Basic Algebraic Geometry
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Module 1
Lecture 2
The Zariski Topology and Affine Space

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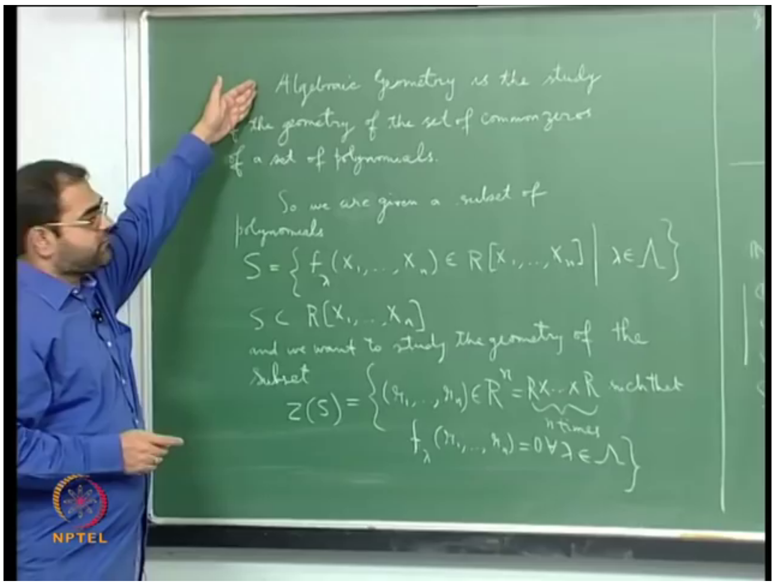

