this map or open sets here ok and similarly for closed sets ok.

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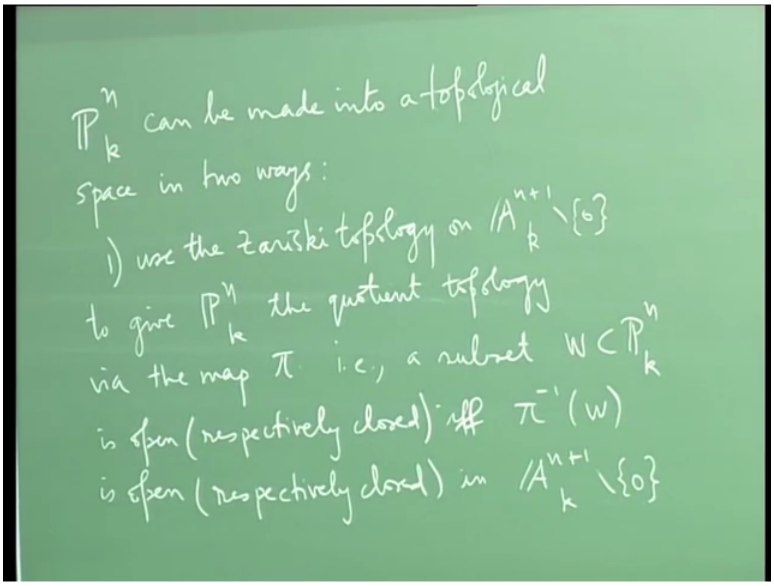

So what we do is that P N can be made into a topological space in two ways, number 1, use the Zariski topology on A n plus 1 minus the origin ok, A n plus 1 minus the origin which is the punctured affine N plus 1 space is an open subset of their affine N plus 1 space and any subset of a topological space automatically gets a induce topology ok. So this has an induced topology from the bigger space which is the whole affine space ok.

And so this has that topology and you can use those are Zariski topology to give the projective space the quotient topology we are the map Pie. You think of Pie as a quotient map because it is quotient by an equivalence relation, the afterall what is projective space ? Is this space of lines. What are the line? They are the equivalence classes enter this equivalence relation. So this is a set of equivalence classes and you always call a map that goes from a set to the set of equivalence classes the quotient map and you call this set of equivalence classes as a set of you call this the you call the quotient of the you call this the quotient of the given set by the given equivalence relation.

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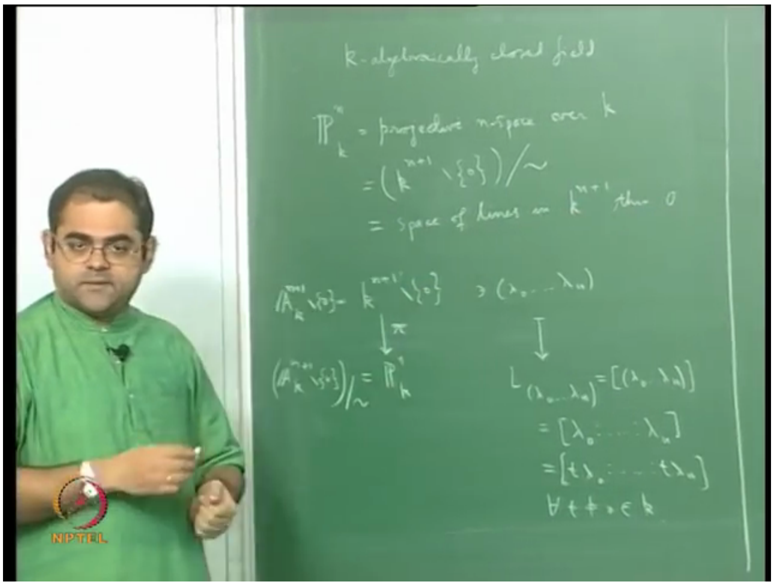

So this affine space here is being this projective space is here being thought of us the punctured affine space modulo going mod this equivalence ok and going modulo this equivalence is geometrically the same as looking at lines ok and every equivalence class here is a line there passing through the origin ok. So offcourse when I say space of lines in K and (())(10:48) through the origin that is important if I simply say is of all lines that is not correct.

I meant space of lines in N plus 1 affine space through the origin alright, and ok so this is a standard thing in topology whenever you have topological ‘and you have equivalence relation on that then you can go to the set of equivalence classes that is a natural subjective map which associates to every point its equivalence class and then you can always give the quotient the set of equivalence classes yet quotient topology using that map ok and we can follow that and what is that?

That is a subset W in P N is open respectively closed if and only if Pie inverse of W is open respectively closed in the space above. Because in the space above which is a punctured affine space you know what closed or open means ok. Open means it is a compliment of closed set and a closed set means it is algebraic set. So closed set here is an algebraic set in the affine N plus 1 space with the origin removed ok. That is what closed sets here are alright for the induced topology.

So this is one way you can give this quotient topology and you see in this quotient topology always in the definition of quotient topology automatically it is a it ensures that the quotient map becomes continuous because you are saying that a set here is open if and only if the inverse image is open. So automatically a set is if a set is open then it is inverse image becomes open that is a condition for verifying the continuity of a map ok. So when you give the quotient topology automatically the map becomes continuous ok.

The, what is the other way of doing things? The other way of doing things is you try to indigenously define the is Zariski topology on the projective space ok and how do you do that you imitate what you did it for the affine space ok for the affine space how did we define the Zariski topology? We define Zariski topology by defining closed sets and whatever closed sets there were sets of common zeros of a bunch of polynomials in the right number of variables.

Now what you do is you just adapt the same definition ok but now you say that you look at common zeros of a bunch of polynomials but not just any polynomials but polynomials are homogeneous ok because if a polynomial is not homogeneous then it is not it does not guarantee that if it vanishes at one point of a line it will vanish in other points of the line passing through the origin ok.

