Basic Algebraic Geometry
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Module 8
Lecture 21
Automorphisms of Affine Spaces and of Polynomial Rings - The Jacobian Conjecture

What we need to now look ahead to in the course is to try to expand the category of varieties ok. So far we have been looking at fine varieties and quasi-affine varieties which are open subsets of fine varieties but then we need to also include a more general varieties and the next in this list are the projective varieties in the quasi-projective varieties and the projective varieties are they have properties very different from properties of affine varieties ok. So you know what I wanted to start with is since we are looking at we have been looking at a fine varieties ok.

The first thing I wanted to say is about the so called Jacobian Conjecture which is a very simply sated conjecture but which is and which is open even at the simplest case ok and the reason it makes sense to talk about that conjecture now is because you know what automorphisms you what I morphisms of varieties are you know what are isomorphism of varieties and you know how to characterize uhh isomorphism of a fine varieties ok.

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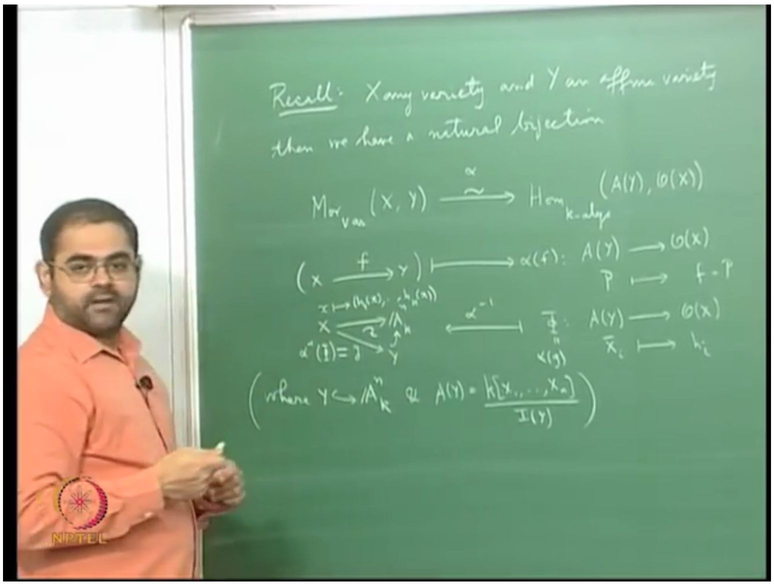

So I will first uhh I first recall the following thing from the previous lecture, so if you recall X any variety offcourse here for us any variety will mean either a fine variety or quasi-affine variety ok and Y and a fine variety which means Y is a reducible close subset of some a fine space over an algebraic close field offcourse then we have here natural bijection so on the one hand you have morphism varieties from X to Y that is bijective to a homomorphism of K algebras from A Y to O X ok.

We proved this bijection and infact what was the if you recall this map a B if I call this map as alpha ten how does is how is this map defined it's the map in this direction is just given by pullback of regular forces ok. So you know in other words if F from X to Y is an element on this side it needs some morphism from the variety X to the fine variety Y then you send it to alpha F this alpha F is going to be K algebra homomorphism from the train of polynomial functions on Y to the regular functions on X and that is very-very easy namely you give me a polynomial function let me put is as P on a, let, me put it as yeah capital P.

Give me a polynomial function capital P on Y ok, you compose it with F to get a regular function on X. Note that A of Y is the affine coordinate ring of Y the coordinate ring of functions polynomial functions from Y it is just polynomial restricted to Y and these are the polynomial functions on the a fine in which Y sits, Y is an fine variety so Y sits inside some A N, the ambient the bigger a fine space and on this bigger ambient a fine space you have the polynomials in N variables.

And each of these polynomial functions by evaluations defines a map into K which can be thought up as a mapping to A1 and it is a regular function offcourse and you restrict such polynomials to any subset in particular to Y and you get a polynomial function on Y only, the only thing is that this polynomial function on Y is not (())(6:22) by unique polynomial it is represented upto addition by a polynomial in the ideal of Y ok.

Which consists of polynomial which vanish on Y ok. So give me a polynomial associated to Y and I just compose it with F first apply P then apply F that is a regular function X because a polynomial function associated to Y is offcourse a regular function on Y and what is happening is I have a regular function on Y and then by composing it with F I get the pullback of the regular function on X and the pullback of a regular function has to be regular function because F is a morphism because that is already built into the definition of a morphism.

So this is how this math is defined ok and there is also uhh the if you look at the map in the reverse direction alpha inverse, how is that defined? Well give me a home of some free from K L algebra home of sum from A Y to O X ok and then offcourse you know if offcourse you assumed that a Y is thought of a sitting inside a fine space and A Y is just well K of is identified with K of X1, X n which is the polynomials on the fine space the Xi being the coordinate functions divided by the ideal of Y ok and so what you do is that you just take the Xi bars which are elements here.

Here the images of Xi in this quotient ok so Xi bar just denotes the cosset Xi plus I Y in the quotient ring ok and you simply send it to a certain function lets call it as Hi it is a regular function on X and the fact is that you will, what will happen is that from X to uhh from X to A n you will have a map which will send any point X small x to this H1 X this (())(9:02) defined by Hs Hn of X because I have N of the Xi's so I have their image is here which are N of the Xi bars so I get N of the Hi's ok and then I evaluate this point at each of these N functions I get antiple which is a point in An this is the map and the fact is that this map will factor through morphism G through the close reducible close of variety Y of A n this diagram will commute and this phi will be nothing but alpha of G, this is the surjectivity ok.

So start with a phi here then you get this Hi's using the Hi's you define a morphism of X into A n the morphism will land inside Y and if you called that morphism as g then alpha g is the phi that you started with ok. That is the surjectivity that gives all other surjectivity and it is also defines that the alpha inverse of phi is this g, ok. G is just alpha inverse of phi. This is how you get the inverse map and that is how this is the bijection ok and as corollarying to this what happens is that if you know if X is also an fine variety then O X can be replaced by A X because for an fine variety the ring of radio functions can be a naturally identified with the polynomials restricted to the fine variety ok.

So and alpha will take your bijection to a bijection ok and therefore what it will tell you is that at if is also a fine then X and Y are uhh alpha infact I should say alpha will take an isomorphism to an isomorphism ok that is an invertible not just a bijection but invertible morphism to an invertible morphism and invertible morphism varieties will go to an invertible ring homomorphism ok. That is a ring isomorphism, isomorphism of varieties will go to an isomorphism of K algebras right ok.

So alpha will carry isomorphism, two isomorphisms provided you know X is a fine alright and what that will tell you is that it will tell you that two affine varieties are isomorphic if and only if their coordinate rings affine coordinate rings that is just the rings of polynomials on those varieties are isomorphic.