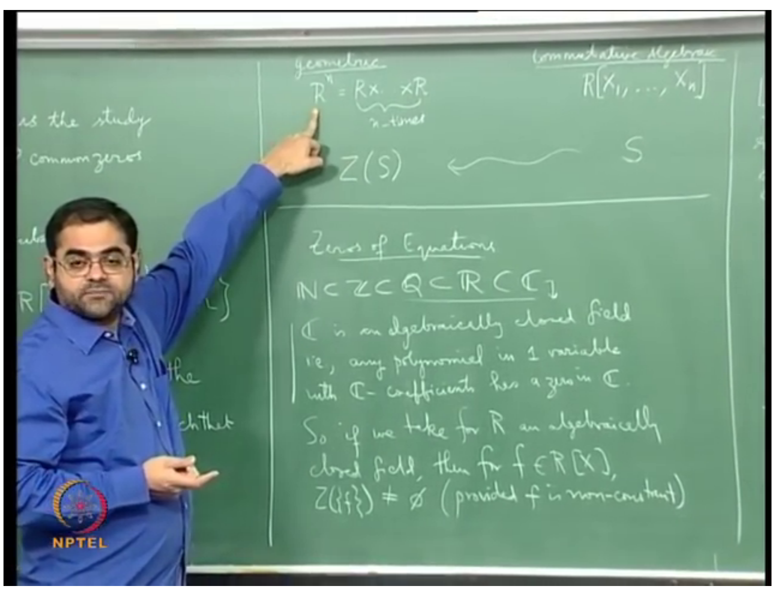

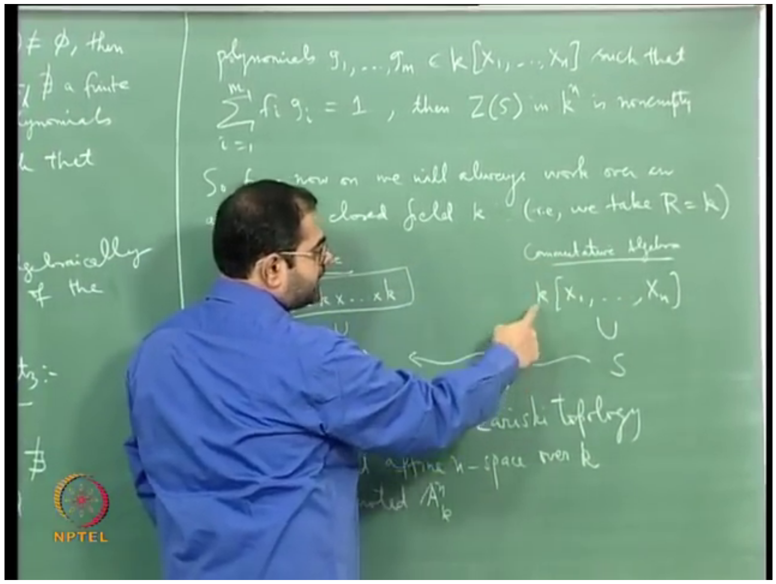

Okay so let me recall from the previous lecture so you know so algebraic geometry is a study of geometry of the set of common zeros of set of polynomials so what you do is that you take a commutative ring R and then you take the polynomial ring over that ring in n variables and you take a subset of polynomials and then you look at the 0 set define by that subset it is subset of the n it is subset of Cartesian product of R with itself n times consisting of all those n tuples which are zeros common zeros of every polynomial in this set S, okay.

And then the problem was that so then you associated to this set S the 0 set in this n dimensional space over R and of course we want to ensure that this set is now this set does not turn out to be empty set so what we do is that we assume that we always work over an algebraic closed field so we take R to be k where k is an algebraically closed field and then our situation is like this we on the commutative algebraic side we have the polynomial ring and n variables over k and we take a subset of that ring which means we take a bunch of polynomials which are the elements of this set capital S.

And then you look at the set of common zeros in n dimensional space over k and of course the important thing is that you should not worry about this as a vector space you not think as a vector space you should think of this as a affine space and which means that you are not worried about the vector properties the additional properties and so on but you are worried about other properties rather topological properties and these topological properties come from what is called the Zariski topology, okay.

So I will tell you what is Zariski topology is, so if you remember I told you that the Zariski topology is declared by declaring certain subsets as closed sets and these subsets are the loci the loci of zeros after bunch of polynomials, okay.

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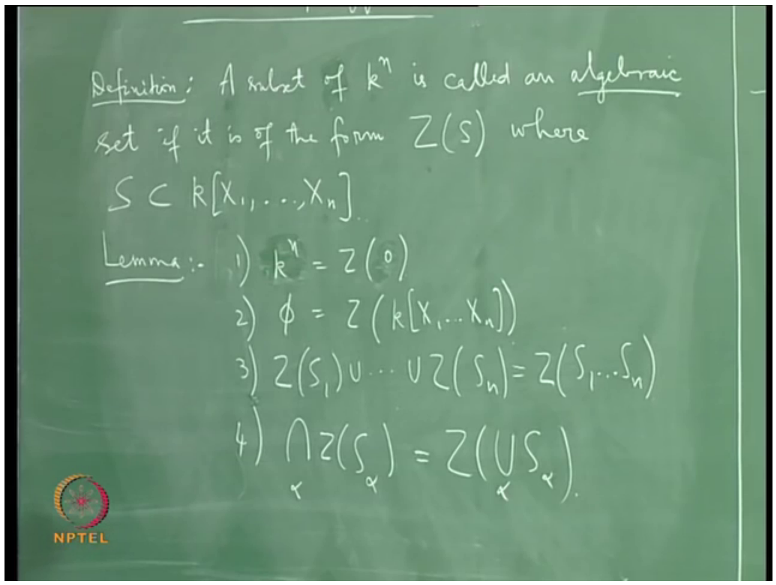

So let me elaborate on that so what I am going to do is I am going to do the following the Zariski topology on kn of course k algebraically closed field let me write it here, okay. So define the definition a subset let me say let me not even give symbols a subset of kn is called an algebraic set, okay if is of the form is Z of S where S is a subset of the polynomial ring in n variables over k, okay.

So look at the definition what it says is an algebraic subset of kn which is k cross k cross k n times the Cartesian product of k with itself n times is nothing but a locus of common zeros of a certain collection of polynomials a subset of polynomials in the polynomial ring with the same number of variables as the number of copies of k here starting with, okay. Now so immediately you have the following I am going to write down some properties and these properties will tell you that you can always declare I mean because of those properties you can declare sets of this forms closed sets and that will give you the Zariski topology.