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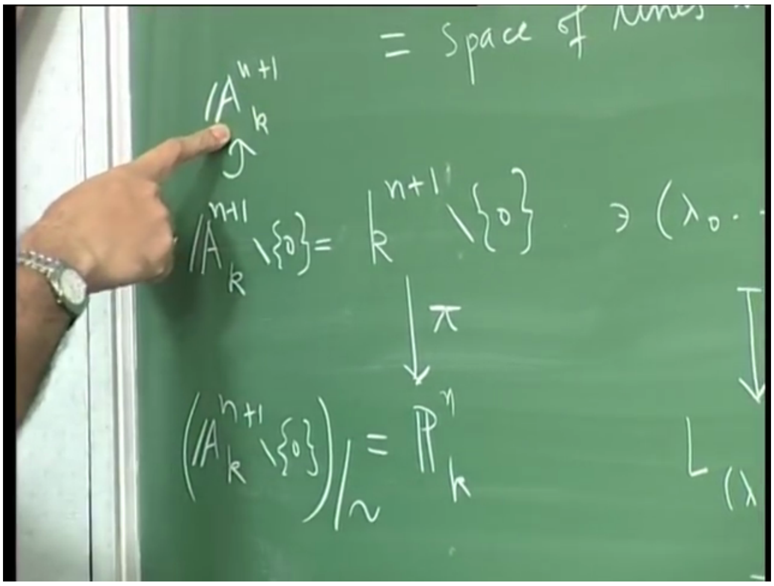

So you know what I am trying to say is that when we define for example a closed set in A n or A n plus 1 offcourse this is sitting inside A n I mean A n plus 1 and how did we define a closed set here?

A closed set here was well we took a polynomial in N plus 1 variables alright and then not one of not one polynomial but several polynomials, a collection of polynomials and look at the common zeros and define such sets to be closed sets here ok. But the problem is that if you take a polynomial ok in N plus 1 variables trying to say that it vanishes at a point here is little tricky which means effectively you are trying to say it is a pol you must think of it as a polynomial ok which vanishes on a point here corresponds to a line here, a line in the affine N plus 1 space passing through the origin.

You want a polynomial to vanish on a line ok. Now that won't happen unless a polynomial is how much it is ok. So atleast I should say that atleast if the polynomial is homogeneous you have hope of the property that if it vanishes at a point on line passing through the origin then it will vanish on the entire line passing through the origin ok.

So that is the reason that we you define this Zariski topology on projective N space you look at homogeneous polynomials and you imitate the definition was a risky topology on affine space by lean a affine space a close the close sets are given by common zero loci of a bunch of polynomials. In projective space the closed sets are the algebraic sets are given by common zero loci or bunch of homogeneous polynomials ok.

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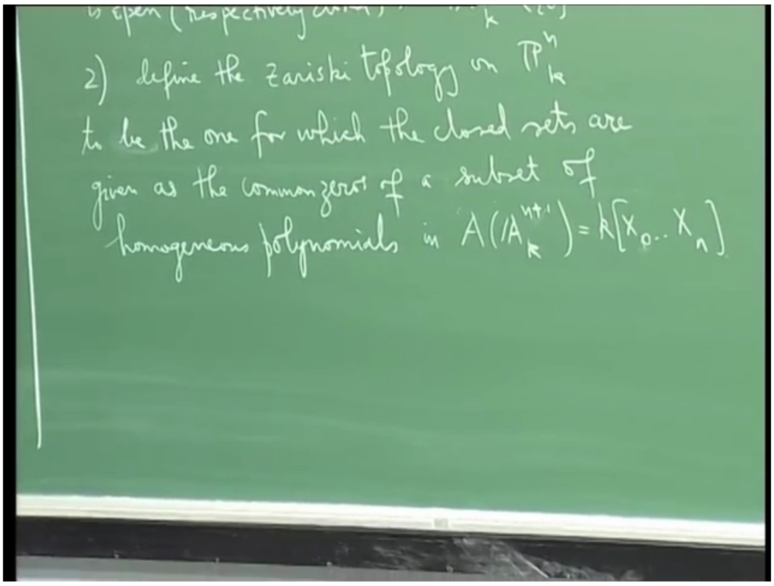

So here is the second definition, define the Zariski topology on P n to be the one for which the closed sets are given as the common zeros of here a collection a subset of homogeneous polynomials in the, in all the you know functions on the affine space above which is just identified with K X knot through X n ok. So you can define another topology on the projective space. This is a Zariski topology on the projective space which imitates it is also given by common zero loci of a bunch of polynomials but this, but in this case we consider only homogeneous polynomials ok.

And mid you homogeneous polynomial means that in the variables if you multiply each of the variables by T then that is if you substitute each variable here constant multiple of that variable ok the same constant multiple then that constant comes out with a power it can be factored out ok and that power that comes out is called the degree of homogeneity of the polynomial ok. So the fact is that actually I think it is a good point to tell you one more thing, there is infact one more way of getting the topology on P n ok.

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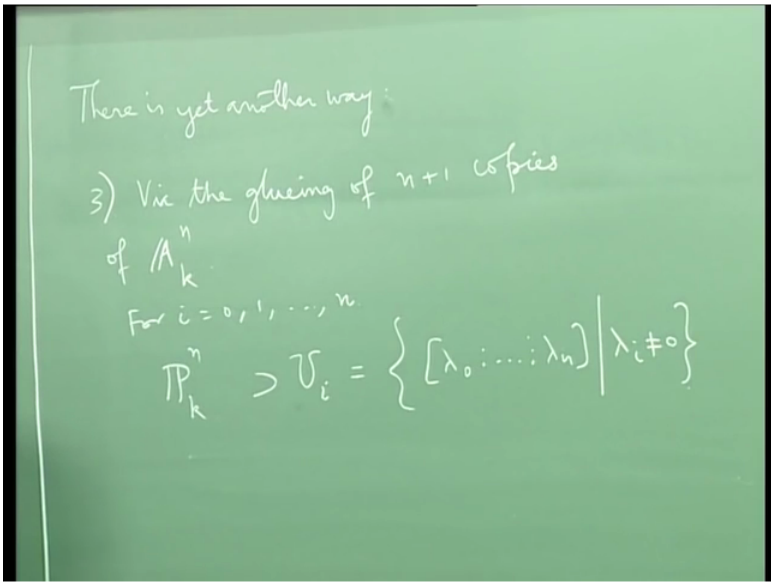

So actually what that one more way there is yet another way ok so there is yet another way of getting the topology on projective space ok and offcourse you know what I am trying to tell you I am just going to tell you that you know you define there are three ways of defining the topology on projective space and they are all the same ok, they all going to give you the same result alright. So what is this yet another way, this is the standard way in which you think of a projective space as being gotten by gluing N plus 1 copies of affine space ok.

So you see I will tell you well let me put it as three in continuation with this via the gluing of N plus 1 copies of A n. Now so I have to explain what this means, so what will do is let me do the following thing. So you take projective space ok take projective space and you know the points in the projective space are given or written in terms of homogeneous coordinates like this ok and then what we will do is lets look at for each of the (coordinate) for each fixed position look at all those coordinates ok for which that particular coordinate does not vanish ok.

