Basic Algebraic Geometry

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Lecture-20

The Coordinate Ring of an Affine Variety Determines the Affine Variety and is Intrinsic to it

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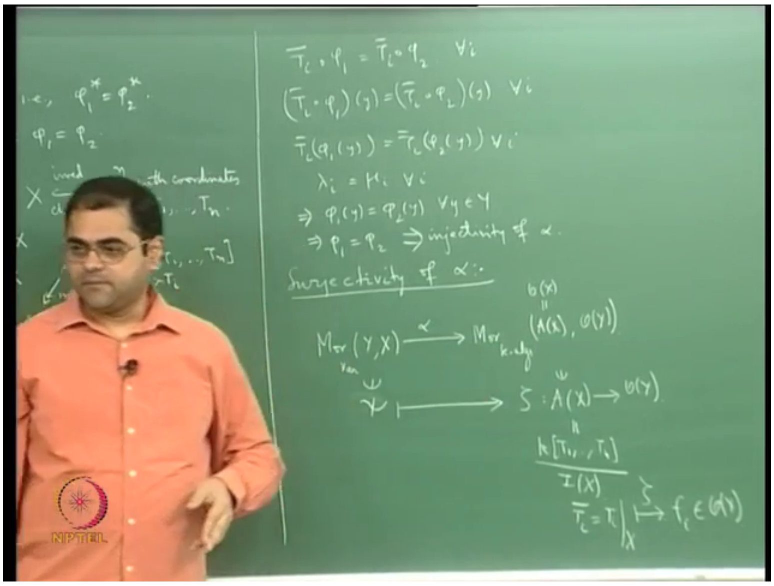

Let us continue with discussion, so what we have just shown in the previous lecture is that if you start with any variety Y and you take an affine variety X then there is a bijection between the set of morphisms from Y to X and the k-algebra homomorphism from the ring of affine functions, ring of polynomials on X to regular functions on Y and this map is just the map induced by pullback of regular functions okay.

So we are just saying that morphisms into an affine via the pullback of regular functions correspond to k-algebra homomorphisms from the ring of functions on the affine, okay so let me repeat that, morphisms into an affine correspond via the pullback of regular functions to k-algebra homomorphisms of polynomials or regular functions on the affine okay, fine so now let me have this done.

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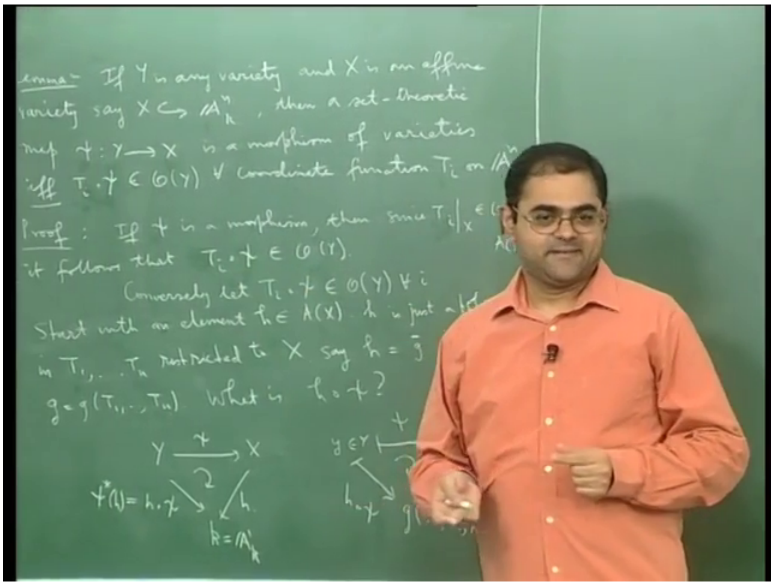

Let me tell you that there is a little bit more to this proof that can be extracted a nice little lemma, here is a lemma if Y is any variety in our case that means either affine or quasi-affine variety and X is an affine variety, say X is in An okay then a set theoretic map psi from Y to X is a morphism, it is morphism of varieties if and only if Y followed by Ti is a regular function on Y for every co-ordinate function Ti on An okay.

So actually this is a very beautiful statement it tells you how to quickly verify that set theoretic map is a morphism okay and so all it is saying is that you know from any variety which maybe affine or not affine if you have a set theoretic map into an affine variety to check that it is a morphism of varieties all you have to is you compose it with the projections which are the co-ordinate projections.