So here is a lemma, number 1 Z of so the whole space is Z of null set, 2 or rather I should write here 0 the null set is Z of maybe I can take the whole ring, third one is Z of S1 union and so on Z of Sn is Z of S1 dot dot dot Sn, fourth one intersection for alpha Z of S alpha is Z of union over alpha S of n, okay. So you see this is the these are four properties and let us try to prove them, yeah I will explain what this S1 etcetera Sn means, it is the enfold products, okay so I will explain this steps one by one.

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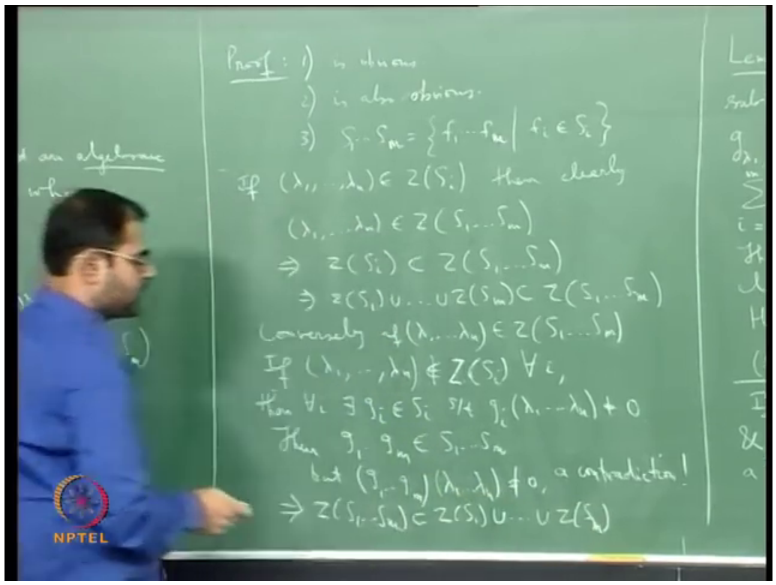

So well they are pretty easy to verify we will do that, so you see so the proof the first one is well what is Z of 0 it is all those points in kn which give 0 when you substitute them in the polynomial 0 but the polynomial 0 is the constant polynomial 0 it always has the value 0 no matter what you substitute in the polynomial, therefore every point in kn satisfies this, okay.

So one is obvious, okay and what you must understand here is that we are thinking of 0 as a 0 polynomial, okay you are treating 0 as a 0 polynomial and the 0 polynomial is also when you substitute the variables in the polynomial the value of that polynomial is the value that you get is also 0, okay. So you must distinguish between 0 as a polynomial and 0 as a value of the polynomial, okay. So the 0 polynomial lives here whereas the value 0 lives on k and so you must understand these two things distinct from each other, okay.

So but in any case this is very obvious and look at the second question look at the second statement if you take the subset S to be well you know if you want to be very particular maybe I can even put I can put this because I write 0 of a subset so I should actually give you a set and if I just write 0 it is just a single element but normally we have this abusive notation that if you have a single element you are just right for a single polynomial f here you do not write instead of f within braces but you simply write Z of f, okay you avoid the braces, so if you avoid the braces this becomes Z of 0 that why what I wrote first but to be very strict to begin with let me write it properly like this, okay.

Now look at the second statement, second statement says that this 0 set of the whole ring is empty set that is obvious because if fact any subset of the ring which contains 1 or for that matter any constant non-zero polynomial will never take the value 0 at any point, okay for example you take the polynomial 1 think of 1 as a constant polynomial it always has the constant value 1 so it is never going to take the value 0. So there is not going to be a single point in this Cartesian product which we substitute is going to give you the value 0 because the value is always non-zero and it is a constant.