So what I mean is you define U I to be the set of all points with homogeneous coordinates lambda knot through lambda N such that lambda A is (()))(20:25).

You define U I like this ok. This is for I equal to 0, 1 etc upto N ok. You define these sets alright and well you know you must guess that the sets are going to be open sets because you know they look like for example U knot is first (coordinate) first homogeneous coordinate not vanishing ok. U1 is second homogeneous coordinate not vanishing, U I is Ith homogeneous coordinate not vanishing and whenever your coordinate does not vanish that should be an open set ok.

So you can see that these are going to be the open sets ok but then let's not worry about them as open sets you know well infact I can say that you know if you take the inverse image of this U I ok under this map then I am going to get the compliment of the zero set of Xi the Ith coordinate infact I should call it I plus 1th coordinate because I am starting with zero, my numbering starts with zero. So you know if I take this U I and take the inverse image above what I will get is I will get all the points with the I plus 1th coordinate not zero ok.

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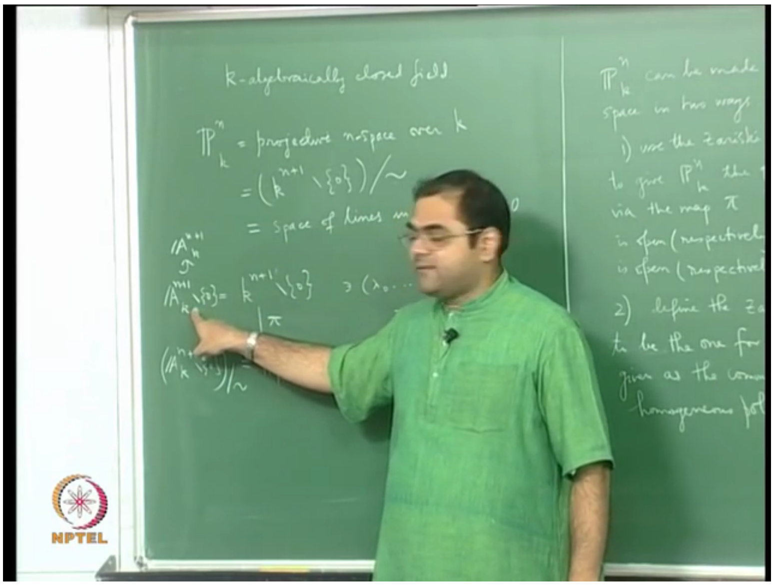

And that is the I will get that set inside the punctured affine space and you know its compliment will be all those points with I plus 1th coordinate I mean with that coordinate zero which is offcourse a closed set ok. So the inverse image of U I under Pie is certainly open set and therefore you know this is certainly open according to the definition of the quotient topology ok I mean according to the definition of the quotient topology.

It is also open according to the definition of the of this indigenous Zariski topology because it is actually the compliment of the zero locus of that coordinate but that coordinate is considered as a homogeneous polynomial of degree one ok.

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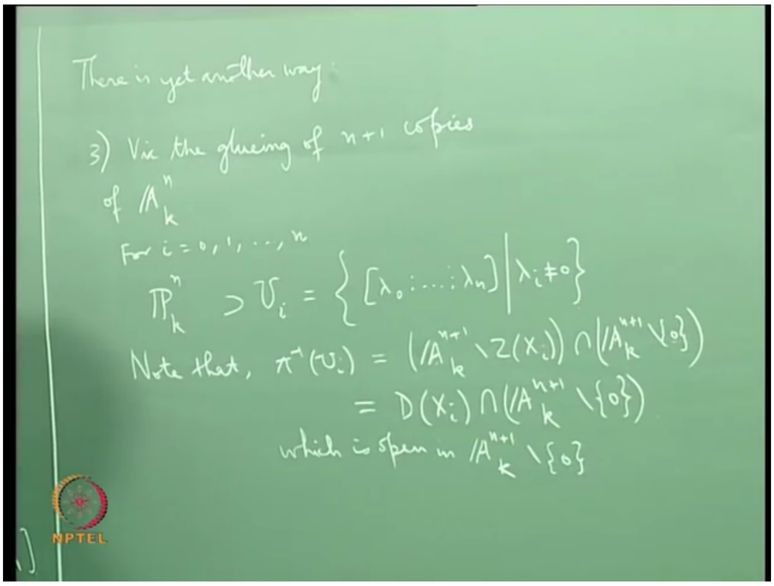

See so note that Pie inverse of U I is actually Z it is A n plus 1 K minus E Z of Xi intersection with A n plus 1 K minus the origin ok, this is what it is. This is actually what? This is just D Xi it is this so the compliment of the zero set of Xi is the basic open set define by Xi alright and that is an open set and if you intersect it with this subset it will give you an open of this ok.

So this is which is open in the punctured affine space above of which the projective space is a quotient ok . So you know according to definition 1 this is an open subset of the projective space if you give it if you look at the quotient topology then they suddenly an open set ok and what you must understand is that you see all these U I's I equal to 0 to N there are N plus 1 of them and they cover the projective space because when you write a point to projective homogeneous coordinates all coordinates cannot be zero because you have thrown out this is the image of a point on the punctured affine space above ok.

So you are not so all the coordinates cannot be zero alright. So whenever you write homogeneous coordinates atleast one coordinate is not zero which tell you that therefore that point lies in one of the U I's so all the U I's is the N plus U I's they are cover for the projective space ok and according to this definition if you give the map Pie the quotient topology then this U I's are open ok according to this definition also it is open because you see.

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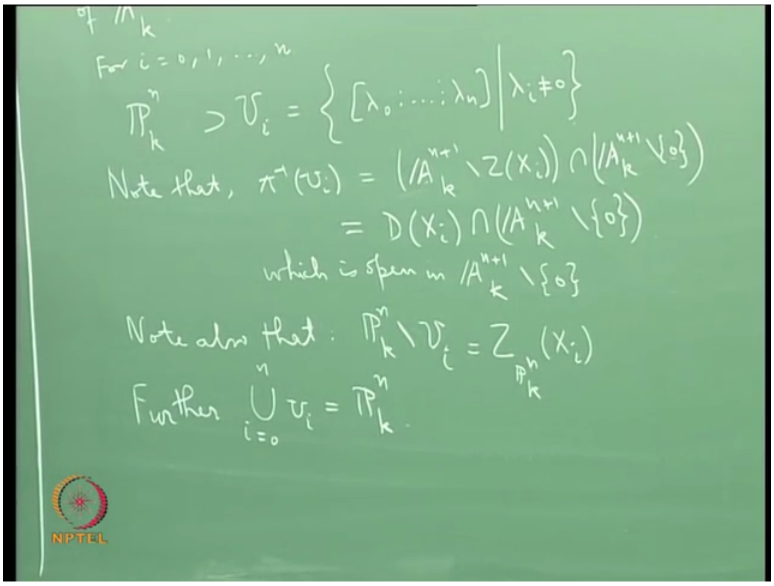

Note also that if you take away U I from P N what you will get is offcourse the zero set in P N of the homogeneous polynomial Xi.