The target variety is sitting inside some affine variety on that affine variety you have the projections it will give you the various co-ordinate functions you compose this map with those projections the resulting functions will be functions from Y to k and all you have to verify is that these are, they are regular functions on Y so it is a very beautiful statement it says a map from any variety into an affine variety, a set theoretic map is a morphism if and only if it pulls back every co-ordinate function to a regular function that is what this is okay.

And this is, the proof is actually, the proof is already contained in the earlier proof that I gave you okay, in the proof of the statement that the morphisms into an affine variety correspond to k-algebra homomorphisms from the ring of polynomial functions on that affine variety, nevertheless let me repeat it okay, so you see if and mind you this is an if and only if condition one way of the proof is very very simple.

If psi is a morphism, then of course, then since Ti restricted to x are regular functions on X it is clear it follows that Ti circle Y, I am sorry I think it should have been Ti circle psi not Ti circle Y okay that is wrong notation Ti circle Y does not make any sense okay I should compose Ti with psi okay please correct that okay so Ti circle psi will of course be regular function on Y, this is very simple, that is just because of the definition of the morphism.

Because of morphism is supposed to pull back regular functions to regular functions okay so T the co-ordinate functions on An if I restrict them to X they are going to be regular functions on X because after all the co-ordinate functions on An are the polynomials these are the polynomial variables okay so when I say co-ordinate function Ti on An it means that I am thinking of the functions on An to be given by the polynomial ring in the n variables T1 through Tn okay.

So each Ti is a polynomial okay and you know polynomial functions are of course regular functions and therefore if you restrict each of these polynomials to X they are in O X and mind you O X is the same as A X, O X is the same as A X because X is affine okay and psi is already given to be a morphism so it will pullback regular functions to regular function and what is the pullback of Ti restricted to X under psi it is just Ti circle psi okay it is just a composition and that is supposed to be a regular function of Y because of definition of the morphism okay.

So one way is very easy, it is the other way that we need to understand, conversely let Ti circle psi if you want of course here when I say Ti circle psi I mean Ti restricted to X circle psi, does not matter that is what it means because I am evaluating Ti on the image of psi which is X because psi goes into X okay, conversely assume that psi is just a set theoretic map with this property that it pulls back co-ordinate functions on the target affine variety into regular functions okay that is this Ti circle psi is in O Y for all i okay suppose this holds.

Then I will have to show that psi is a morphism okay, now how do I show that psi is a morphism I have to take check two properties of psi namely the first thing I have to check is that psi is I will have to check that it is a you know continuous, then the other thing I have to check is that it pulls back regular functions to regular functions, so the idea is very very simple so let me say tell you the idea in a very qualitative way.

How do I check psi is continuous? I check psi is continuous by checking that the inverse image of closed sets are closed but what is a closed set, a closed set on X is just given by common zero locus of a bunch of polynomials okay and what is the inverse image of such a set under psi, it is a common zero locus of the bunch of regular functions on Y that I got from these polynomials by composing with psi and a set of common zeroes of a bunch of regular function is a closed subset of a variety, therefore psi pulls back closed sets to closed sets so psi is continuous, it is very simple.

So let me write that down, so you know it is actually this it is actually this argument, it is actually this argument but let me write it here if Z of J in X is a closed subset then psi inverse of Z of J is so I am just reproducing what I have written there, if psi inverse of Z of J is a intersection of Z of h where h belongs to J and this is the intersection h belongs to J of Z of h circle psi which is closed in X okay.

The zero set of a single regular function is a closed subset okay and an intersection, an arbitrary intersection of closed sets is again closed sets therefore this is a closed set, so what I have proved is that the inverse image under psi of a closed set is a closed set so this implies that psi is continuous okay, now the only other property that I have to check that, check in order to ensure that psi is a morphism is that it pulls back regular functions to regular functions.

So there is a small hitch here in the argument I will need to, I have already used the fact that h circle psi is a regular function this is something that I will have to show okay and that is purely a matter of verification, so let me do the following so the correct way to do it is to use the fact that these are regular functions yeah so basically it is a calculation. So what I need to do is a following, so let me rub this of just to say it in sequence so I need to tell that start with, so I have to make a small calculation.