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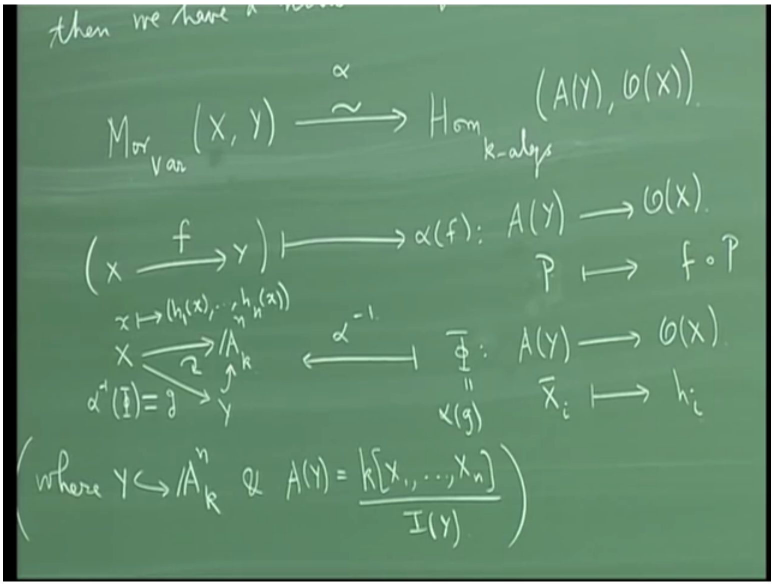

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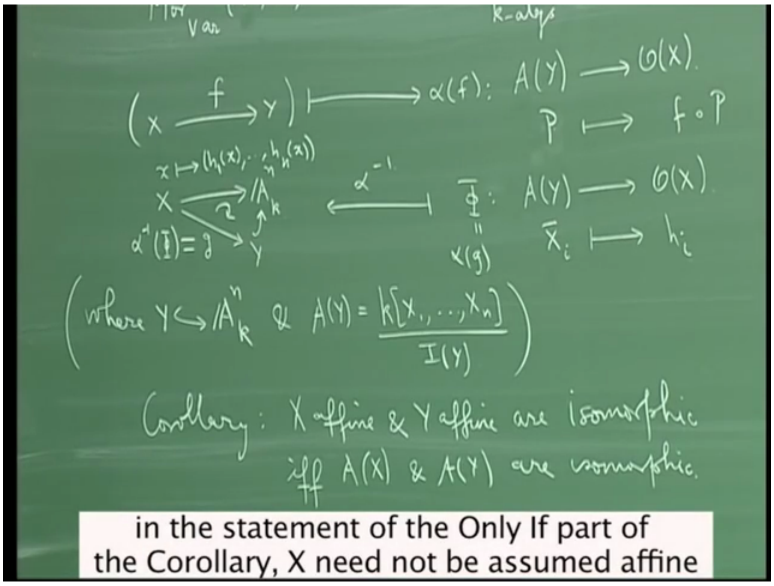

So to the corollary to this is the corollary to this is X affine and Y affine are isomorphic if and only if A X and A Y are isomorphic. Offcourse this isomorphism when I say X and Y are a fine and a isomorphic there I mean isomorphism is varieties and when I say A X and A Y are isomorphic I mean isomorphism as K algebras ok.

So to a fine varieties isomorphic if and only if the their rings of polynomials, the rings of polynomial functions on those a fine varieties isomorphic is K algebras ok, and how many such isomorphisms are there, they are as many isomorphism here as there as isomorphism here ok. So you know as a particular case what you can do is well you know I can take uhh I can take for X and Y just A N itself.

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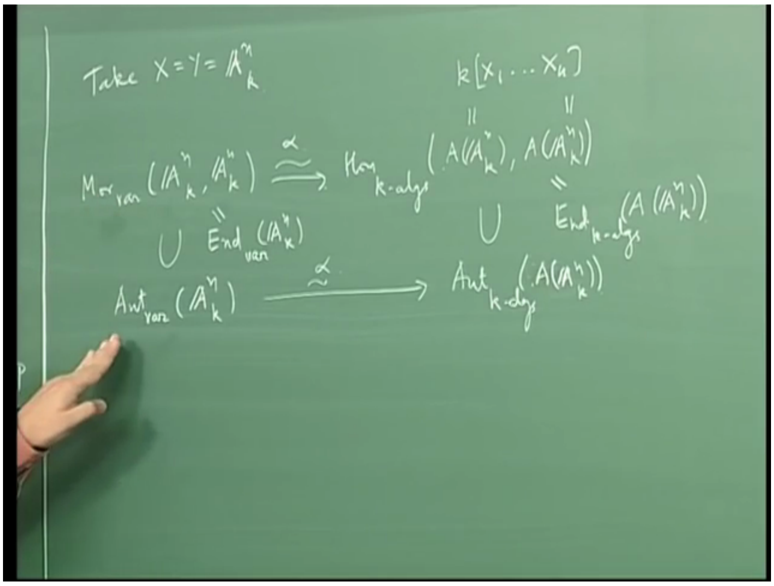

So you know take X equal to Y is equal to A n, so what you will get is you will get morphisms of varieties from A n to A n is bijective to the K L algebra of homo-morphisms from Af An to Af An and offcourse you know Af An is the is just polynomial ring.

So you know if you want well both of these are equal to K knot X1 etc Xn ok, if you take them to be the ring of polynomials if you take the ring of polynomial to be the ring of polynomials in determinant X size ok and in particular if I look at the automorphisms so the word automorphisms means a self-isomorphism it is a morphism is from one object to another object an automorphism is a morphism from the object back into itself and infact uhh morphism of an object back into itself is general is called endomorphism ok and an invertible endomorphism is called automorphism ok.

So here offcourse when I say morphisms from An to An actually I am looking at the endomorphism of An. So this is all the endomorphisms these are maps from A n back into itself they are endomorphisms so offcourse the you know the other notation is endomorphisms as varieties of A n, and offcourse here what I have is endomorphisms as K algebras of the polynomial ring Af An ok, and what are the automorphisms?

Automorphisms are the invertible endomorphisms they are nice they are endomorphisms which are also isomorphisms ok. So they are self-mapped they are the morphism of the object pack into itself which can be invertible ok. So if you look at the automorphisms that is offcourse a these if you look at it carefully this is a group because a composition of two morphisms is again a morphism therefore a composition of two automorphisms is again automorphisms so this is a group and on the other hand you also have a group here.

This is automorphisms as K algebras of K well let me write as A of A n, ok. So and then alpha carries automorphisms to automorphisms ok, so this alpha will also give you a map like this, ok. You start to the morphism a morphism is an isomorphism which means it is here if and only if will see here and conversely ok and now the you see the Jacobean Conjecture is connected with automorphisms of the polynomial ring right, so let me make a statement uhh so what I want to look at is a, let's take any morphism from A n to A n ok.