Therefore this set turns out to be empty, okay and look at the third one so for the 2 is also obvious as for the third one I should explain what this S1 through Sn means. So this S1 through Sn means mind you all these Si's are the subsets of polynomial ring and in the polynomial ring you have multiplication, okay. So S1 through Sn actually means products, one you take products consisting of n factors that the ith factor comes from Si so let me write that S1 Sn is actually the set of all f1 fn such that fi is in Sn this is what it means, okay you take one member from each Si then multiply them out, okay and then you take the zero set of that and my claim is that this and this are the same.

So as usual how do you check two sets of the same you take something here show it is there to take something there so is here. So let us try to do this if you have a point lambda 1 etcetera lambda n, oops I think I do not want this n to be the same as that n so let me change this to m because this could be any number of subsets but it has to be a finite number, okay so let me change it here as well if you take an n tuple which is in z of Si, okay what it means is that you take any polynomial in a Si and you plug in for the variables this n tuple then you get 0 and therefore any such product will also vanish, so it is clear then clearly this point is in Z of S1 etcetera up to Sm, okay that is obvious because you know to get a 0 of an element of this means you have to get a of an element which is a product like this product of polynomials and for that it is enough just one or it is a zero of just one of them so that when I plug in this n tuple into each of these even if one of them vanishes then the product vanishes , okay.

Therefore what it tells you is that is Z of Si is always in Z of S1 Sm so this tells me that Z of S1 union is in the 0 set of the product, okay this is obvious what (())(14:10) little not so obvious is the other way around but this is also very easy to do by you can do it contradiction. Conversely if lambda 1 etcetera lambda n is in Z of S1 etcetera Sm, okay then you have to show that it is in at least one of the Z of Si for some i, okay.

So how do you prove it? You prove it by contradiction you assume it is not that in any of the set of Si if lambda 1 lambda n does not belong to Z of Si for every i, okay then what does it mean that it means then for every i there exist a gi in Si such that gi of lambda 1 etcetera lambda n is not 0 the fact that a point is not in the 0 locus of a set of polynomials means there is at least one polynomial in that set for which this point is not a 0 that is if you evaluate that polynomial at this point you do not get 0, okay.

So for and I can do this for every Si, okay then if you take the product g1 etcetera up to gm this belongs to S1 etcetera up to Sm but you know g1 etcetera gm of if I evaluate it on lambda 1 etcetera lambda n is not 0 a contradiction, okay. So what this will tell you is that this n tuple has to belong to some Z of Si, okay so that will give you the inclusion of this into this and we are done, okay so it is a pretty simple argument, okay and well then there is only the fourth one left which is pretty easy to see and once we do that we have this Zariski Topology on affine space.

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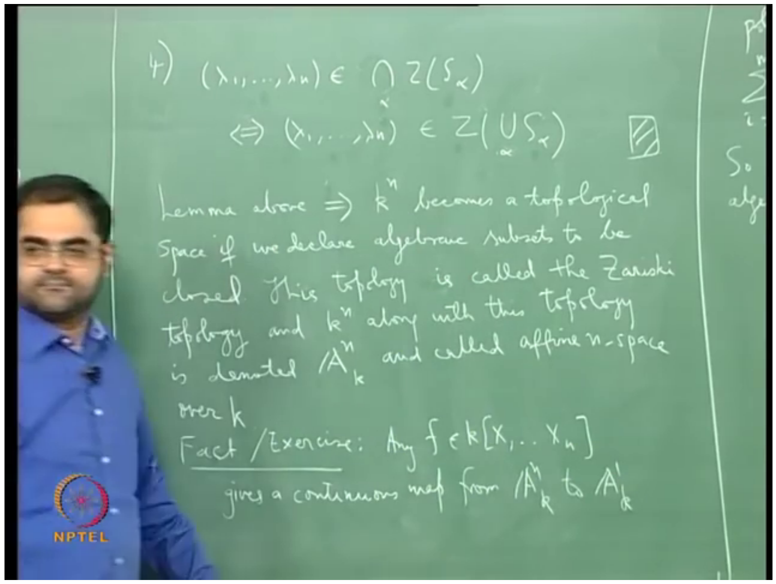

So fourth one you take a point lambda 1 etcetera lambda n is in the intersection of over alpha z of S alpha so here of course you are assuming that S sub alpha is a collection of subsets of the polynomial ring alpha is index and of course alpha varies over some indexing set which I do not want to be expressive about is that is not what we so we are not worried about that indexing set because there is not (())(17:48) on the indexing set it need not be finite it could be infinite, okay.