Start with an element h in, start with an element h in A X, h is just a polynomial in T1 through Tn restricted to X this is what it means because X is an affine variety in An and the ambient affine space An in which X sits is supposed to have these co-ordinates, so the polynomial ring generated by this is a set of all polynomial function on the affine space in which X sits and any element in the polynomial function sit, the ring of polynomial function on X is the restriction of the polynomial to X okay.

But of course the function is uniquely defined but the polynomial is not uniquely defined it is defined only upto addition by an element of the ideal of X okay so choose h instead of h choose a polynomial okay that represents h and call that also as h okay then say h is if you want let me call this as g bar where g is g of T1 etc Tn, g bar is the image of g in A X which is the quotient of the polynomial ring in n variables in these n variables okay which is the ring of all polynomial functions on A okay.

Now look at, so I want to calculate what the pullback of h under psi is this is what I want to do okay so what is h circle psi let us calculate this so basically what is happening is that I have Y, I have psi, I have X and here I have a polynomial h okay and this is my map h circle psi okay it is just pullback of h under psi, it is a psi star of h okay, it is just this, what is it if I evaluate it.

So if you take a point small y and capital Y that goes under psi to psi of Y and that will go under h and you know the psi of Y will have some co-ordinate lambda 1 etc lambda n okay and that if I apply the function h of course the function h is going into k which is A1 if you want okay with the zariski topology alright so if I apply h, applying h is the same as applying g okay so what I will get here is g of lambda 1 etc lambda n this is what the map this is this map which is h circle psi okay.

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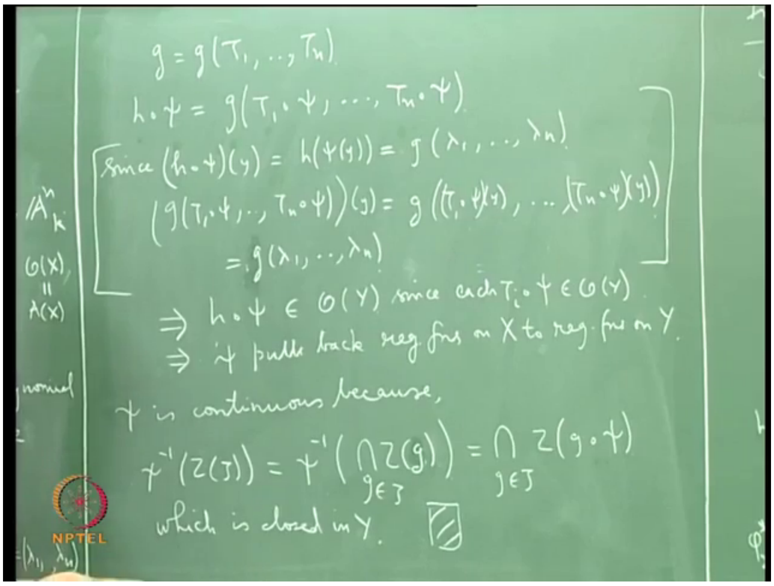

And if you look at it now it will be clear that h circle psi is regular function on certainly a regular function on capital Y because it is actually a polynomial in the Ti circle size in exactly the way in which g is a polynomial in the lambdas, in the T’s, see g is g of T1 etc Tn okay so if you calculate h circle psi it will be actually just g of T1 circle psi etc Tn circle psi you will get this, because you know evaluate it and check.

Since h circle psi if I evaluated a point what will I get, I will get h of psi of Y and h of psi of Y, psi of Y is lambda 1 through lambda n and then I evaluate h on that h is represented by a polynomial g so I will get g of lambda 1 etc lambda n okay and you will also see that g of T1 circle psi etc Tn circle psi evaluated at a point, at the point Y is just evaluating g on T1 circle psi of Y comma dot dot dot Tn circle psi of Y, but then you see psi of Y is the point lambda 1 through lambda n and if I apply Ti to that I will get lambda I so I will simply get g of lambda 1 etc lambda n okay.

So this calculation tell you that h circle psi is just a polynomial in the Ti circle psi so what is h circle psi it is a polynomial combination, it is a polynomial in these functions but what are these functions they are already given to be regular functions on Y each Ti circle psi is already a regular function on Y and so if you write, if you give me a regular functions on Y and write a polynomial in those regular functions on Y with co-efficient in the field, the result is again a regular function because sum and product of regular functions is again a regular function okay.