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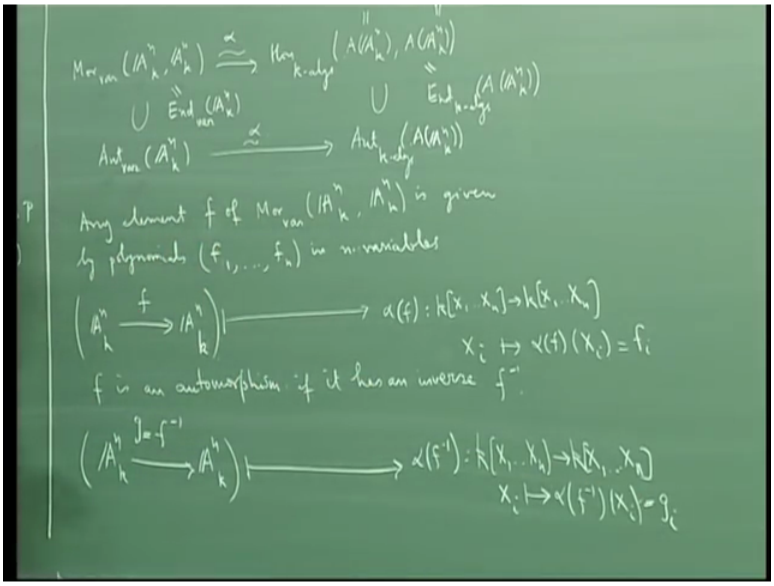

How is it going to be given? It is going to be by N polynomials in the N variables ok, any element of phi of a or let me use F of morphisms of varieties from A n to A n is given by a polynomials infact so let me write this F1 etc Fn in N variables. So N polynomials in N variables. That is (like) that is just because of the this bijection ok. So you know what are all this you know I start with so here is my F from A n to A n then you know I have this is mapped that is its point that is an element here it is mapped on element here, this is alpha of F, and this alpha of F is a map from well K X1 etc Xn to K X1etc Xn.

Where I am identifying the ring of polynomials ending with K X1 etc Xn ok and how is a from a polynomial ring a map is dictated by the images of the variables. So you know if I take alpha F of Xi this is what alpha F takes Xi to and this (())(19:08) dictates the K algebra homo-morphisms alpha of F because universal property of the polynomial ok. So what I want to tell you is that and this alpha F of Xi that I am calling as F i's so let me write that here. I need a better duster.

So I have Xi into alpha of F of Xi ok, and I am calling this a Fi ok, so essentially what you doing is that corresponding to this morphism A n to A n that corresponds to giving me N polynomials ok and that is what I have written in the previous line that a morphism from An K n is simply given by N polynomials in N variables ok and when is this morphism an isomorphism where it has an inverse ok F is an automorphisim that is it is a invertible morphism namely an automorphisim.

If it has, has an inverse ok and so it has an inverse F inverse so you see what will happen I will have an F inverse which again will go from A n to A n ok and that will be mapped under alpha to alpha of F inverse that will turn out to be again mapped K algebra homo-morphisms from the polynomial taking N variables to again to itself, back into itself and that is again going to be dictated by the images of the Xi's under this, so alpha of F inverse of Xi ok and (())(21:30) if you if I call this as you know if I call this as Gi which means I am just calling F inverse as G ok by my previous notation if F goes to F1 etc Fn then F inverse equal to G will go to G1 etc Gn alright.

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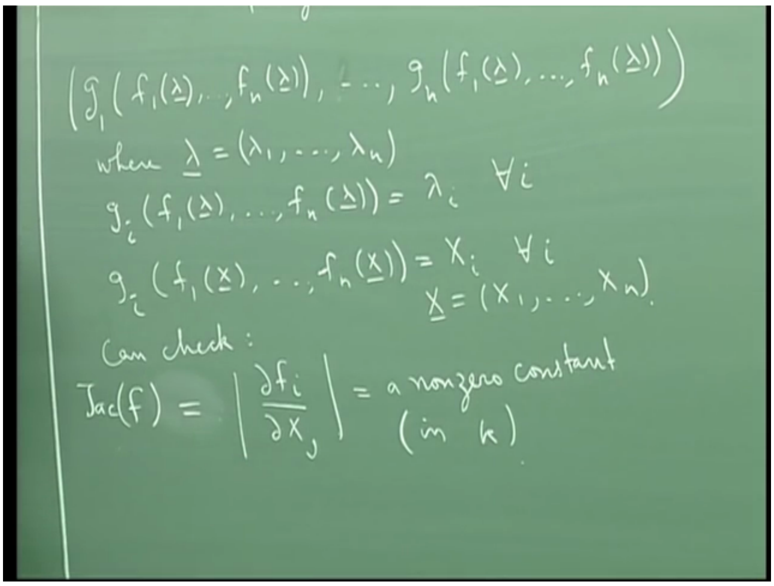

And what you can say immediately is that see you will have if I start with a point X1 etc if I start with a point with coordinates lambda 1 etc lambda n that will go under uhh if I apply the map to F to it what I am going to get is F1 of lambda 1 lambda N and so on Fn of lambda 1 it is a lambda N this is the point it is going to go to ok. That is what it means to send Xi's to Fi's and ralph of F ok and now this point is if I apply F inverse which is G this point has to go back to this but under F inverse where will this point go.

See this point will go to G1 of F1 of the lambda's I will put a I will put it like this dot-dot-dot Gn of F1 lambda Fn lambda, this is what it looks. So what where offcourse you know where lambda underline is just lambda 1 through lambda N ok. So what you get is Gi of F1 lambda etc Fn lambda is simply lambda I for every I. this is what happens if you have a an automorphisim right, and the now what you can do is you know this holds for all lambda's ok so you can write this in variable form as Gi of F1 of X Fn of X is equal to Xi for every I.

You can write this in variable form ok, which makes sense alright and then you know for example if you want you can take the partial derivative on both sides with respect to any X J you will get offcourse if it is X if you take partial derivative with respect to Xi you will get one on the right side if you take partial derivative with respect to X J, J knot equal to I you will get zero then what you can check is that you can check the Jacobean of Dou Fi by Dou X J is a non-zero constant in K.

So what you must understand is that you see I take each of these polynomials each of this Fi's each Fi is a polynomial in N variables and if I differentiate partially with respect to each variable I will again get a bunch of polynomials ok. I have N polynomials I have the N Fi's I differentiate each N of them with respect to the N variables so I get a, I will get this Jacobean matrix ok, that is in general going to be a matrix of polynomials again. Because you take a polynomial in N variables and differentiate it partially with respect to one of the variables resulting is, the resulting thing is again a polynomial in N variables ok.

And then therefore if you look at this Jacobean determinant then infact I think the correct maybe it is better to write it as offcourse here X underline stands for X1 to Xn and maybe it is particular write this as Jacobean of F and call it like this ok and there the point I want to make is that you see if you calculate this Jacobean determinant it is going to only be you expected only to be a polynomial because a every entry is a polynomial gotten by taking a partial derivative with respect to certain one of the variables.