See you take the polynomial Xi ok that is offcourse homogeneous polynomial it is homogeneous of degree 1 alright. So by definition each zero set is you should define closed subset of projective space and therefore it is compliment which is precisely U I is open ok. So this U I is open according to this definition also open according to this definition alright. Now further all the U I's union of all the U I's I equal to 0 to N is actually P N offcourse. The projective space is covered by this sets ok. But the third topology on P N is something that I not yet defined. So but I need these U I's to define here.

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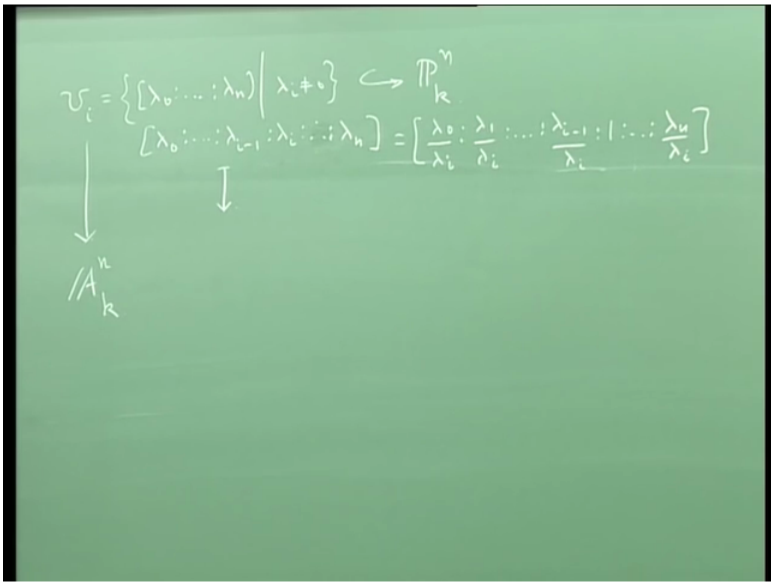

So what I am going to do is now you take this U I which is the set of all homogeneous coordinates points in P N such that the Ith coordinate is not zero which is subset of P N and what you do is you define a map like this into A n ok. The point is that each U I is automatically an affine space. Each U I is actually an affine N space ok and beautiful thing is its compliment which is Z Xi is a projective space of dimension one less ok.

So the beautiful thing is that each of these U I's is actually an A n ok and its compliment is a P N minus 1 alright. So let me explain that and that compliment P N minus 1 is called the hyper plane at infinity ok. So let me explain that so you know this map is very easily define so what you do is you take a point it coordinates lambda knot etc so I will write this down there is a lambda I minus 1, there is a lambda I and then there is a well.

There is lambda I plus 1 I wont write lambda I plus 1 because it might be the last then lambda N ok and you know what I am going to do, see I know lambda A is not zero alright and in the homogeneous coordinate is not going to be effected if you know I multiply the whole set of homogeneous coordinates by non-zero number and what I am going to do is, I am just multiply it by 1 by lambda I which makes sense because lambda is not zero alright, and then that will give me a 1 in the ith position ok on the in the place of lambda I will get a one.

So what you must understand is that this is the same as let me write this it is lambda or not by lambda I, lambda 1 by lambda I bla-bla-bla here I get lambda I minus 1 by lambda I then I will get a 1 and so on I will get a lambda N by lambda I these two ratios are these two are one and the same point on projective space. That is because this is just this multiplied by this the same lambda I which is a non-zero element of K ok, and but then you see this point 1 less I mean this 1 is redundant so what you do is, you sent you get only the remaining N distinct scalars and this are dense scalars to which I mean this are the coordinates to, this define the coordinates of the point which we are going to send it.

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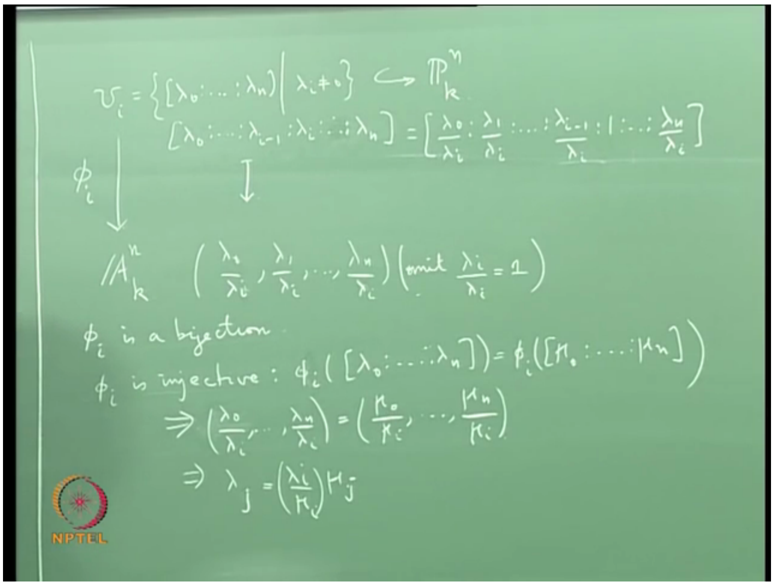

So what you going to do you going to just send it lambda knot by lambda I, lambda 1 by lambda I and lambda N by lambda I where you know the way I have written it is you omit lambda I by lambda I which is 1 ok. So this is N plus 1 entries I omit that 1 and I get N entries ok. Now because of the way I have defined it you can check that this is a bijective (map) this is a bijective map the reason is because you know the moment one of the coordinates becomes 1, you see then the other coordinates are unique for that given line.

Because if you try to change by scalar ok, then this you can't change it by scalar without changing the 1 to something else ok. So if you freeze the Ith coordinate as 1 then the remaining then for each line for is Ith coordinate is not zero ok. If you rationalize so that the Ith coordinate becomes 1 for a representative point on the line then the remaining N coordinates are unique. So the fact is that this is a bijective map so I know and let me call this as Phi I, Phi I is the bijection. This Phi I is the bijection and offcourse you know it is a bijection why it is surjective well.