So all this will tell you that h circle psi is in O Y okay since each Ti circle psi is in O Y okay you will get this, so what this will tell you, in principle it will tell you that psi pulls back regular functions on X to regular functions on Y okay so this is one of the conditions for the map psi to be a morphism, the only other condition now I have to check is the continuity of psi, which will follow as I gave in the earlier proof, so only I have to check psi is continuous and that is because of the following thing.

Psi is continuous because psi inverse of Z of J is psi inverse of intersection of Z of h, let me put Z of g, g belong to J this is intersection g belongs to J, Z of g circle psi which is closed in Y, so I am using the fact that if you give me a bunch of regular functions and look at the set of common zeroes that gives a close locus and the reason for that is every regular functions locally quotient of polynomials are looking at the zero set is actually the same as looking at the zero set of the numerator polynomial okay.

So and we are done, so I verified that if psi pulls back co-ordinate functions to regular functions then not only is psi pulls back any regular function to a regular function but it is also continuous so psi is a morphism okay so it is a very beautiful statement it tells you very quickly how to verify that a map from a given variety into an affine variety is a morphism okay, it is very simple all you do is you just show that the pull backs of all the co-ordinate functions are regular functions, that is all you have to check okay fine.

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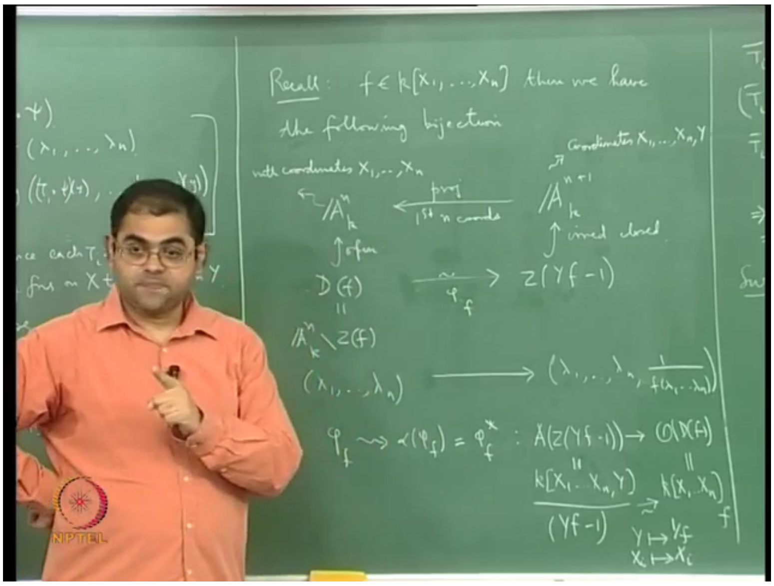

So this is a fact that is kind of buried in the proof of the theorem that I gave in the previous lecture so I just wanted to bring it out, now what we will do is let us try and apply this and reconcile some of the results that we had earlier, that we had already talked about earlier, so if you recall that you know if you have f in polynomial ring k1 etc upto k of x1 through xn then and yeah then we have the bijection the following bijection.

So you see you have An and this is just, this is with, this here is with co-ordinates X1 through Xn okay and you had D f the basic affine open set defined by f this is just, it is just a compliment of the zero set of f, it is An minus zero set of f okay, so it is a locus where f does not vanish and that is an open subset and I told you that this is a basic open subset, it is basic because every open subset can be written as union of such open subsets and because of quasi compactness in fact it can be written as a finite union of such basic open sets.

And I told you that this, the other beautiful thing about this subset of this type is that they are actually themselves isomorphic to affine varieties and I told you that they are isomorphic to a closed, irreducible closed sub variety of affine space one dimension more okay so how does that come about, that comes about by taking An plus one of k with co-ordinates, the same X1 through Xn but I add an extra co-ordinate Y okay.

And what I do is well I look at the zero set of Y f minus 1 okay so f is a polynomial in the first n variables and Y times f becomes Y minus 1 becomes a polynomial all the n plus 1 variables okay and this polynomial is irreducible and its zero set therefore gives an irreducible closed subset of affine space mind you this is open and this is irreducibly closed of course you must remember that this being an open, non-empty open subset of affine space of course I am assuming this is non-empty set okay, f is a non-constant polynomial okay.

And mind you that it is an open subset of affine space non-empty open subset means that it is also irreducible and it also dense mind you this set here is irreducible and dense okay and this set here is irreducible and closed and we have this projection, you project onto first n co-ordinates that is this projection map okay and under this projection map you have an isomorphism under the projection.