But the fact is F has an inverse G ok, and therefore you know if you write down everything F followed by F inverse is identity ok and for the identity function if you take the Jacobean you will simply get the identity matrix ok. So finally what will happen is that you will see that Jacobean polynomial of F into Jacobean polynomial of G will if you take the product polynomial it will be equal to 1, which will be the Jacobean polynomial of the identity map which is just the identity matrix.

You have two products of two polynomials equal to 1, so each of them has to be constant it has to be a non-zero constant that is why Jacobean of F will be a non-zero constant. So what this a simple argument shows so far is that you know if I start with an automorphism ok I end up with uhh you know bunch of functions I end up with the N polynomial functions sort of the automorphism F I end up with N polynomial functions whose Jacobean is a non-zero constant ok. That the converse of this is true is a Jacobean conjecture ok.

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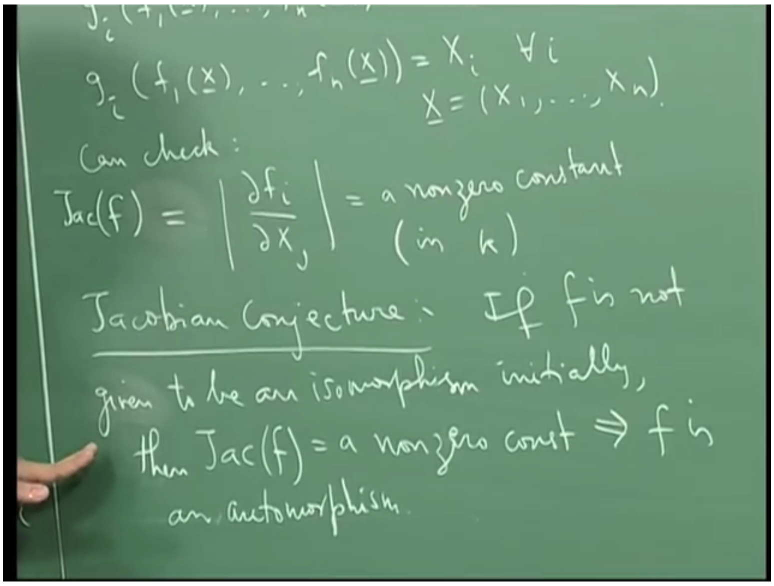

So the Jacobean conjecture is if F is not a known to be is F is not given to be an isomorphism initially does then the condition that the Jacobean of F is a non-zero constant implies F is an automorphism ok. This is the Jacobean conjecture, the Jacobean conjecture is that if I start with an F which I don't know is a invertible I don't know it is an automorphism, but suppose I have the condition that if you take the Jacobean of F namely you take the Jacobean of this N polynomials that specifies F ok under this correspondence.

If this Jacobean is a in generally you expected only to be polynomial in N variables but suppose it turns out to be a non-constant I mean here non-zero constant polynomial then the Jacobean conjecture says that F should be invertible that means you should be able to find another set of polynomials N polynomials which if you plug in to F the F's will get back identity ok, that is the Jacobean conjecture and the point is that somehow so another way of stating that is that you know there is a map from here to a the polynomial ring given by taking determinant of the Jacobean ok.

Every endomorphism is given by N polynomials and you take the Jacobean determinant of the N polynomials you will get a polynomial. So you get a map from this endomorphism to your ring of polynomials and what we have seen is that the inverse image of K-star that is non-zero elements of K that uhh well that contains this ok, namely every automorphism for every automorphism the Jacobean is a non-zero constant but the question is whether if you take the inverse image ok then it will be exactly this, that is the question, ok.

Namely if you give me N polynomials ok for which the Jacobean determinant is non-zero constant, do those N polynomials actually corresponds to an automorphism is the question. So the question is whether the inverse image of the non-zero constants under the Jacobean determinant map from this endomorphism is exactly this we know it contains this, but what is required to show is that this is exactly this ok, that is the Jacobean conjecture and the beautiful thing is that even for K equal to complex numbers at even for N equal to 2 just polynomials in two variables this is open ok and it is a very difficult problem.

And the beautiful thing is that in the case of complex numbers we have also complex analysis holomorphic functions, we have active area also but it doesn't seemed to have helped ok. So this is a very deep problem and working being able to solve this or being able to give a counter example to this is a worth being awarded by fields medal which is the equivalent to the noble prize in mathematics. So that is the depth of the problem.

It is a very hard problem called the Jacobean conjecture and the point is it can be stated now because you people know that there is a bijection between you know (morphism) (iso) morphism of a fine varieties and K algebra homo-morphism of their coordinates rings. So that is these why we want to state said it here. So maybe I hope that many or atleast some of you will go ahead and try to tackle this problem in your future carrier ok, alright. So now ok so now this is just to begin this Jacobean conjecture what I wanted to do now is I want to go to the question of more general varieties ok.

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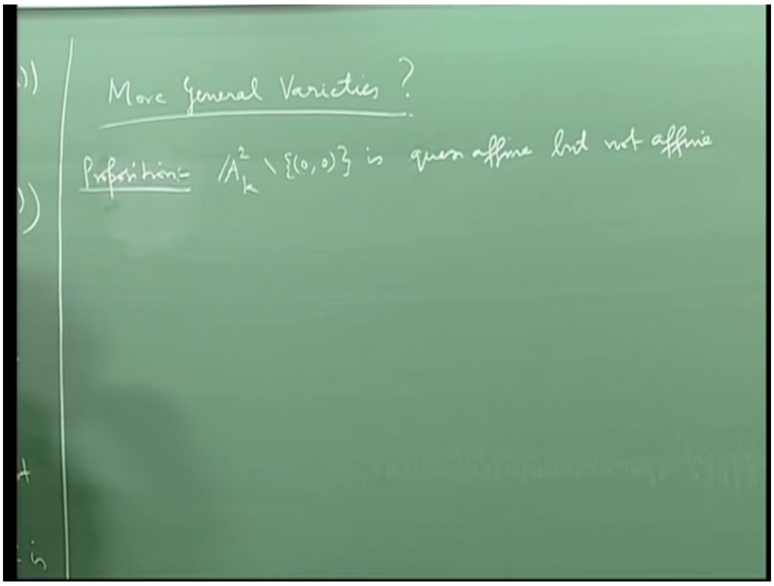

So this is what I want to do next right, now so let me give first an example, so what are the varieties that we know so far? We know affine varieties and we know quasi-affine varieties ok and quasi-affine varieties are open subsets of affine varieties and you know that there are quasi-affine varieties which are actually affine and for example basic open subsets they are all quasi-affine varieties but they are actually isomorphic to affine varieties in a fine space one higher dimension ok, because of this so called Rabinowitsch trick ok.