Phi I is injective it is injective and surjective for the reasons I just said but anyway let me if you want you can write it is not a big deal. You know if you have if Phi I of (one point) given point lambda knot through lambda N is equal to Phi of another point Mu knot through Mu N then, this will imply that lambda knot by lambda I etc lambda N by lambda I is the same as Mu knot when Mu I by Mu I and so on, Mu N by Mu I ok and what this will tell you is that it will just tell you that lambda knot, so it will tell you that lambda knot is lambda I by Mu I into Mu knot.

So each lambda J is lambda I by Mu I into Mu J and this lambda I by Mu is one and the same it is fixed, I is fixed ok. Lambda is fixed, Mu I is fixed ok. Therefore this quotient is a fixed scalar and what you are saying is that every lambda is affixed scalar times every Mu the corresponding Mu and that means that these two points define one and the same line that means they gong the same point on projective space.

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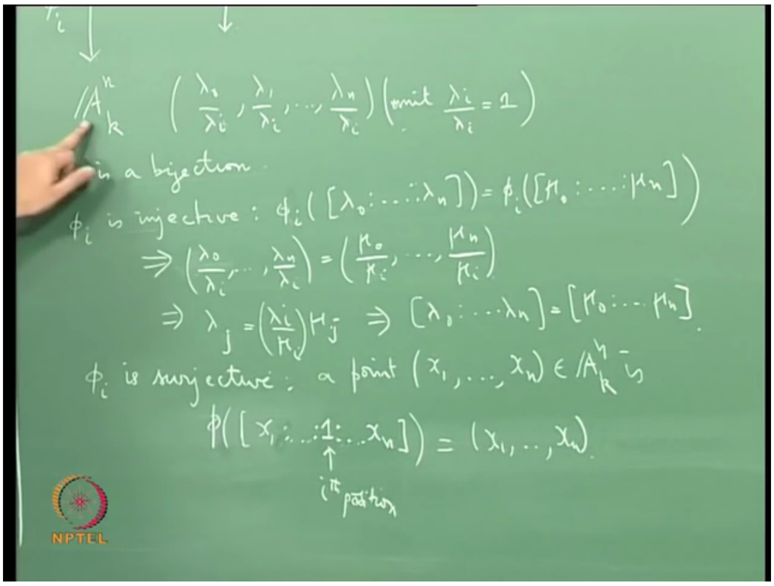

So this implies that the these two points have the same ok this is just injectivity I mean and offcourse surjectivity is very-very simple again. Y is that, so give me a point in affine space here point well X1, etc X and in K n is phi of well I can write this point you know what. I am just going to put yeah I put X1 etc and then I put 1 in the Ith position and then I put offcourse I put (semi) I have to put colons here because this is the, this is just homogeneous coordinates. So if take this point this point will go to the point is sorted with. So it is surjective.

So if you want a map from here to here, you simply write these coordinates in the same order an N plus entries homogeneous coordinates but put don't feel the Ith position. The Ith position you put 1 the remaining N positions you out this in the same order that is the map in reverse direction. Therefore you know this Phi I's are all bijective maps the phi I's are all bijective maps and the fact is that now what I can do, I can do the following thing. So you know in very much the same way in which you get a manifold by getting a local charts into utility and space.

You think of trying to make projective space you know in a sense like a manifold by taking this as local charts which identify these essentially open sets with affine spaces. So you have so you think of so in other words I am saying that you know I am saying use this Phi I's which are bijective maps to transport the topology on the affine space back to U I. See each of affine each copy of affine space has a topology it is Zariski topology.

So I can simply transfer the topology here, namely what is it? You give me a subset here, subset here is open or respectively closed if and only if its image here open respectively closed for Zariski topology on affine space ok. So with that definition also I can give you a topology on each U I ok and the U I's cover the U I's offcourse cover the affine space and the only thing that you will have to worry about is whenever there is an intersection the topologies agree, ok.

So that will happen ok that is analogous to the compatibility of charts which involves the transition functions being good ok and in this case the transition functions will be just excite by X J ok and therefore what will happen is that you will get a all this topologies on each U I which are transported via the Phi I so each of these topological spaces U I they will glue well on the intersection U I intersection U J to give a unique topology on P N ok and that is yet another topology on the projective space ok.

Which comes from which comes via the gluing of N plus 1 copies of affine space is N plus 1 copies of affine space are given by the U I's which have been identified with the corresponding affine spaces via Phi I's ok. So there are and the big deal is that the big deal here is that you know that all this topologies agree on the intersections ok.

That is a very important thing. So you know if you have a topological space ok and suppose you have two, if you have sub if you just have a set ok and if you have two subsets and suppose on each sub sets I give you a topology alright. Then on the union of those two subsets this will give a topology provided on the intersection the topology should agree ok. The intersection a set in the intersection considered as a subset of 1 is open if and only if it is open with respect to when it is considered as a subset of the other ok.

So you know if you have two sets two subsets of a space of a set you give them topologies separately ok then when you get a topology on the union you will get a topology on the union only when on the intersection the topologies agree that is a subset of the intersection as open respectively closed in one if and only if it is in the other ok. This is the situation for two sets but if you have collection of sets. For example even a collection of subsets which is a cover for the given set.

On each subset if you give me a topology when will all this subsets glue together to give a global topology on the set that requires a compatibility ok. It requires a compatibility that whenever you take a certain sub collection of subsets and you take their intersection, subset of that is open if and only if are closed if and only if it is so in each topology coming from the corresponding members ok. So and the fact is that is a big fact, the fact is that all this topologies on each U I ok which come from the (topo) Zariski topology on affine space, they all glue together and give you a topology on the projective space.

This is the third way of getting a topology on the projective space and the fact is al, the three are one and the same ok. So there are there ways of giving a topology on projective space and they are all the same ok. Now what I am going to do is, so let me write that down so maybe I will write it here.