So if there is a point in this that means you are saying that this point vanishes in every polynomial in each of S alphas, so it means that this polynomial will also vanish I mean this point will also be a 0 for the set of polynomials gotten by simply taking the union of all these alphas, right? This is obvious what are the first what are the top line it means that you take any polynomial in any S alpha for any alpha and you plug in this evaluate it at this point that is plugin this with the variables then you get 0 but that is exactly what this means if you take all the polynomials in all S alphas then they all vanish in this point, okay so this implies to us.

And the other way is also obvious if you have a point here then certainly that point is going to be a 0 of every polynomial in S alpha for every alpha so it is going to be onto each ZS alpha and therefore it is going to be (())(19:24) intersection therefore this is just obvious we provided you think about it for some time this is basic this is very simple logic, okay. So this is quite clear, okay this is (())(19:41). So that is the well that is the end of the proof it is pretty simple exercise but what is the import of this lemma import of lemma is that you can declare subsets of kn of this form namely the algebraic subsets you can declare them to be closed sets, if you declare them to be closed sets the lemma says that the whole space is a closed set the empty set is a closed set the union of finitely in many closed sets is again a closed set the intersection of the arbitrary collection of closed set is again a closed set these are exactly the schemes for the closed sets in a topology.

So the lemma tells you lemma about tell implies that kn becomes a topological space if we declare algebraic sets to be closed, okay. And so this means that therefore the Zariski Topology as it is defined here is defined where closed sets and of course the open sets are the compliments of closed sets always. So let me mention with this as a Zariski Topology this topology is called as the Zariski Topology and kn along with this topology is denoted Ank and called affine n space over k, okay. So you see we strip off the vector space structure but then to make it into a different thing we called the affine space we added the Zariski Topology, okay and this is the affine end space over k, right?

So things are quite good if you look at this picture there is the so the space on this side has at least become a topological space. So the moment it becomes a topological space you can at least start thinking of continuous functions, okay once you have a topological space you can at least think of continuous functions and of course the first question that one would ask is these are of course polynomials are of course functions on this space and then you would ask are they continuous the answer is yes, okay it is an exercise but we will also see it in some other way later.

So it will happen maybe you want if you want you can try it out as an exercise that you take any polynomial here and tint of it as a map from kn to k evaluating every point on that polynomial we produce a element of k so every polynomial becomes a function on this affine space and then check that function is actually continuous for Zariski Topology because the source space is kn with this a Zariski topology is Ank and the target space is k which is A1k, okay. So fact Ri even I put it as exercise but we will see this later in detail any f in k X1 etcetera Xn gives a continuous map from Ank to A1k that essentially enough it is essentially enough to check that the it is essentially enough to check that the inverse image of any point is a closed subset, okay but it is an exercise you will have to do some work you have to do some work, okay.

So you see there is some you know there is some cyclicity going on here, okay see you know in the usual topology when we study suppose you study real valued functions or complex valued functions then the set of points where the function vanishes is a closed set of course I am only worried about continuous functions the set of points were a continuous function vanishes is a closed set, right? And well the reason for that is topological because a single point is closed as a subset is a closed set and the set of points where the function is where the function takes the value 0 is the inverse image of the singleton consisting a point 0 and that is the inverse image of a continuous function and continuous function of a close set so it has to be closed, okay.

So a set of points where continuous function takes a fixed value is always a closed set I mean 0 is a special value but could be any other value and we use that kind of intuition to define the close sets here to be given by the loci the common loci of you know where a bunch of functions vanish, okay but then you know what this exercise tells as that it indicates our stat I mean it tells you that our original intuition is correct, okay.

