Basic Algebraic Geometry
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Module 9
Lecture 22
The Various Avatars of Projective n-Space

Alright so you see so just now we have seen we have these examples of quasi affine varieties is not affine alright and then the what I am going to do is to tell you about more general varieties which are called projective varieties and open subsets of such projective varieties which are called quasi projective varieties ok. Then our definition of variety will include affine, quasi-affine, projective and quasi projective ok, and these projective varieties are completely different class of specimens ok, whose properties are very different from the properties of affine varieties ok.

But to tell you where they come from I just want to tell you that they come from a process of gluing ok, so let me give you some motivation. So you see lets look at the usual topology ok and look at well for example look at the complex plane or the real plane alright and so you know let's do the following thing.

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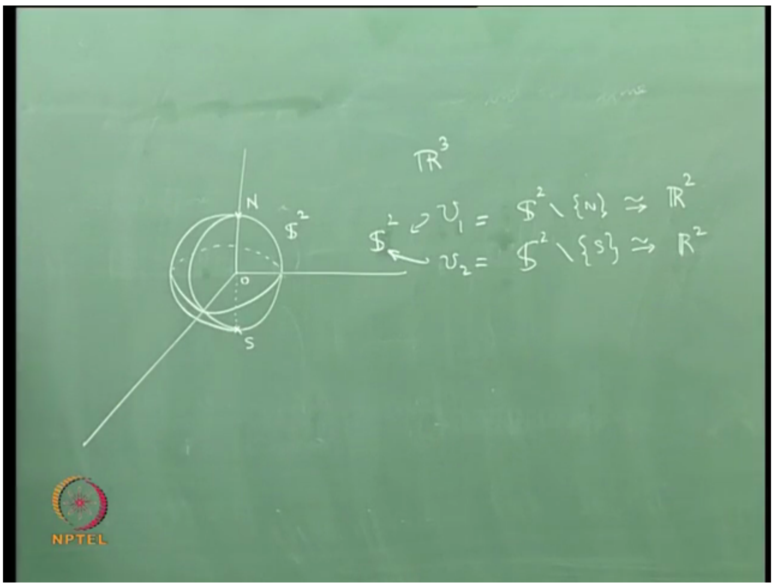

We take the usual plane and then suppose I draw this sphere here ok, so here is my sphere. So this is unique sphere in three space ok. Assume that your in R3 ok, assume you are in R3 and what you do is well you have this is the origin this is the point 1, 1, 0 ok and or rather 1, 0,0 and you know.

So what I do is if I take the if I call this point as a north pole and if I call this point as a south pole ok you would have heard of the so called stereographic projection in complex analysis which identifies so if I call this sphere as S 2, so S 2 minus the north pole can be identified homo-morphically with R2 ok by projecting from the north pole and S2 minus the south pole can be identified homo-morphically with R2 by projecting from the south pole ok.

So these are the so called Riemann stereographic projections from the Riemann sphere ok to the plane and therefore the sphere minus the north pole is compactified the one point compactification of the sphere minus the north pole is a sphere and that corresponds to that will correspond to the extended plane by adding a point at infinity under this homo-morphism. So what it tells you is that the real plane can has a one point compactification which is just the sphere and so there are these two stereographic projections.

Now what you must understand is that if you consider each of this things they are you can call them as two open subsets of the sphere both are open subsets of the sphere ok and they cover this sphere and each open subset looks like R2 because looks like R2 means it is a homo-morphic to end the homo-morphism is via the stereographic projections so what has happened is so what we say is this a standard example of what is called as gluing.

So what you do is you take two copies of the plane ok and you glue them together ok, so basically you take a copy of the plane and then you fold it to get the sphere minus the north pole take the other copy of the plane fold it to get the sphere minus a south pole and you know just glue them together and you get the sphere ok.

Now this is standard procedure you have some spaces you glue them together to produce new spaces but the point is when you do this the new space that you get will have completely new properties. So for example in this case if you take the sphere the new topological property that you get is that it is compact whereas you know neither of the two copies that you originally started with to glue to get this sphere is compact of course you know both R2's are both copies of plane are non-compact because you know the nucleon space something is compact a subset is compact depend only if its both close and bounded with respect to the usual topology.