So the basic open set defined by non-vanishing of a polynomial is a quasi-affine variety in that affine space but in an affine space of dimension one more it becomes a close reducibly close sub-variety ok. Now the question you can ask is are the quasi-affine varieties is not affine? So I want to give an example of such a specimen ok and just to tell you that there are quasi-affine varieties is not a affine varieties so here is a fact so here is a claim or I can just put it as lemma, A2 K minus the origin is not affine, is quasi-affine but not affine.

So here is the lemma actually it is more than a lemma it is you can call it a theorem because you going to use this result and you know that this result is in it's a grand version of the Nullstellensatz so it is very bad to call it but anyway I call it a lemma alright. So and it is usually or maybe I will have second thoughts and atleast call proposition, so what are we going to do, see first we will understand this statement it is quasi-affine because it is an open subset of affine variety namely it is an open source set of A2 and it is an non-empty open subset because I only deleted the origin alright.

So compliment to the origin so it is an non-empty open subset so it is certainly a quasi-affine variety but I want to show it is not affine. What you mean by saying it is not affine? What I mean by that is it cannot be isomorphic to any affine variety that means you cannot find an isomorphism of this punctured plane this is a punctured plane, this punctured plane you cannot find isomorphism of that with an irreducible close of subset of any affine space that is what it means ok.

That is what it means to say that it is not affine, correct, so it means that if I take any map from the punctured plane into any affine space certainly it is never ever going to be a closed embedy I can never expected to be a closed embedy alright. So you see it is like if I if you stated in this generality it looks the proof looks very difficult to you know verify because you are trying to say that I will have to just try to think of look at all possible you know maps morphism of this into a various of affine spaces in and I will have to check that each one of these is not a embedding onto a close subset.

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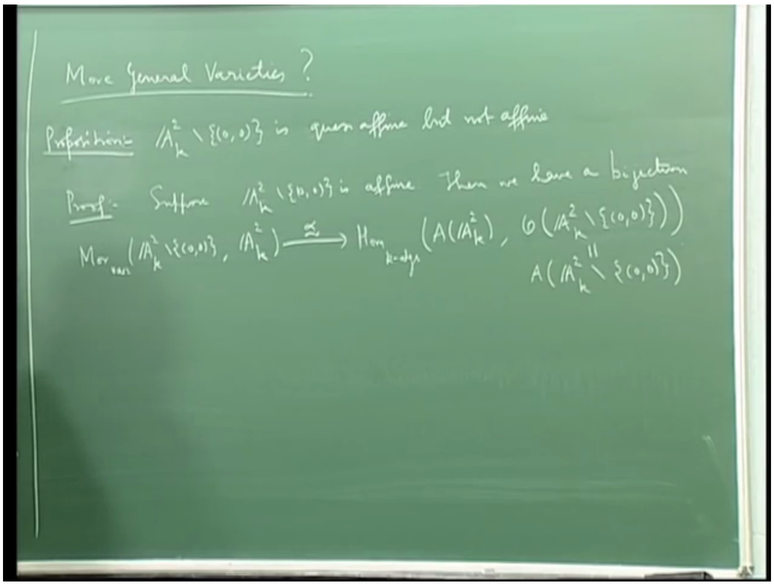

It is not an isomorphism onto a close subset, that is how that is what it means ok. (whether) but the way we prove it is rather we use all the techniques that we have developed so far that is some of them so we prove by contradiction you go by contradiction so suppose A2 minus a point A2 minus the origin punctured plane is affine ok. Now again what that means is that I am assuming that is isomorphic to affine variety ok. So see if it is a affine then it has affine coordinately alright, so then we have a here bijection as we have seen there is this bijection alpha which is morphisms of varieties from A2 minus a point minus the origin to A2 so I have this map from this into uhh homo-morphisms of K algebras from A of A2 to O of A2 minus a point ok.

So I have this bijection that I have already written down here, ok the set of morphisms from any variety into affine variety is given by is in bijection with the set of all K algebra homo-morphisms from the coordinately the target affine variety to a regular functions of the source variety right. So I am just applying that here but I am noting that uhh well you know since I have assumed A2 minus a point to be an affine variety I this can be replaced by A of that ok and offcourse so this is this can be replace by A of A2 minus, minus zero comma zero and what this means is that what does this means, see because I have assumed A2 minus a point is affine there is some embedding of it as a any (redu) uhh with an some embedding of it into some affine space some big affine space.

I don't know what dimension and certainly dimension get (()))(39:52) equal to two alright and in that affine space since it is a realize as a closed sub-variety I take the ring of the polynomial functions there and that is what this means and these two are one and the same ok that is something that we have proved for affine variety the regular functions and the ring of polynomials are one and the same ok. Offcourse by polynomials I mean polynomials restricted to the affine variety right.

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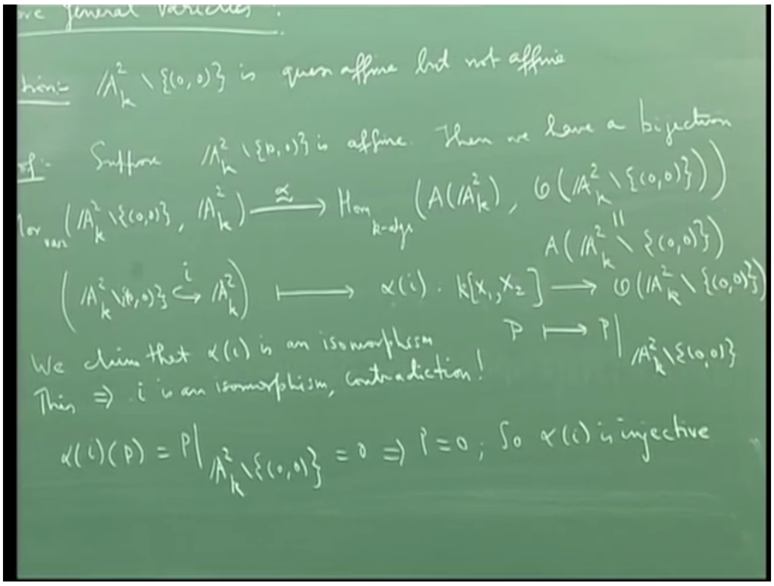

So now in particular you know I will look at the canonical inclusion I will look at the inclusion map A2 K minus the origin this is a sitting inside A2 and there is a natural inclusion map there is this map so let me put the I here ok then I will get this will go to alpha of I ok and what is this alpha of I? this alpha of I is going to be a map from K of let me write this like this K of X1 etc Xn to you know O of A2 K minus the origin ok. I will have this. Now the you know that under this bijective correspondence ok, oh sorry I should not write X1 etc Xn it should be only X1 X2 sorry offcourse yeah thank you yeah.