If our original intuition to declare 0 sets 0 loci as closed sets is correct then our functions using which we declared this 0 sets as closed sets they should of course be continuous and in the fact is yes, okay so this is an exercise it is pretty easy to check but I will come back to it later in a more general form, okay.

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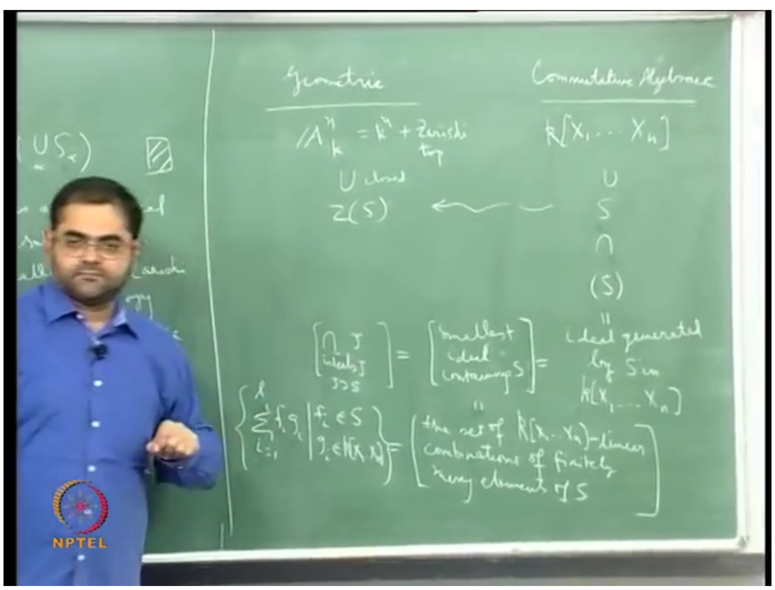

So the point I wanted to make is the next point I want to make is again I think I will need this diagram again so I will so let me again draw the diagram I will keep needing the diagram again and again so here is the so let me do that this is geometric side and this is commutative algebraic side.

So on the geometric side you have Ank which is actually kn plus Zariski topology, right? And on the commutative algebraic side you have the polynomial ring in n variables the same n as this affine space, okay and what we get is the we started with the subset here and then we associated to this an algebraic set which is a set of 0's and this is actually closed by definition of algebraic sets are the closed sets, okay. And how do you get more general closed sets? You get more general closed sets by either taking arbitrary intersections of such closed sets or by taking finite lenient of such closed sets that gives you all the closed sets.

Now the what I am now next going to worry about is this side of the picture, see on this side of a picture what we have is just a subset, okay and a subset of a ring is not a very interesting object by itself, okay usually so the way I am asking you to think is whenever you look at an object you try to think of sub objects in the suitable sets, so for example you know if you thinking of groups if you think of subgroups, okay if you are thinking of for example you are thinking of vector spaces the interesting subsets are subspaces, okay.

So the same way if you are thinking of subsets of a ring, what are the interesting subsets? The interesting subsets well one could be ideals the other could be subrings, okay but there is no part there is no point in taking a subring here, the problem is if you take a subring then it contains 1 and the moment t contains 1 Z of 1 will becomes empty because you know Z of S will always become empty the moment S contains a unit because a unit is a constant polynomial it is a non-zero constant polynomial which never vanishes, so if your subset S contains a unit which in this case is a non-zero constant then the 0 set is going to be empty, so there is no point in taking a subring here.

So the fact is you do not want any units to begin with because if you take units here with set is empty there is nothing to study. So what are the other sub objects you can think of the ideals and you know an ideal is non-trivial if and only if contains if it does not contain a unit the moment ideal contains a unit is the invertible element in the ring then the ideal has to be the whole ring, okay.

So what this tells you is that it is the interesting things here to look at from the commutative algebra point of view are the ideals, okay but what we solved out is just a subset, now how do you go from a subset is ideal there is always a philosophy for this, whenever you have an object of a certain type when you have a subset and you want a sub object what you do is you take the so called sub object generated by the subset, okay.