So moral of the story is that you know actually you are able to by gluing spaces you are getting spaces with your properties ok and so this gluing process is the process that is used to produce new spaces from old spaces alright and basically another good example of gluing is well you know there are several examples.

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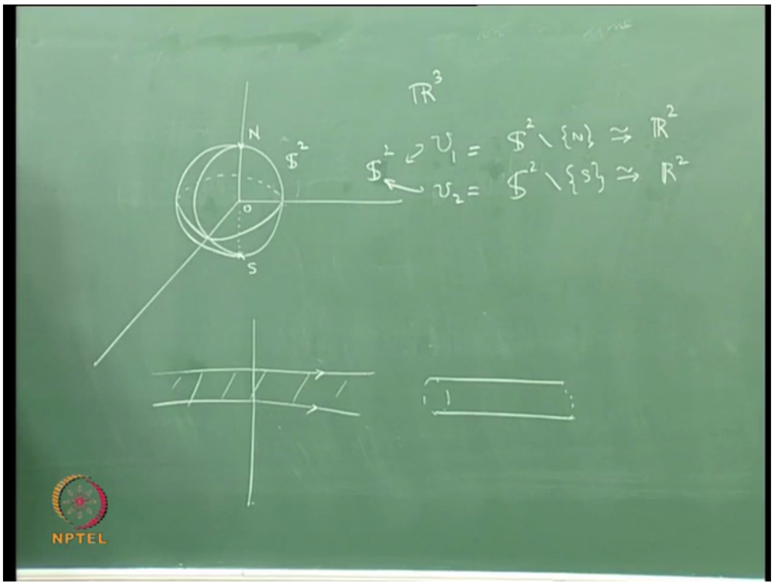

For example you know I can take yeah so you know I can just take I can take the compact I can take the plane ok and thinking of this as a complex plane let me say something if you glue it correctly ok then you can make sense instead of like thinking of S2 as just the real sphere.

You can think of S2 as a surface on which you can do complex analysis you can make it into a Riemann surface ok and you can make sense of holo-morphic functions and then (())(7:51) theorem will tell you that there are no global holo-morphic functions it will tell you every global holo-morphic function will be constant. So the beautiful thing is that on the plane you will have so many holo-morphic functions ok you have so many entire functions whereas this glued object there are no entire functions the only entire functions namely the functions of holo-morphic are everywhere or constant.

And you know so you are basically having two affine spaces you have glued them together to get the space and this space is compact and it has no global non-constant functions ok. The same thing happens in algebraic junction. A projective space is gotten by gluing bunch of a fine spaces ok and on the projective space you will see that there are no global regular functions. The only global regular functions are in the projective space will be constants ok and it is a complete analogy to what is happening here.

So it is a gluing process. So projective spaces are gotten by gluing affine spaces ok just like this sphere has gotten by gluing two copies of R2 alright. Of course some other examples of gluing or for example you know if you take horizontal strip or a vertical strip for that matter and then you know if you or for that matter you know well if you glue the top edge or the bottom edge what will happen is that you will get a cylinder and that is by identifying the top edge and the bottom edge you cut of you cut the strip and then you identify the top edge of the strip with a bottom edge of the strip and fold it out you will get a cylinder you will get an infinite cylinder.

Now the original strip is topologically different from the cylinder because the original strip is simply connected. Any nice any simple closed curve in the original strip can be completely you know shunt to a point whereas the cylinder is not simply connected because the any loop that goes around the cylinder that cannot be continuously shunt to a point. So you see again you have produced by a gluing process you have produced a new topological space with topological properties are very different from the original space alright.