There is a bijective map from here to here this is something that I told you and in fact I told you that this bijective map is actually an isomorphism of varieties okay in fact the map is very very simple to define you give me an element lambda 1 etc lambda n at which, which is here it means f does not vanish at this element you simply associate to it this element lambda 1 through lambda n and then you add 1 by f of lambda 1 to lambda n then you know this satisfies the last co-ordinate Y multiplied by f applied to the first n co-ordinates minus 1 equal to 0.

So it satisfies the equation and I can invert f of lambda one through lambda n because lambda 1 through lambda n is in the locus where f does not vanish okay, so this is very easy to check that this is a bijective map but I told you actually to check as an exercise that is it is a homeomorphism of topological spaces I hope you done that but now the time, and then I told you that it is even an isomorphism of varieties, I have told you that this is an isomorphism of varieties and you know I told you that this is something that we will see later because when I told you at that time I had not defined what a morphism of varieties was.

But now that I have defined what a morphism of varieties is, I can go back and justify this statement okay so if you look at this statement now okay what you can see immediately is that this is a quasi-affine variety because it is an open subset, non-empty open subset of affine variety this is an affine variety okay and if you take this map I have a set theoretic map now, it is even a bijective map, I have a set theoretic map from here to here okay and let me take the map in this direction okay.

Let me take the map in this direction, it is a bijective map okay. So you see I have a map from a variety to an affine variety, this is an affine variety okay and it is a set theoretic map how do I check that it is a morphism? I check that it is a morphism by checking that if I compose it with the co-ordinate projections okay then the resulting things give me regular functions on the source variety okay.

So you see if I compose this map with the co-ordinate, if I compose this map with the first co-ordinate projection I simply get the first co-ordinate projection here if I compose this map, the second co-ordinate projection and so on upto the n th co-ordinate projection I simply get the n th co-ordinate projection here, which are of course regular functions okay because a polynomial is always a regular function and a regular function restricted to an open set is also a regular function there is no problem okay.

What about projection under the last co-ordinate, okay if I project, if I well I mean so that is it I mean I just have to I just check that if I project on to the last co-ordinate I get 1 by f okay if I take a point here I take the image there and I project on to the co-ordinate, last co-ordinate what I get is a point here going to 1 by f of that point so I get the function 1 by f but 1 by f is also a regular function on D f okay.

So I have verified the condition that every projection of this map with co-ordinate functions is a regular function therefore this becomes a morphism so it is a bijective morphism now okay now the only thing you have to worry about is that if it is a bijective morphism so you would be worried as I had warned in one of the earlier lectures that a bijective morphism need not be an isomorphism because the inverse map need not be a morphism okay.

But the fact is that what this induces at the level of regular function okay is the following you will see that if I call this map as let me call this map by something, let me call this is as p, no I should not call it, this way is projection so I should not call it projection, I should call it something so let me let me give it some name phi if you want okay, let me call this as phi sub f because this f is involved okay then you know this phi sub f will induce alpha of phi sub f which is a pullback map, it is phi sub f star, it is a pullback map.

And what is this, what is a pullback map? It will go from the regular functions on the target so it will A of Z of Y of f minus 1 to the regular functions on the source this is what you will get okay this is the pullback map okay and see this is, this co-ordinate ring, ring of functions is given by simply k of X1 through Xn, Y divided by f Y of minus 1 this is what it is, this is the ring of functions on that irreducible closed subset, it is just the ring of functions of a ambient affine space which is the polynomial ring in this n plus 1 variables, the first n being given by the axis and the n plus 1 variable given by Y modulo the ideal of this closed subset.

The ideal of this closed subset is the ideal generated by Y of minus 1 because the Y of minus 1 is an irreducible polynomial and it is a an irreducible element in this polynomial ring which is a UFD, unique factorization domain, so the ideal it generates is prime so actually I should write here the radical of this ideal okay because if I take Z of J, then I will have to put I of Z of J okay but I of Z of J is rad J so I will have to put radical of Y of minus 1 but radical of Y of minus 1 is Y of, is the ideal, a radical of the ideal generated by Y of minus 1 is the ideal generated by Y of minus 1 because Y of minus 1 is a prime ideal.