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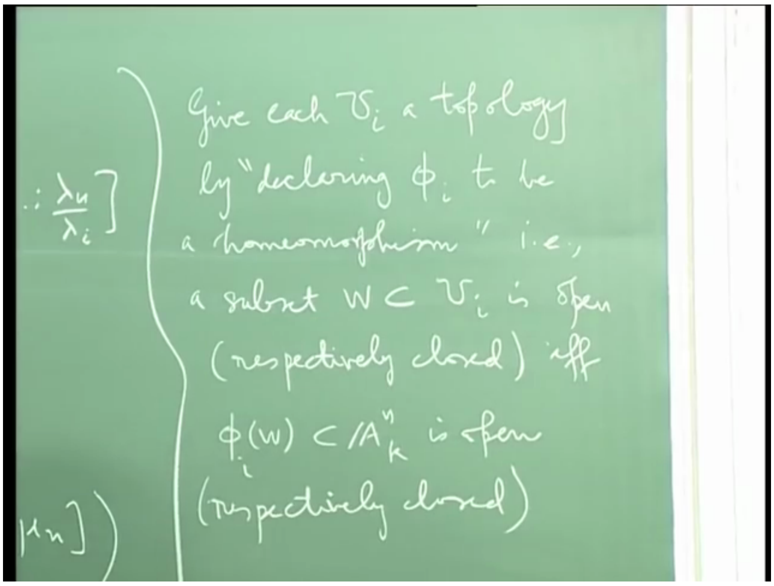

So I will write here, give each U I your topology by declaring Phi I to be a homo-morphism that is a subset W in U I is open respectively closed if and only if Phi W in A n is open respectively closed ok.

So if you give this topology to U I where Phi I then it becomes it is automatic that this Phi I becomes the homo-morphism of U I too affine space ok. So you know what you doing is for every point on this U I you are giving me affine you are just taking N affine coordinates ok. So each one is a coordinate map each one should be thought of a chart the analogue of a chart for a manifold ok.

So each one is a coordinate map alright and what we are saying is that once you have a manifold structure each chart in the atlas if you take the corresponding coordinate map if it offcourse a good function it is offcourse an isomorphism ok. So in this case what is happening is we are gluing topological spaces all the U I's are glued together to give the topology on P N.

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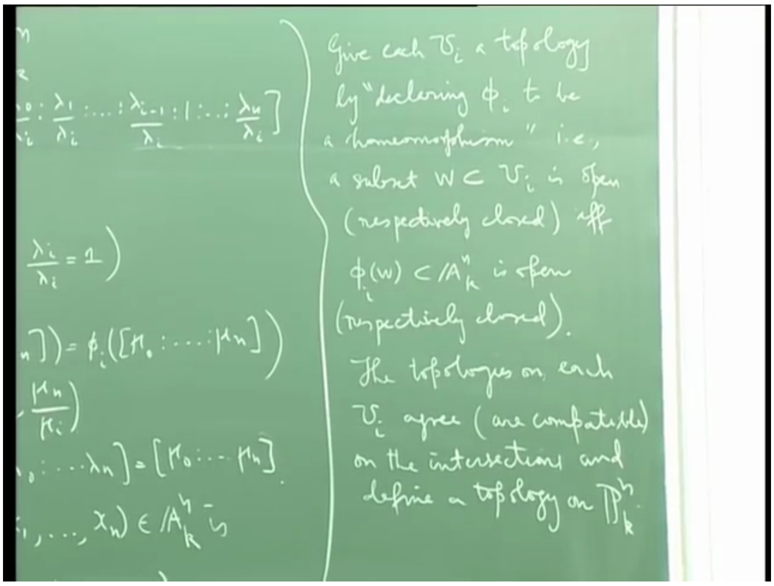

So the topologies on each U I agree are compatible on the intersections define a topology on P N and for that topology and P N when do you say a subset of projective space is open or closed it is open or closed if and only if its intersection with each U I is correspondingly open or closed in that U I. and how do you check it? You check it by taking its image under Phi I and see whether the set you get in the corresponding affine space is open or closed ok.

So this has to be checked that you have to check that the topologies on each U I they are all compatible alright so you know for example if I take two different U I's U I and U J it should not happen that a subset of U I intersection U J is an open subset of U I and it is a not an open subset of U J, such a thing should not happen, it won't happen that is the compatibility you have to check ok. So in other words U I intersection U J has a topological subspace structure from U I it also has a topological subspace structure from U J and my claim is that these two are the same ok.

That is what has to be checked alright. It only if you check that then you are saying that these topologies glue together to give a global topology on the ambient space which is the full projective space ok, and what is the big deal? The big deal is whether that you use 1, 2, or 3 the topology you get on projective space is one and the same and that is the Zariski topology on projective space ok. So let me write that down, fact, so here is the fact which needs a little bit of checking that it is pretty easy to do that you can try it out.

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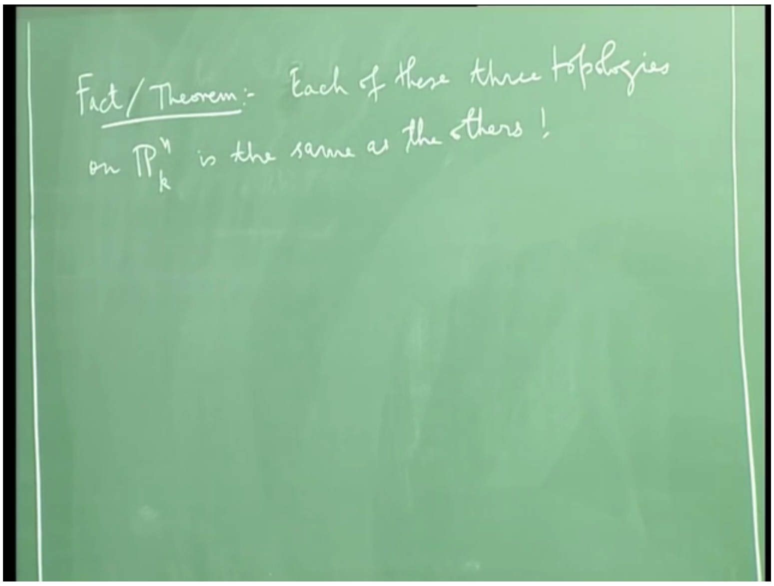

Fact or theorem if you want (or) each of these ways each of these three topologies on P N is the same as the others that is a fact ok. So the what you must understand is that all these Phi I's are actually homo-morphisms, so if you think of projective space with Zariski topology if you think of this as an open subset and you take the induced topology the each Phi I is automatically a homo-morphism ok by this fact by this theorem alright.

Now so this is something that I leave it you to check but probably portions of it we will check as we go ahead now what I have got to do is I want to you know the purpose of offcourse this course is to translate from geometry to algebra and back ok.