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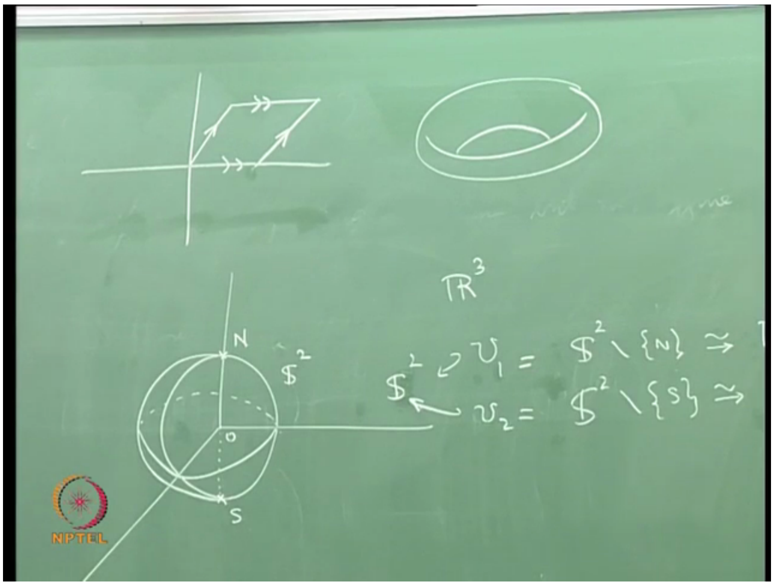

Another example is of course you know you could have taken you could have done also something like this you could have taken just a parallelogram ok and then you could have glued the parallel opposing edges and the result is that you will get a torus because if you glue the upper edge with the lower edge will get a cylinder with two circles on the two ends which need to be further identified if you identify them you will get a torus and the beautiful thing is that this is simply connected but this is not ok.

So the gluing process is a very standard process it is a process that allows you to produce new spaces with new object with new properties ok and you must think of projective space also as coming out of a gluing process ok. So I will explain how projective N dimensional complex space is gotten by gluing N plus 1 copies of N dimensional affine space ok and on what we are going to do is that we are going to define as Zariski topology on the projective space ok.

So we are going to define algebra subsets you're going to define irreducible sets we are going to define closed subsets of projective space call them projective varieties and then whatever we did for affine varieties lot of similar results like the Nullstellensatz etc will also work for the projective case ok. But of course certain things will go wrong alright and I will explain in the coming lectures this and the coming lectures what is going to wrong and what is not going to go wrong.

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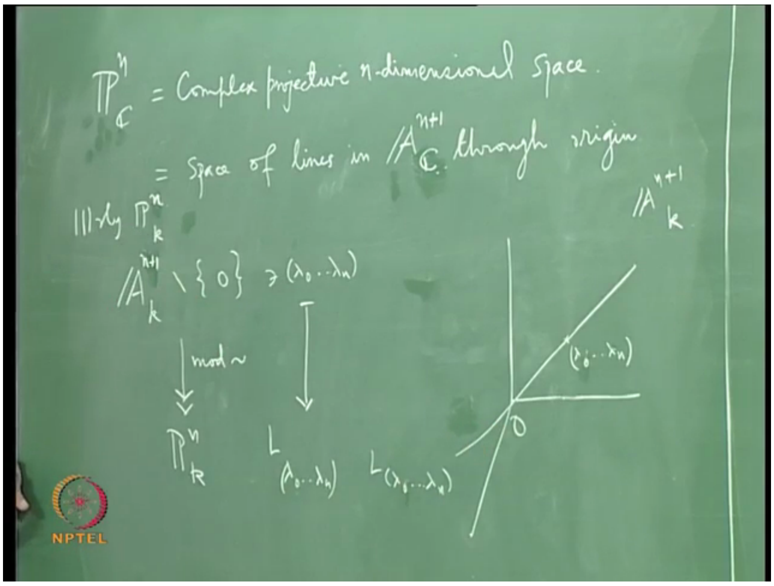

So let me start with the definition of projective space so the, so P N c so this is complex projective N dimensional space here is complex projective N dimensional space and how is it define it is the space of lines in affine space so complex projective N dimensional space is a space of lines in A n plus 1 ok and through the origin and that is true not only for complex numbers for any Phi and how do I get it as a space the reason I have to consider lines in N plus 1 space is because you see I want an N dimensional object ok and if I take lines in N space ok by taking lines I am cutting down by one dimension so the resulting space will be N minus 1 dimensional.