And that is because Y of minus 1 is an irreducible polynomial, okay and it is sitting inside this polynomial ring which is UFD okay, so this is the ring of functions and you can I want you to checked that this is the same as polynomial ring in n variables localized at f okay so this is an exercise which I wanted to do that to check that it is very clear that if you take an element here, what is the element here in the localization? It is of the form g by f power m where g is the polynomial in these n variables, these n axis divided by some power of m.

That is of course a regular function on D f because it is a quotient of polynomials regular function is something that is locally a quotient of polynomials in this case it is a globally quotient of polynomial so it is very clear that these guys are certainly here, the fact is they are all, you can check that the inclusion of this inside that is actually surjective map therefore this is actually equal to that okay.

This is, this checking can be done and now after you do that checking you check that this map is the natural isomorphism you get in commutative algebra, it is the map that sends Y to, it is a map that is given by sending Xi to Xi and send Y to 1 by f okay this map comes from this direction because of the universal property of the polynomial ring in n plus 1 variables and the map comes in and this map is an isomorphism because the map in this direction comes from the universal property of the localization.

So you can check that this pullback map is actually this isomorphism, this isomorphism which is completely commutative algebraic okay so what this will tell you is that this map phi f which is a bijective morphism, if you look at the pullback map, the pullback map gives you an isomorphism okay and now you can if you believe that whenever you have this isomorphism of whenever you have an isomorphism then you know the object, I mean this tells you that it is correct to define the, it is correct to think of D f, of D of f as an affine variety okay and to call this as A of D of f which is what we started with, we defined A of D of f like this okay.

And I had given you, I had proved that if you take the whole polynomial ring then the O of that I mean if you take the whole affine space then the O of affine space is same as A of affine space, I told you that holds not only for affine spaces it holds for affine varieties, it holds for basic open sets okay and that this checking you have to do and so all this tell you that what we originally start with namely with defining this as A of D f if the definition is correct because we define A only for affine varieties and D f is an affine variety because actually isomorphic to this.

The fact that this bijective morphism is actually an isomorphism namely the crucial fact that the inverse map is a morphism comes from the fact that this is an isomorphism okay it comes from this fact, it is reflected in this fact that this is a isomorphism alright, so this is justification for defining A of D of as this set, as this ring okay that is one thing and then the final thing that I wanted to say is about the equivalence of categories of affine varieties and finitely generated k-algebra which are integral domains, so for that I need a little bit of functoriality.

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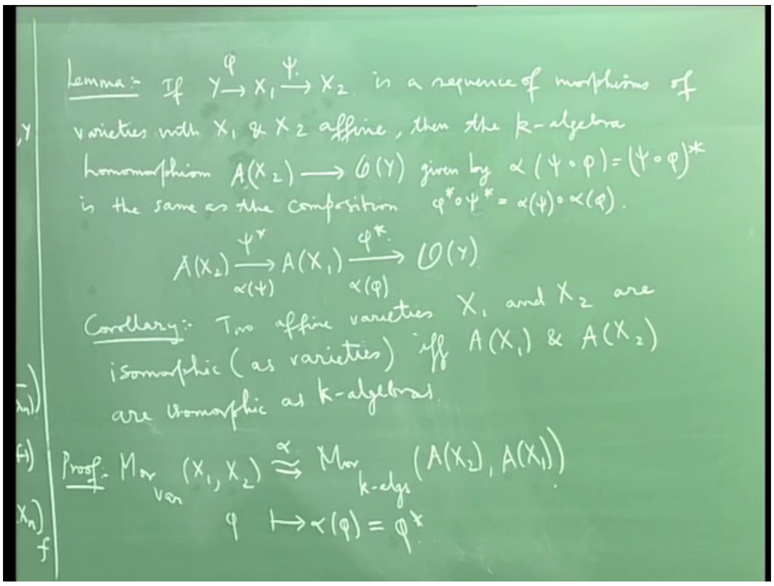

So here is a lemma that you can easily check if Y to X1 to X2, phi, psi is a sequence of morphisms of varieties with X1 and X2 affine okay then the morphism let the k-algebra homomorphism k, sorry A of X2 to A of Y given by the pull back of side circle phi which is side circle phi upper star is the same as the composition, I should be careful here it is A X so I should not write A Y, I should write O Y because as I told it is an convention that we write A only if it is an affine variety okay, Y is just a general variety, it need not be affine.