So alpha is from K X1 X2 because it is just affine space two dimensional affine space is just polynomials in two variables, so that N may root below before was equal to 2 right. So I have this map see ok so what I told you is that under this bijection you say I told you isomorphism corresponds isomorphisms ok. What I will prove what you can actually see is that alpha of I is actually an isomorphism. Therefore it will mean that I has to be an isomorphism but I cannot be an isomorphism because it is not even surjective because there is a point missing ok. So that is how you will get the contradiction ok.

And this contradiction will prove that A2 minus a point cannot be identified with any closed sub-variety of any affine space so it is not a affine ok. So (claim) we claim that alpha of I is an isomorphism this would imply that I is an isomorphism and that is a contradiction. So this contradiction prove that our assumption our original supposition that bunch of plane is affine is wrong so this proof proceeds by contradiction right. So (how it) to show that alpha of I is an isomorphism ok there are two things that need to be done it is a K algebra homo-morphism I have how its injective then I have to show its surjective ok.

So let me explain why alpha of Y is injective so first of all lets understand what alpha phi is. Alpha phi is a you know his alpha is just the map that is induced by pullback of regular functions ok. So what is the meaning of the alpha of I it means you give an element here it emits an polynomial in two variables so it is a function on this A2 ok and if you compose it with I which amounts to just restricting the polynomial to the punctured plane that is what it goes to so it is just a polynomial going to polynomial restricted to A2 minus a point, A2 minus the origin.

This is what the map is, because pullback means you take a regular function on the target you compose it with the map. In this case the morphism is I but composing morphism with I is as same as restricting that morphism. Because I is just the inclusion of this subset ok. So pulling back a map under inclusion is just restriction to the subset corresponding to the inclusion. So what is this alpha of I it is just take a polynomial and restrict it to the punctured plane ok.

Now how would I show that alpha of I is injective homo-morphism by showing its kernel is zero because after all its K algebra homo-morphisms to check it is injective I have to just show its kernel is zero so if I have polynomial function ok if I have polynomial which if I restrict to a 2 minus a point vanishes ok then it vanishes everywhere because you see, see if a polynomial vanishes on a set it will also vanish on the closure of the set because of the continuity of the polynomial for this risky topology ok.

Therefore if the polynomial vanishes if this is zero then you are saying that the polynomial restricted to the punctured plane is zero but the punctured plane is a dense open it is an open subset it is an dense open subset any non-empty open subset is reducible and dense ok therefore this polynomial is going to vanish on a dense open subset therefore it will vanish everywhere by continuity. So what will tell you is that this polynomial as a function vanishes everywhere and in this case this because you are working with a an infinite field the field is an algebraic close field so it is infinite.

So if a polynomial vanishes uhh as a function then it has to vanish as a polynomial. So see so that tells you that alpha of I is injective ok. So alpha of I of P is equal to P restricted to A2 minus the origin is equal to zero implies P equal to zero so alpha of I is injective it says that alpha of I is injective the only thing that I will have to now prove is that alpha of I is surjective. If I prove that then I would have then I am done then I proved that alpha of I is an isomorphism and we would get the contradiction that we want alright.

So how do I prove is surjective? So we do it like this, so well alpha of I surjective how do I prove this? Well so what I do is I start with a regular function on the punctured plane and I have to show that it is given by the restriction of a polynomial ok. So what is the statement? I sort the regular function on the punctured plane and I have to prove that this regular function is nothing but restriction of a polynomial in two variables ok.

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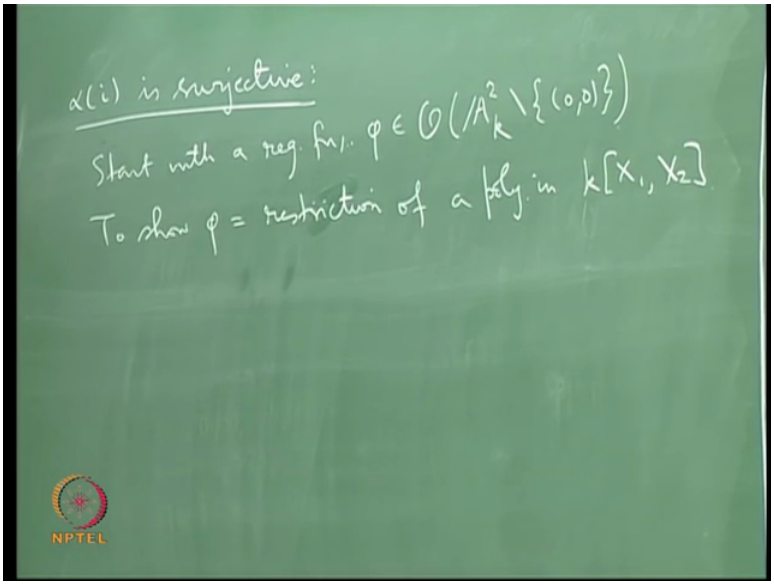

So start with a regular function phi be on the punctured plane to show phi is equal to restriction of a polynomial in K X1 X2 this is what I have to prove alright, his is what I have to prove.

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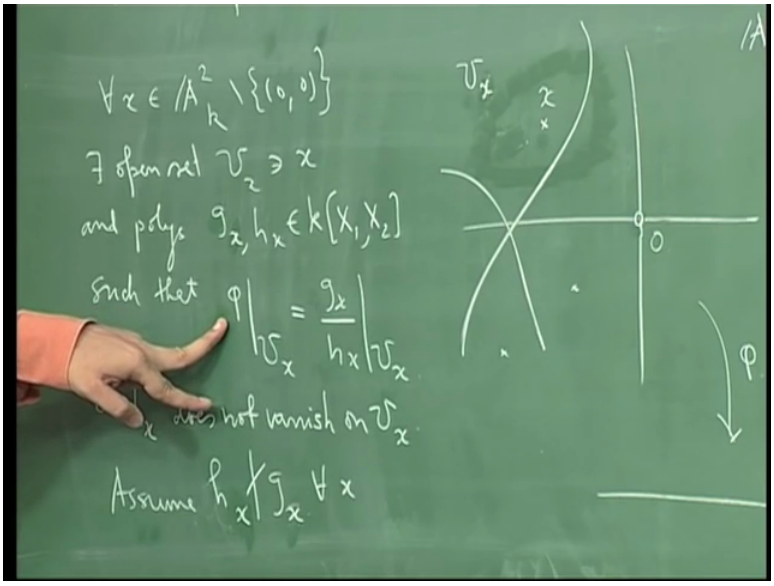

Now so you know so let me draw a (dia) a diagram so you have something like this you have the origin this is A2 and I have thrown out the origin so I will put a circle here ok. So this is the punctured plane and I have regular function on this namely well I have a phi which takes values in A1 right. So this is my phi and I have to show this phi is actually coming from a polynomial right.