So if I take space of lines in N space I will get an N (dimen) N minus 1 dimensional space ok. Because I am actually moding out by scalars. So if I want an N dimensional projective space I should take lines in n plus 1 space ok. So how does one get it one takes points in A n plus 1 and then you go modulo equivalence relation, what is an equivalence relation? equivalence relation is very-very simple so you know if you give me a point lambda 1 etc lambda N ok then the that point defines the same line as some other point if and only if the two points have coordinates which differed by a non-zero constant multiple ok.

So you know if I take a point lambda 1 lambda N so I will call them as lambda knot to lambda N right so I am in A n plus 1 so the coordinates are N plus 1 coordinates which I am not labelling the coordinates 1 through N plus 1 I am labelling them from zero through N which is the standard convention whenever you do studying projective space. So the line passing through this point so this is the line the line passing through the point lambda 1, lambda 0, lambda N this is the line passing through that and this is what I am going to do I am going to take this point lambda 0, lambda N and simply going to map it to the line passing through lambda 0, lambda N.

And of course through the origin that is the line that joins the point this point to the origin I am simply mapping this point to that line ok and what I want you to understand is that this is an equivalence relation in the sense that if you take this L lambda 0 is the same as L Mu 0 etc Mu N that means both these points lie on the same line and you know both these points lie on the same line if and only if this is a non-zero multiple of that by a single scalar non-zero scalar ok.

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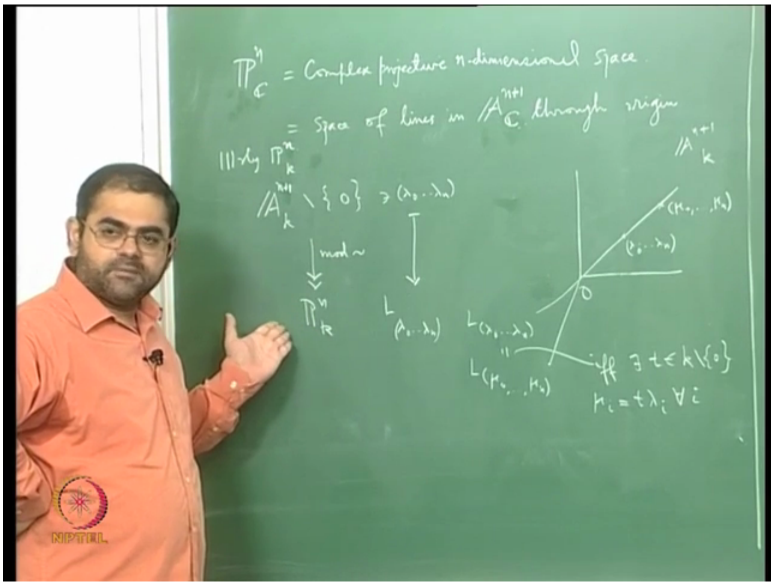

So this is if and only if there exists T non-zero element of the field such that Mu I is equal to T lambda I for every I ok. So you know between N plus 1 triples of points you put this equivalence relation is an equivalence relation that two points, one point is multiple of the other point by an non-zero element of the field that is an equivalence relation and if you go model of that equivalence relation what you are going to get is possessive T projective space which is the space of lines ok.

Two points here if they are they are equivalent namely they will differ by their coordinates differ by one and the same scalar non-zero scalar multiple if and only if the lines that they define through the origin and the same ok. So space of so what has happened is that we have gotten the projective space as a quotient ok it is a set modulo and equivalence relation so it is a quotient and this is very-very good because once you have this for example if I have a topology here I can transport the topology here by giving this quotient topology alright.

So it is always good that whenever you have a quotient kind of situation then you can transport from the source lot of things to the target right. So well you know I have this a Zarisky topology on this because this is after all this is just after all sitting inside A n plus 1 which is a Zarisky topology which is affine N plus 1 space and therefore this has the Zarisky topology and I can put the quotient topology on this and that will give me as a Zarisky topology on P N alright and what it will happen that this map will be an infact an open map ok and of course it will be continuous for the Zarisky topology alright.