But only X1 and X2 are assumed to be affine yeah so let me continue with the statement given by alpha of psi, psi circle phi is same as the composition A of X2 to A of X1 to O of Y which is psi upper star this is the same as alpha psi and this phi upper star this is alpha phi so I should write it as, first apply phi upper star then apply sorry first apply psi upper star then apply phi upper star which is the same as first applying alpha of phi then apply alpha of psi okay.

This is this is just functorial, this is functoriality, I am just actually you know this looks a little wrong to write down but all I am saying is that the quotient of pullback of maps is very functorial which is I mean if you have a composition of morphisms pulling back the function on X2 all the way to Y is the same as first pulling it back to X1 via psi and then further pulling back the resulting function via to Y via phi okay this is a very, this is something that you can check very easily okay, there is nothing much about it okay.

Now this a very easy lemma to check okay so I will leave it you but then the important corollary to this lemma is the following, two affine varieties X, X1 and X2 are isomorphic and when I say isomorphic it means as varieties if and only if A X1 and A X2 are isomorphic as k-algebras okay, so you know this is basically because morphisms so the proof is it follows from the fact that morphisms of varieties from X1 to X2 via this alpha map which is pullback.

This set can be bijectively identified with morphisms of k-algebras from A X2 to A X1 mind you here I should have put O X1 but I can replace O X1 by A X1 because X1 is affine okay if X1 is not affine then I have to replace this A X1 by O X1 that is what we proved alright and of course this map is as usual you sent a morphism to alpha phi which is phi upper star this is just a pullback alright and what I want you to understand is that if this morphism phi is, has an inverse then there is a psi such that phi circle psi and psi circle phi are corresponding identity maps okay.

And because of this lemma it will follow that phi star and psi star will be inverses of each other and that will tell you that phi star phi is an isomorphism of varieties if and only if that induced map on pull back of regular functions phi star is an isomorphism of k-algebras okay so you get this corollary as a result of this lemma and the earlier theorem that morphism into an affine variety are in bijective correspondence with k-algebra homomorphisms from the polynomials on the affine variety polynomial functions are (())(43:49) on affine variety okay.

So this completes the statement that we have an equivalence of categories so let me write that down, that is a final statement which is, which as I mentioned couple of lectures, at the end of lecture before the last one that it is a grand you know statement of the Hilbert Nullstellensatz namely it is an equivalence of categories so let me write that down here.

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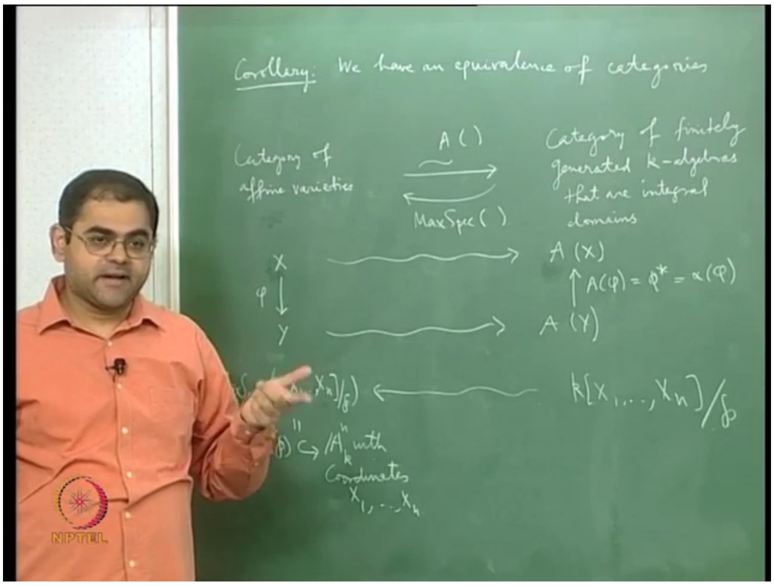

So you have corollary, we have an equivalence of categories, so here is the equivalence of categories on this side we have category of affine varieties okay these are the objects and when I say category of affine varieties, it means that the objects are affine varieties the morphisms are morphisms of varieties and on this side we have the category of affine co-ordinate rings over k which should be defined as category of finitely generated k-algebras that are integral domains.

And of course the morphisms there are k-algebra homomorphisms and you have an equivalence of categories namely you have functor in this direction okay a functor is a generalisation of function okay but since the source and target are not sets but they are categories we do not use the word function we use the word functor, so you have functor like this you have, which has a kind of inverse functor in this direction okay and what is the equivalence? You give me any X, you send it to A X so in this direction it is A okay.