And what is the meaning of saying that this is a regular function lets go back into that give any point X or any two ok, then you know I can find a an affine neighborhood I can find a neighborhood of X U X ok so you know so this is a neighborhood of X ok and this is I must admit that this diagram is not accurate because the neighborhood does not look like that any neighborhood of a point is going to be anonym to open sets so it will be dense ok. So more ideally I should think of the neighborhood well as a compliment of some curves ok right.

This is how it should like but then you know so you know if I take a neighborhood U X of the point X it is the compliment of bunch of curves alright. Because the only close subset the close subset here a curves and this are the one dimensional closed subsets and the zero dimensional closed subsets will points so it will be the compliment of some curves and some maybe finitely many points ok that is how an open set here will look like.

So given a point X I have this neighborhood U X and then what I have is the fact that it is a regular function means that on this U X is given by a quotient of polynomials with the denominator polynomial not vanishing on U X ok. So let me write that there exists an open set open offcourse Zariski topology U X and taking X and polynomials G I, G X, H X and K X, X1, X2 such that Phi restricted to U X is the same as G X by H X restricted to U X and H X does not vanish on U X ok. This is the definition of what are the regular functions.

Regular functions is something that is locally given by quotients of polynomials and to make a quote make sense of the function defined by a quotient of polynomials the denominator polynomial should not vanish because you can’t divide by zero ok. So this is what it means alright. But now notice that the you know the what I need to prove is that phi comes from a polynomial alright and if phi came from a polynomial then that polynomial restricted to U X will be equal to this quotient of polynomials restricted to U X ok.

And you know if I cross multiply it what I get is that I will get H X times that polynomial equal to G X everywhere ok, because the fact is if two polynomial function agree on an open set they agree everywhere. If two regular functions agree on an open set they agree everywhere ok. The reason is because open sets are dense and polynomial functions, regular functions they all are continuous and if a continuous function is zero on a non-empty set is identically a zero.

So if two continuous functions are equal on a non-empty on a dense set ok then they have to be equal everywhere if they if two continuous functions are equal on a dense set then they have to be equal everywhere by continuity alright. So the what this will tell you is that you know finally I have to prove phi is a polynomial ok so it will tell you that H X has to divide G X ok it will tell you that the polynomial will be equal to G X by H X on U X ok but if I cross multiply it will tell you that polynomial into H X is equal to G X on U X but then that will mean polynomial into H X equal to G X on whole affine space.

Because if two polynomials coincide on an non-empty set they are the same ok. So what it will finally tell is that H X divides G X ok so to obtain a contradiction I will assume H X does not divide G X and I will try to obtain a contradiction. So assume H X does not divide G X for every X I assume that offcourse I know I am in the polynomial ring which is a unique factorization domain and it makes sense to talk about when one divide the others because you have unique factorization. Any polynomial can be uniquely factor into reducible columns.

So this assumption is, this assumption will be true only if phi does not come from a polynomial function please understand I have to prove phi comes from polynomial function ok. Otherwise I have to show phi is just restriction of a polynomial alright but if phi is a restriction of a polynomial it will mean that H divides G ok. Conversely if H divides G alright then phi will be the restriction of the polynomial alright. So if your assume phi does not come from polynomial which means if you are assumed surjectivity is false then your actually assumed mean that H X is does not divide G X that is what you assume ok.

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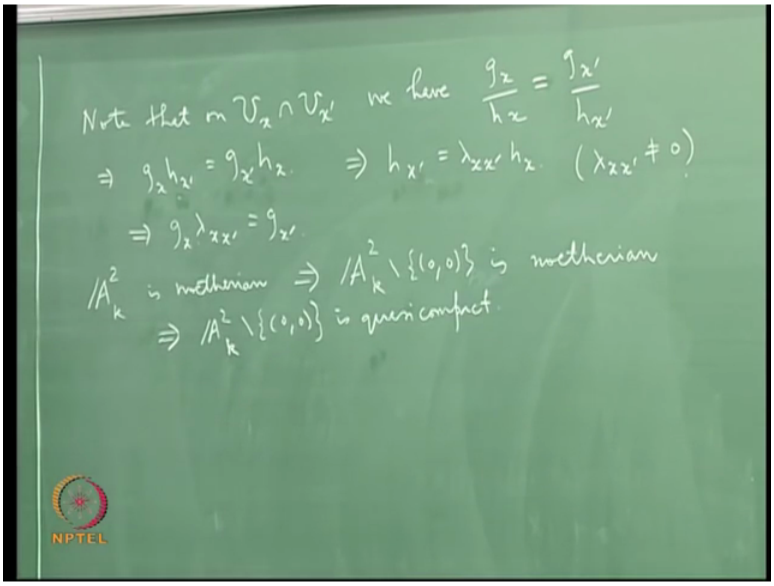

So let's assume that it will come to a contradiction. Now note that on U X contradiction U X prime we have well G X by H X is equal to G X prime by H X prime ok you will have this alright and that will tell you that you know G X H X prime is equal to G X prime H X right and now you see H X divides the right side so H X divides this but H X does not divide G X so it just divide H X prime ok and you will similarly get H X prime divides H X so the moral of the story is that H X and H X prime will differ by a non-zero constant ok.

So this will tell you that H X prime is equal to some lambda X X prime in to H X this is what you will get ok. That is because you see H X let me again repeat the argument H X divides the left side so H X divides this product so it has to divide but it does not divide G X that is our assumption so it has to divide H X prime but this argument is symmetric in X and X prime so you will also get H X prime divides H X ok and therefore if two polynomials divide each other they have to just be constant multiples of one another.

And that constant offcourse should be a non-zero constant ok. So this lambda X X prime is not zero and well if you put that back into this what you will get is that you will get that G X into lambda X X prime is equal to G X prime you will get this ok. So it also tells you that the G's are they differ by a non-zero constant multiple of one another alright. Now what I am going to do is I am going to do the following thing you know A2 K is noetherian this is noetherian and you know any sub space of a topological space it is noetherian is also noetherian so this will tell you that A2 minus the punctured plane A2 minus 0 0 A2 minus the origin is also noetherian and you know noetherian topological space is quasi-compact ok.

Therefore what will happen is that so what you will get is at this quasi-compact so you will get A2 minus is quasi-compact infact one characterization of noetherian and topological space is that every (subs) a topological space is noetherian if and only if every subset is quasi-compact ok. So this is cozy compact but you know if I take all the U X's as X varies in A2 minus a point I get an open cover for A2 minus a point and that quasi-compact so finitely many of this should be enough to cover A2 minus a point.