But then there is another way of defining the Zarisky topology on this you know very in a slightly analogous way to the definition of the Zarisky topology on affine space so you see so the point I want to make is that you know of course I can for example you know I could have put K equal to forget algebra get closed I could have taken K to be real numbers and then I will get real projective space ok and then I can study things on this I can simply take the usual topology on top and then you know give this quotient topology alright.

I could have done that and similarly I could instead of K I can take complex numbers and then will get complex projective space and on the complex projective space I could have again put the usual topology I have the usual topology here I could have given the quotient topology ok and because the complex numbers are also an algebraic close field I have another topology here which is a Zarisky topology that also I can use to give a quotient topology here.

So complex projective space has two topologies one is the topology which comes as a quotient topology for the usual topology in the other topology comes as a quotient topology for the Zarisky topology on this punctured affine space above ok.

So now let me go back to how we define this Zarisky topology on N plus 1. See the Zarisky topology on affine space was defined by giving closets and the closets were given as zero sets of I mean common zero loci of a bunch of polynomials ok. Now what we are going to do is we are going to do implement the same thing here what we are going to do is we are going to look at common zero loci of a bunch of homogeneous polynomials ok and then you see it will make sense you see if you take a polynomial function for a polynomial function of course if you take a polynomial in N plus 1 variables if I change if I evaluate that polynomial on a line alright of course a polynomial the values will change ok. But if the polynomial is homogeneous the property of it vanishing or not will not change ok.

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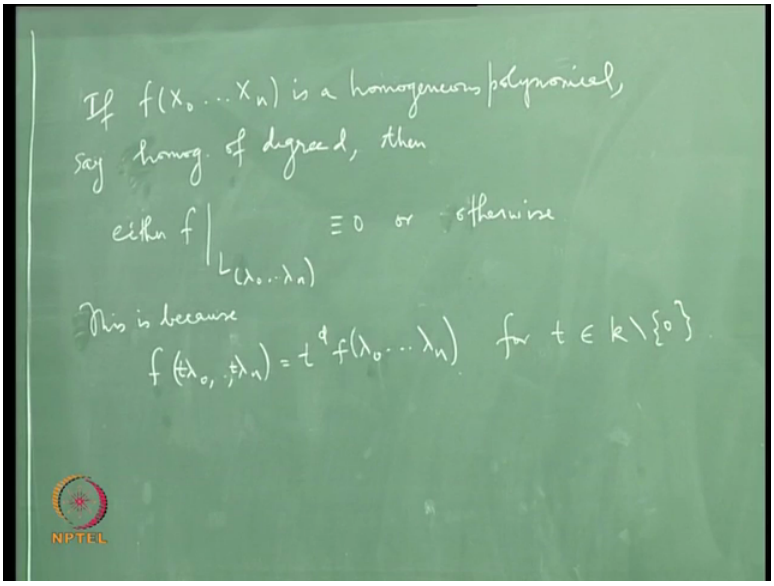

So this is the first observation if F of X knot X n is a homogeneous polynomial ok say of homogeneous of degree D then either F restricted to line through lambda knot lambda N is identically zero or otherwise ok. So what I am trying to say is because this is because you see F of lambda knot lambda N suppose I put T lambda or knot etc T lambda N is T4 D F of lambda knot etc lambda N for T non-zero constant.

So you know if the polynomial vanishes at one point of the line then it will vanish at every point of the line ok. So in other words so I can make sense of whether a polynomial vanishes on a line or not but what is the line? A line is a point here. So I can make sense of whether a polynomial vanishes at appoint here or not ok and then I define the closets here to be common zero loci of the bunch of points where the polynomials vanishes. All those points where a bunch of homogeneous polynomials vanishes ok.