So all this that we have been saying is more or less completely geometric and topological but then to go to the completely algebra side will have to worry about the polynomials a (())) (45:42) ok and you know that that is where the second definition comes in the, second definition which says that Zariski topology on projective space is given by closed by taking the closed sets to be common zero loci of a bunch of homogeneous polynomials in the right number of variables.

The number of variables will be one more than the subscript the super script appearing in the projective space ok, so we have to study these homogeneous polynomials. So for the question is why homogeneous polynomials? Ok and then that will lead to looking at the theory of homogeneous ideas ok. So the fact is when we did just like when we did translation from the Zariski topology and affine space to the polynomial ring we were worried about ideals ok.

We will when we do studies Zariski topology on projective space you will also have to translate everything to ideals but the only thing is the ideals now will be so called homogeneous ideals ok and one way of thinking of homogeneous ideals is that these are ideals they are generated by homogeneous polynomials ok. So first of all I wanted to realize that the homogeneity property it can be defined in more than one way and to explain that lets look at an example ok.

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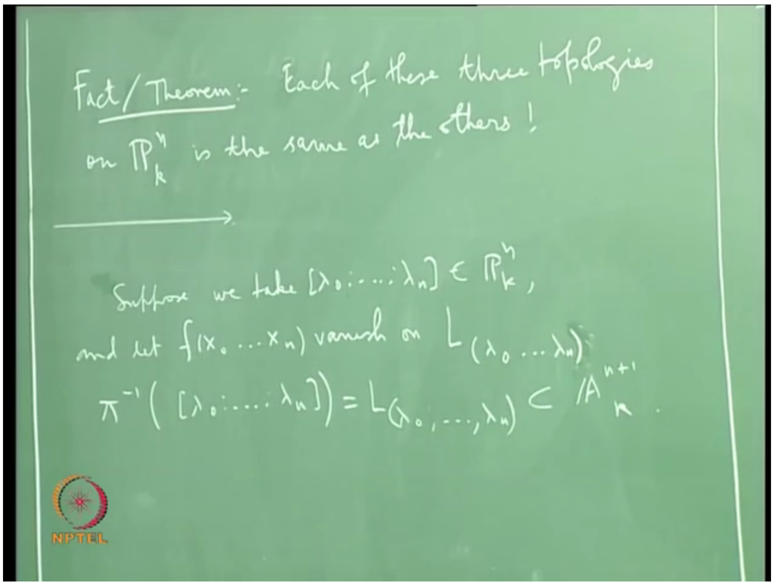

So suppose you take a suppose we take a point lambda knot lambda N in projective space and let F X knot to X n vanish on the line through lambda knot etc lambda N which is actually this point ok. The point in projective space is the same as its inverse image which is the line ok. So you know well the way I have written it is that when I say that this point on projective space corresponds to this line, this line can also be thought of as leaning its actually being thought of as leaning above.

So actually you know this the inverse image of this point is this line thought of as a line above ok, here you are thinking of the line as a point here ok but if you think of it as a subset of A n plus 1 ok then the inverse image of this is this line itself ok. So you know Pie inverse of this point is actually the line through lambda knot lambda N considered as a subset of A n plus right that is what it is. So you know you get another nice picture of this whole of this map you get the picture what is called as a line bundle.

A line bundle is something for which you have a map from one topological space to another topological space. For every point below the inverse image is a line above ok so you think of this as a bundle of lines above this and what is every line above goes to the corresponding point it represents in the projective space ok. So you think of this vibration as it is called as a it is thought of as a line bundle.

You think of given a point here, a point here corresponds to a line above, a line through the origin above and what is that line through the origin above in terms of the point here, it is well if you take the inverse image of this point that is exactly that line above that is what I have written here alright.

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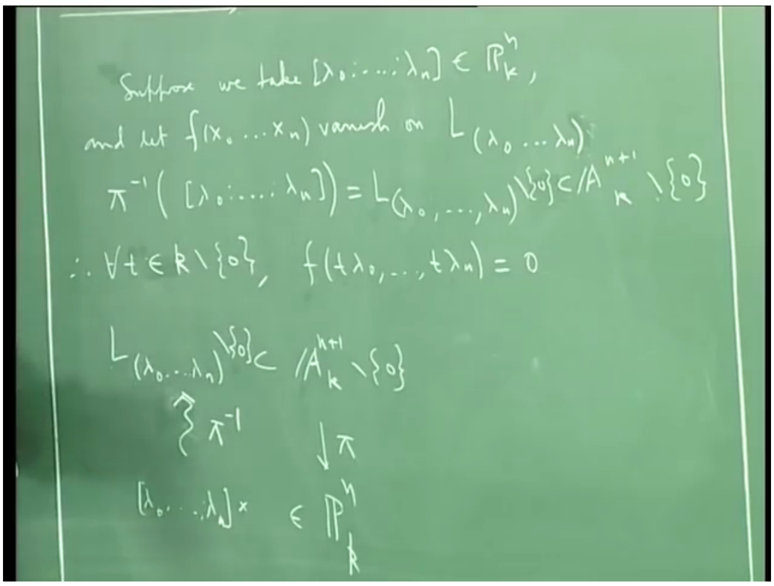

Now you see if the polynomial vanishes on this line ok what it means is so this implies for every T which is a non-zero scalar the field ok well F of T lambda knot etc T lambda N should be zero, this should happen right because you see I have told you that the so you know if I draw a diagram, here is my projective space ok and I have taken a point here with homogeneous coordinates lambda knot through lambda N and under this map Pie I go to the affine punctured affine space above what I get is well I get the line through lambda knot etc lambda N minus offcourse the origin ok.

So if I take Pie this is what so Pie inverse gives me this inverse image don't confuse Pie inverse for inverse map I get this line offcourse I the origin is thrown out because I am considering it as a subset of the punctured plane then you see when I say the polynomial vanishes on this line it means that it should vanish at every point on that line and every point on that line looks like this ok, but then you write this out, what it will tell you is the following. Yeah I should here also when I say Pie inverse of the point is this line I should say this line minus the origin or inside (affine) inside the punctured affine space because Pie is not defined at zero alright.

So this is the right way to write it, but you don't worry about the point at zero which has been intentionally removed ok.