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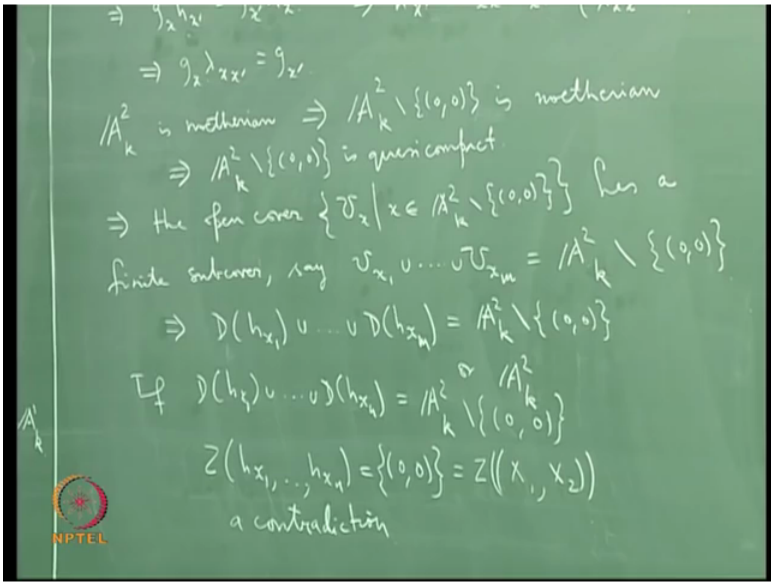

So this implies that the open cover U X, X belonging to A2 minus a point A2 a punctured plane has a finite sub-cover say U X 1 union U X N, U X M is equal to A2 minus a point ok.

And mind you each open set is being taken A2 minus a point ok, right, and for every point in this I am looking at a open set there alright. If you want the only problem is that this open subset might contain the origin but then you can throw it out you can simply throw it out from the open set and replace it with you can puncture it at the origin if it contains the origin. So you see this therefore this union will be this alright. Now well you see what this will tell you is that if you look at it the, it will tell you that all the see the H X does not vanish on U X ok.

So it will tell you that U X is contained in D H X ok, so it will tell you that U X is contained in D H X alright and therefore what it will tell you is that all the D H X or the corresponding D H X is will certainly contain this alright. So it will you that D H X 1 or union D H X N, X M will be two minus this point or it may even be A2 itself ok. So in any case every D H X contains U X so you know if you take the union of all this D H X I say it has to contain this punctured plane it might even contain the origin alright.

Now what I want to tell you is that each of this has a meaning if you exhaust each case you will get the contradiction if D H X 1 union D H X N is equal to A2 if that is equal to the punctured plane what this will tell you if you take complimentual it will tell you Z of H X 1 H X N is the point ok. Because you know if I take compliment of D H I I will get Z H and intersection of all the Z H is just Z of this ok. But you see for a point this is actually Z of the maximal idea X1, X2 because the point 0, 0 corresponds to the zero set of the maximal ideal X1, X2 alright.

So but you see and mind you that you know all the H X they are multiples of constant multiples of one another. So if you look at the zero set of all the H X it is essentially zero set of a single polynomial and you are saying the zero set of a single polynomial is a point which cannot happen because the zero set of a single polynomial has to be a curve ok because you know that the you know there are this we have seen this is one of the earlier lectures that there is a notion of geometric hyper surface and there is a notion of commutative algebraic hyper surface it is a locus which is given by vanishing of single polynomial and it has a dimension one less ok.

So if all the H X are just multiples of one polynomial then this is just the zero set of one polynomial and you are saying a zero set of one polynomial in two variables is just a point that cannot happen the zero set of a polynomial in two variables has to be a union of hyper surfaces. So in case it should be a union of curves it cannot be a single point. So that is the contradiction, here contradiction ok the zero set of a single polynomial in two variables cannot be a single point. On the other hand so the other possibility is the union of all this is A2 ok. If the union of all this things is A2 it will tell you that all the, this is something that we have already seen.

It means that all the H X will generate the unit ideal, the ideal generated by the H X will be a whole polynomial ok.

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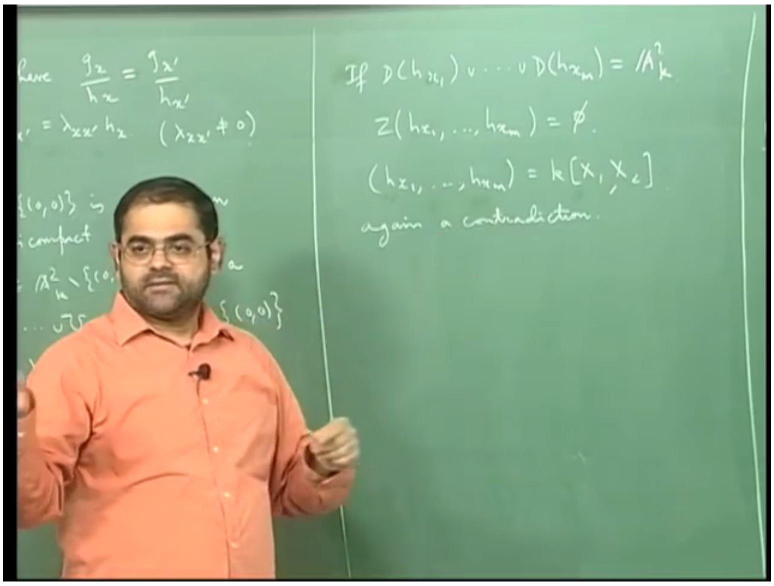

If D H, H X 1, D H X M is A2 what is will means is that Z of H X 1 with X M is a null set ok and this means that the ideal generated by H X 1 H X M is the whole polynomial, because one version of the notion says that if you take a (non) if you take a proper ideal a non-trivial ideal then the zero set of the ideal cannot be the null set ok.

So zero set of an ideal is a null set if and only if that ideal contains is unit ideal right. So but then you know but all the H x are all multiples of one another. So you are saying that this polynomial ring into two variables is generated by a single element and that is again a contradiction ok. So again a contradiction. It is a contradiction because you know all the H Xi's are multiples of one another. So this is the ideal generated by single polynomial and you are saying the ideal generated by a single polynomial contains 1.

So it will mean that, that polynomial multiplied by some other polynomial is equal to one which will mean that, that polynomial itself is a non-zero constant but then offcourse I assumed all the polynomials the H x to it will finally reduce to a assuming that all the H x are constants but then that will tell you that phi's are all polynomials but I already assumed that phi does not come from a polynomial so again get a contradiction ok. So both this contradictions demonstrate that you know if you assume that this phi does not come from a polynomial then you get a contradiction.

So it means Phi comes from polynomial every regular function on the punctured plane is the restriction of air, polynomials in two variables therefore the map is surjective and we are done ok. So that finishes the proof and therefore the moral of the story is the punctured is an example of a quasi-affine variety which is not affine ok. So we do have quasi-affine varieties which are not affine right. So I will stop here and what I am going to do in the next lecture is I am going to tell about projective varieties and quasi-projective varieties which are the more general varieties than affine and quasi-affine varieties ok. So let me stop here.