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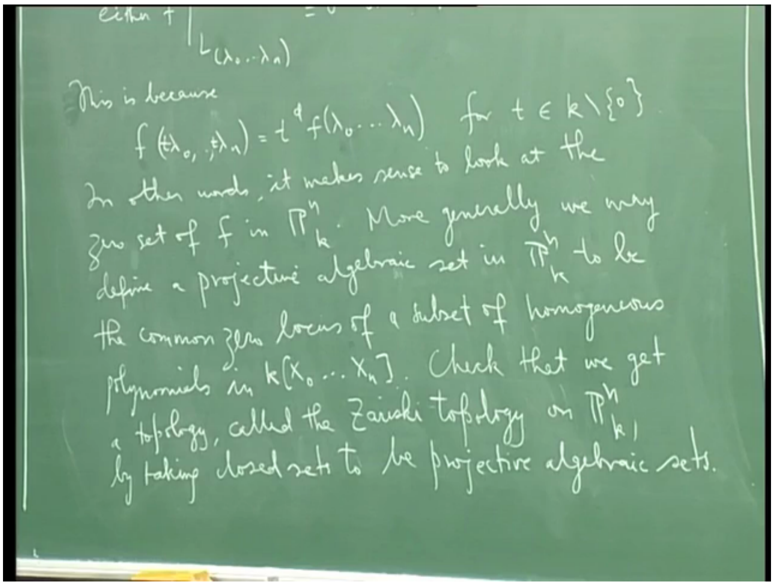

So what you do in other words it makes sense to look at the zero set of F in P N ok. What is the zero set of F and P N? Point in P N where F does not vanishes corresponds to a line on which F does not vanish. Point in P N where F vanishes corresponds to a line on which F vanishes ok. So let me repeat, I can make sense of a zero set of a polynomial in projective space namely it is all those lines on which F vanishes. So all those lines on which F vanishes, and I though to have to just do it for one polynomial I can do it for any collection of polynomial.

So what you do is, more generally you may define a projective algebraic set in P N to be the common zero locus of a subset of homogeneous polynomials in K X knot etc Xn ok. The fact is that we get a topology on projective space that topology will be the so called (risk) so called the Zarisky topology ok, you can check that this gives the (topo) check that we get a topology called as the Zarisky topology on projective space by taking close sets to be projective algebraic sets ok.

If you take projective algebraic sets to be close sets you get a topology on projective space and that is called as the Zarisky topology and the fact is that is the following the fact is that topology is the same as the quotient topology that you get from this a Zarisky topology on the top ok. This topology is the same.

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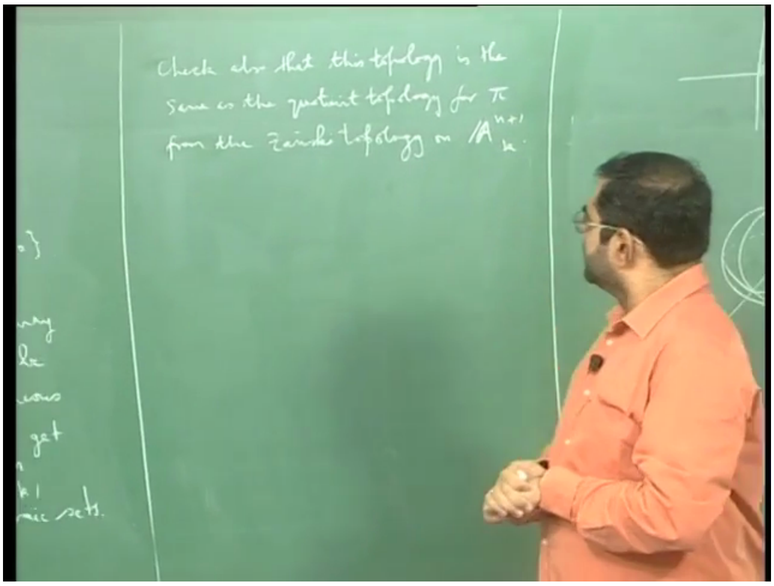

Check also that this topology is the same as the quotient topology for Pie I will call this map as Pie the projection using quotient topology for Pie from the Zarisky topology on A n plus 1 ok. So the moral of the story is that you can get as a Zarisky topology you can get a topology on this the Zarisky topology on this that came here in two ways either you take the quotient topology I mean it is a topology that you put that makes this map continuous ok and that is one definition ok. The other definition is that you define the topology directly on this to be given by closed sets which are given by common zero loci of a bunch of homogeneous polynomials.