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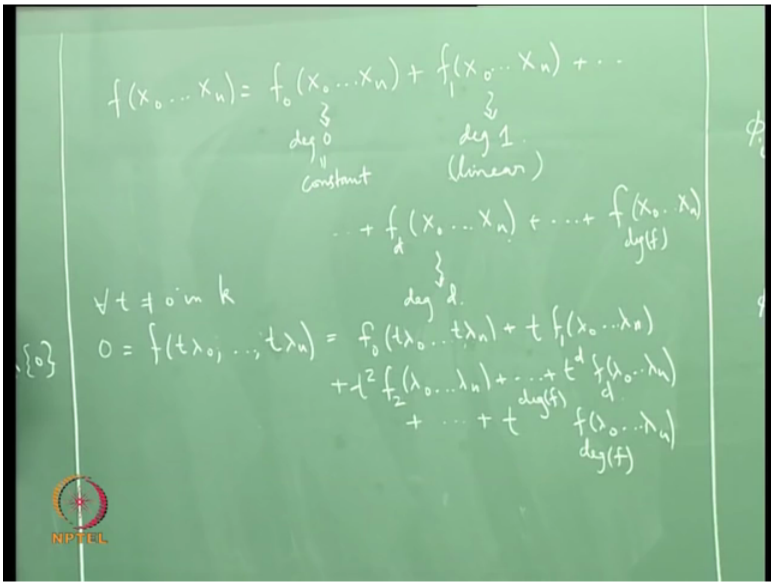

So well in any case now you see you write this polynomial F of X knot X n you break it down into its various homogeneous components, you see a polynomial in N variables can be broken up into pieces a some of pieces each of which is homogeneous of a particular degree, so there is a degree 1 term which will be linear expression in the Xi's ok.

Then there will be a degree 2 term which is a linear expression in product of two Xi's namely it will involve Xi squared and you will get X i X j ok and so on, you will get, so the polynomial breaks down into F knot X knot etc X n this is degree 0, so this actually a constant this is the 0^{th} this is a constant term of the polynomial which s what you will get when you put everything zero ok, plus I will get F of F1 which is degree 1 term this is the linear term ok then I will go on like this and I will get F D which is a degree D term this is the degree D term and offcourse when D is 1 its linear D is 2 its quadratic, D is 3 is cubic and so on and so forth.

And so on you go on upto F to the degree F is the highest degree of the polynomial ok it is a degree of the highest weight monomial that monomials that occur ok and the point is that each of these guys is homogeneous of a corresponding degree. So if you take F D this homogeneous of degree T F1 is linear its homogeneous of degree 1 alright and now you know lets look at this condition F of T lambda knot through lambda N is 0 for every T non-zero.

So what it will tell you is 0 is equal to F of T lambda knot T lambda N that will be that will tell you that you know when I plug this then I will get, this will just remain F knot T lambda N plus when I plug T lambda knot through T lambda N here the T will come out because its degree is 1. Similarly when I plug it in F D, T power D will come out because it is homogeneous of degree D so the expression will look like T times F of lambda knot etc lambda N plus T square F of lambda knot etc lambda N and so on T to the D F of lambda knot etc lambda N plus bla-bla-bla you will get T power degree F times F of lambda knot etc lambda N, this is what you get.

You get for every T not equal to zero in the field you get this expression sorry yeah I should put sorry you are right I should put F1 should put F2 F D, F sub-degree yeah thanks. This are not they should not be F they are the corresponding homogeneous components thanks. So this is T times F1 T square times F2 T power D times F D and so on ok.

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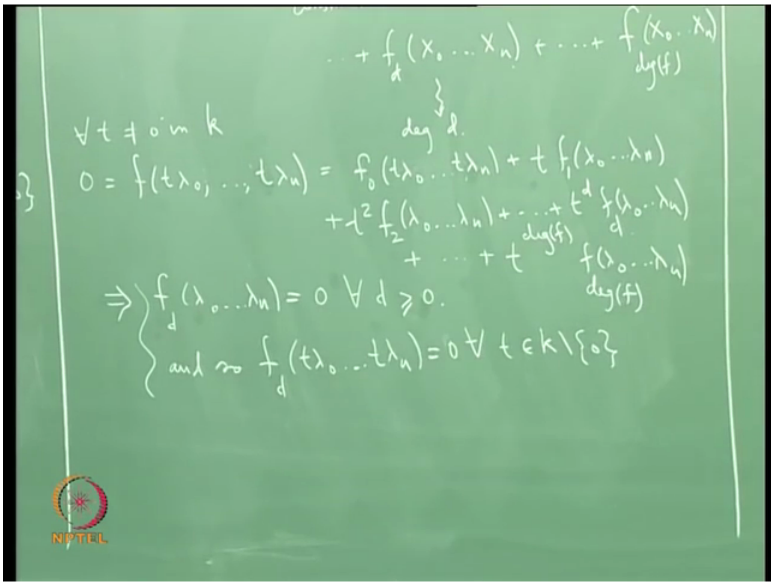

Now you know I want a, I claim that this implies that you know I claim that this implies that F each F D of lambda knot etc lambda n is zero for every D greater than or equal to zero ok.

I claim that each F D vanishes and so F D of T lambda knot etc T lambda N also vanishes for all T in a scalar which is non-zero, this is my claim ok. You know see therefore what I am trying to say is that if your polynomial vanishes on a line then each of its homogeneous components also vanishes on that line ok, each of its homogeneous components also vanishes on that line ok, that is what I am trying to say and this is the key to defining a homogeneous ideal alright and why is this true is because you know let me call this scalar as A knot afterall this is a scalar.

Offcourse here this is independent of T because it is a first term is constant so let me call this, so what I will get is, I will get A knot plus T A1 plus T square D A to plus T power D A D and so on upto plus T power degree F A sub-degree A f is 0 so I will get a polynomial in one variable ok with this coefficients K coefficients and what I am saying is that polynomial identically vanishes on the affine line minus the origin ok but you know if a polynomial vanishes on an open set it vanishes everywhere ok.

If a polynomial is zero on an open set it is zero everywhere that is just because of continuity of the Zariski topology therefore and offcourse mind you here is the first place where you are using K is algebraically closed because I want K to be an infinite field ok. So probably I don’t ok I need K an infinite field alright for that. So I am just saying the fact that you have polynomial in one variable you know if it vanishes on an open set ok then it is so identically then it is identically the zero polynomial.

So therefore this polynomial is identically the zero polynomial therefore every quotient is zero that is exactly what I have written here ok. So the moral of the story is I wanted to remember is the, this is the key fact defining what a homogeneous ideal is. The fact is if a polynomial vanishes on a line then every each one of its homogeneous components that also vanishes on the line and in particular it is a non-constant (pol) I mean the constant term is zero in particular, the constant term cannot be non-zero, the constant term has to be zero ok.

So this is the point I wanted you to remember when we go to the next lecture right. so I will stop here.