And if you give me and a functor is not only supposed to map objects to objects, it is also supposed to map morphisms to morphism so if you give me another affine variety I get A Y and if you give me morphism phi from X to Y, I get A of phi, this A of phi is nothing but phi star pullback of functions which is just alpha in all our previous notations okay, so this is a functor, for every object here you get an object there, for every object here which is an affine variety I get an object there which is a finitely generated k-algebra.

And that is an integral domain because it is a polynomial ring modulo a prime ideal alright so it is a finitely generated k-algebras (())(47:12) and given any morphism in this direction, on this side I have pullback functions which is a k-algebra homomorphism okay and this and that the set of all such psi is bijective to the set of all such k-algebra homomorphism in this side is the statement that we have seek there is a corollary there okay.

I mean it is part of the theorem that we proved and what is more, what you should see is that the arrows are reversed, an arrow in this direction gives rise to an arrow in the other direction because you pull back functions from the target to the source, so we say that A is a contra variant functor because it changes the direction of the arrows as it goes, an arrow in this category is converted into a arrow in the reverse direction in the target category so it is called a contra-variant functor.

And what is the inverse functor in this direction, the inverse functor in this direction is given by max spec okay it is given by max spec and namely if you start with the finitely generated k-algebra if you start with the finitely generated k-algebra which is k of X1 etc Xn modulo some prime ideal okay then what you can do is that you can take max spec of that and this can be identified via the Nullstellensatz to the zero set of p as an affine variety in A of An, in An with co-ordinates these Xi's okay.

And so if you go like this what happens is that you get a functor in this direction this functor is also contra-variant okay and if you have bijection between sets what usually happens is that we start with an element here, you go this way and then when you come back you should get the identity on this side and similarly for the other side but when you have a bijective equivalence of categories you will not get the identity what you will get is something upto isomorphism.

So if you start with something here you go you take A X and then if you take max spec of A X what you will get is something that is isomorphic to X and why you get the isomorphism is because A of that and A of your original X will be isomorphic because of the choice of co-ordinates, the isomorphisms comes because you are choosing a bunch of co-ordinates okay (())(50:11) comes because it depends when I write A X okay you are choosing co-ordinates.

So here is the very very important subtle technical point the technical point is of following, an affine space, if you take an affine variety, the affine variety can sit as a close subset in any affine, in so many affine spaces for example take the plane, the plane can simply sit inside A2 by the identity map, it can sit as a plane, it can sit as a 2 plane in A3, it can sit as a 2-plane in any An but you know if you, but how is affine co-ordinate ring defined, affine co-ordinate ring is defined based on the embedding, based on the ambient affine space in which you are affine variety is sitting.

So if X is your plane sitting inside A, if X is a plane A2 then A X will become the polynomial ring in two variables but if X is say the X-Y plane sitting in three space then A X will become k X Y Z mod Z which is again k X Y okay so the beautiful thing is, this affine variety no more matter in what affine space it is sitting in as a closed sub variety the A X that you get will always be the same upto k-algebra isomorphism okay that is the beautiful thing, so the co-ordinate ring, the ring of functions on the affine variety is a very nice object, it does not depend on the embedding of the affine variety in a certain affine space.

If it sits in some other affine space also the ring of functions will change only upto isomorphism which is a nice thing which goes on to tell you that the fact that something is an affine space is a very intrinsic thing okay that it tells you that trying to characterise an affine variety by its ring of functions is a very intrinsic thing because it does not depend on the extrinsic choice of embedding, of putting that variety as a irreducible closed subset of some affine space, no matter in which affine space you put it as an inducible closed subset.

If you calculate the affine co-ordinate ring you will still get the same ring of upto an isomorphism which means essentially a change of variables okay so you will get an isomorphic ring, so what this tells you is that the affine co-ordinate ring or the ring of polynomial functions on an affine variety is defined upto isomorphism and depends only on the variety, it does not depend on which affine space in which you are considering this as an affine variety okay.

So it is a very intrinsic object, so this A is a very intrinsic object from affine variety and that is the subtle point, so we started with this definition then we are, it is nice that we have come to the point where we are able to say that this does not depend on the affine space in which you are considering X as an invisible closed subset okay so with that I will stop this lecture.