The only difference with the projective case in the affine case is that in the affine case you consider all polynomials but in the projective space the projective case you consider only homogeneous polynomials and you know why you have to consider homogeneous polynomials because only for a homogeneous polynomial you can say for sure whether it will vanish uniformly on a line passing through the origin ok. If it is a non-homogeneous polynomial it could vanish at some points on the line it could be non-vanishing at other points on the line ok.

If you take a non-homogeneous polynomial and take it zero's set that zero set will be hyper surface ok which will be N dimensionally in N plus 1 space and the hyper surface could hit the line at not at all points it need not contain the line. So it could hit the line at some points and it could not hit the lines at some points. So a non-homogeneous polynomial could vanish at some points on the line and not vanish at some other points of the line.

But if you have homogeneous polynomial it either completely vanishes on the line or it vanishes at no point on the line ok. So if you take a homogeneous polynomial it is very easy to define the zero set of that in projective space and then if you take a bunch of homogeneous polynomials then the common zero locus of this bunch of homogeneous polynomials is what is called an algebraic set and that is how a closed set is defined ok and this gives us the risky topology on the projective space. And now you know lot statements that we know for the usual affine space the same statements will carry over for projective space.

The only thing is for example in the affine case you deal with ideals general ideals and general polynomials in the projective space you will deal only with homogeneous polynomials and you will deal with ideals which are generated by homogeneous polynomials and this are special they are called homogeneous ideals ok. So just an in the affine case you have a bijection between radical ideals and algebraic subsets in the projective case also you will have a bijection between radical homogeneous ideals of this ring polynomial ring in N plus 1 variables and algebraic projective algebraic subsets but you will have to throw out one ideal which is called the irrelevant maximal ideal and that is the ideal generated by all the variables.

That is the one that you have to throw out and you have to throw it out because on top you have thrown out the zero of the zero set of that which is the origin ok you have to throw that out. So and just like in the affine case where you have the affine Nullstellensatz which says that if you take an ideal which is you know a proper ideal then the zero set in non-empty. Similarly you will also see here that if you take a homogeneous ideal which is not which is essentially not whose radical is not the irrelevant maximal ideal then its zero set will be non-empty.

So you get a projective version of the Nullstellensatz. So lots of this correspondence between ideals and closed subsets that you had for a fine space will also hold for projective space and we will review that in the next lecture but this one point of caution the point of caution is the following. You can go and start defining regular functions on a projective variety ok on an open subsets of a projective variety we will call an irreducible closed subset of P N as a projective variety alright and it will turn out that it will be closed subset will be irreducible if and only if it is ideal is prime.

The ideal will be homogeneous ideal but you will require that it has to be prime and what will happen is of course that you know in the projective case also if you look at so in the projective case also you can define regular functions the only thing is that in affine case your regular functions where quotients of polynomials ok. Now you have to define them as quotients of homogeneous polynomials ok.

If we take quotients of two homogeneous polynomials and assume that the both polynomials are homogeneous of the same degree then that will define a proper function on (affine) on the projective space because you know if I divide two such polynomials then the T power D is and if they have the same degrees the T power D's will get cancelled and therefore a quotient of homogeneous polynomials of the same degree will define a nice function on the projective space.

Functions that look locally like this will be called relay functions on the projective space and then beautiful thing is that if you try to look at any global regular functions on a projective space it will turn out to be constant.

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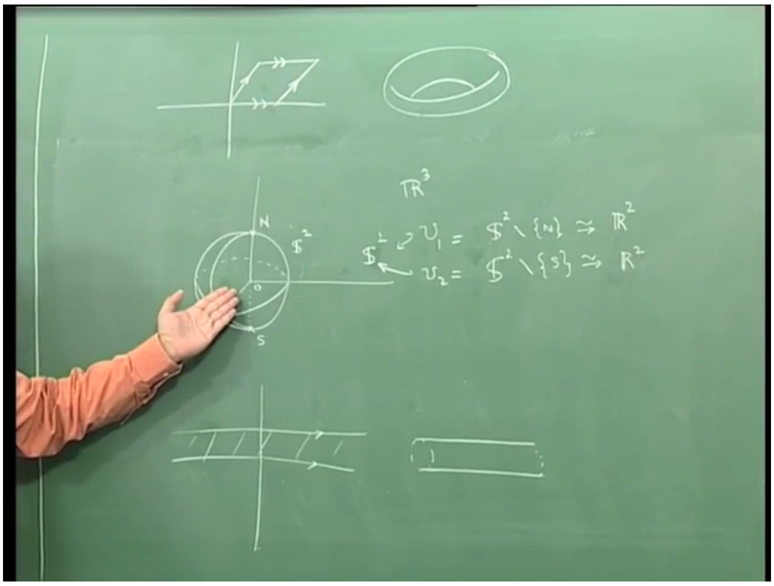

Just like if you try to look at a global holomorphic function on the Riemann's sphere it has to be constant ok. So these projective and it is true for any projective variety if you look at any global regular functions it will be a constant will prove that ok and this is in sharp contrast with the case of affine variety when the global regular functions are given by all the polynomials restricted to that affine variety and there is so many of them.

Whereas if you go to projective varieties there are no non-constant regular functions ok. So and of course I also forgot to tell you just like in this case S2 is a union of two R2's.

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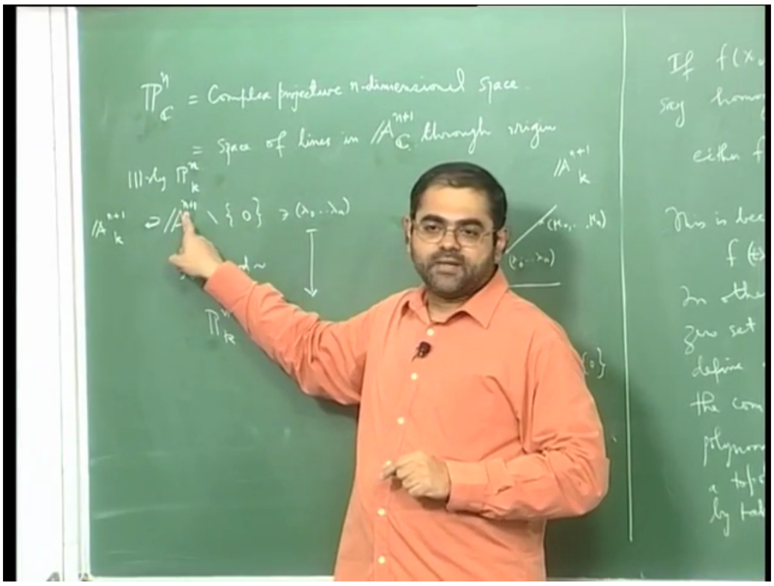

I will show in the next class that P N is union of N plus 1 copies of A n, so the projective space locally it is covered by N plus 1 open sets. Each open sets looks like A n affine N spaces. So what you have done is you have taken N plus 1 copies of affine N spaces and glued them in a nice way to produce the projective space and the beautiful thing is that though each of the pieces that you glued with have lots of you know regular functions polynomials.

On this glued object there is no global regular functions which is not one such ok. So we will see all this aspects in the forth coming lectures and let me also tell you one more point of difference that is that you know for affine variety the coordinate ring of the affine variety is invariant namely two affine varieties are isomorphic if and if only if their coordinate rings are isomorphic and now this will completely going to be false for projective varieties ok. So the same projective variety can be embedded into different projective spaces and if you try to define the ring of functions on that as the this polynomial ring modulo the ideal the homogeneous ideal we will see that ring is capable of changing.

So the embedding of a projective variety in some projective space could be very different I mean alright so you don’t have the beautiful analogue of coordinate ring for affine varieties you don’t have the correct analogue in that sense for projective varieties ok and for that matter that is what leads one to study line bundles and sections of line bundles etc on projective varieties which are probably the content of a second cosine algebraic geometry ok but I will stop here and we will continue in the next lecture.