So if you have a group and you have a subset a subset (())(31:23) is not too interesting but you can always take the sub group generated by the subset which is the smallest subgroup which contains that subset. Similarly, if you have a ring and you have a subset if ideas are what you are interested in as sub objects and not subrings then you can look at the ideal generated by the subset, okay. So what you can do is you can replace you can think of replacing this set S by the ideal that is generated by S.

So what we will do is let us do the following thing let us write S let me use the following notation I will S with a round bracket and S with a round bracket means the ideal generated by S, okay. So this is the ideal generated by S in this polynomial ring, okay. Now what is this ideal generated by a subset in the polynomial ring of course there are two ways of describing it, one is this is equal to the smallest ideal containing S this is one definition, okay and that is and what is the smallest ideal containing S how will you define a (())(33:03) containing S it should well it is first of all it has to be an ideal which contains S and it should be smallest among these which means that if there is a any other ideal which contains S that should contain this as well which tells you that this is intersection of all the ideals which contain S so there is another way of stating this set theoretically these intersection of all ideals J over J where J contains S, okay.

And of course you know when you want a smallest sub object containing a subset this is what you always do you take the intersection of all the sub objects which contain that subset and usually in good situation the intersection of all these sub objects will again give a sub object, okay. So if you take the intersection of family of ideals that is again an ideal the intersection of a family of sub groups is again a sub group, okay.

So if you want a subset generated by a subgroup generated by subset over group all you have to do is simply take the intersection of all the sub groups which contain that subset. Similarly if you want the ideal generated by a subset of a ring all you have to do is just take the intersection of all the ideals which contain that subset, okay. So that is small (())(34:25) containing S and there is another way of writing it out the other way of writing it out is just take S linear just take the ring linear combinations of finitely many elements of S just take that, okay so let me write that down this is equal to the set of R so you know k X1 etcetera Xn linear combinations of finitely many elements of S, that is what it is that this is the same as this is an exercise having it is a simple exercise which you should have across in a first course in commutative algebra but if you have not done it, it would not take you probably more than a couple of minutes to do it.

And of course what does it means? This means that you know this is the set of all elements of the form sigma fi gi i equal to 1 to l where fi are in S and the gi are in the k X1 etcetera Xn. So you are taking finitely many fi's from the set S and then you are multiplying them with coefficients which come from the ring when you are taking a linear combination, okay and you can check that this is same as this and that this is same as this that is a very simple exercise, okay.

So well the fact is that the picture here becomes better because this is an ideal and that is a nicer object than just a subset because on the left side you are looking at commutative ring you better the interesting sub objects are ideals. And then of course a question is you take Z of this take because that is after all it is an ideal there but ideal is also a subset you can look at the set of common zeros and the fact is that you do not get anything new you get the same thing, that is the beautiful point the beautiful point is see if there is a point here then it is a common 0 of every polynomial in S therefore if you take any combination like this it will also be a 0 of that because every term contains factor from S and there are only finitely many terms and every term looks like this.

So something here is always here, okay and because this is the subset of that, okay something here is also here, okay because you know when I take ring linear combinations I can set all the gi's to be whatever I want so I could set all but one gi to be 0 and that one gi I can set it to be 1, okay then I will get that a point here will be a 0 of each element of S so this will also be there and that will also be here so the moral of the story is the same. So what is significance? If significance is that instead of looking at 0 sets of bunch of polynomials because a bunch of polynomials a subsets of polynomials is very not so very attractive it is not so interesting from the ring theoretic point of view if you look at the 0 set of an ideal of polynomials things look better because an ideal is something that is the ring theoretic object.

And passing from a bunch of polynomials so the ideal does not do you any harm because the 0 set is not affected, okay. So this tells you that somehow you know now the story is getting better on this side we have closed subsets and on that side you just do not have subsets you have ideals, okay. So you see slowly you are getting some kind of a correspondence here you have closed subsets which are the algebraic sets and they are supposed to be 0 sets of bunch of polynomials but you can as well take them to be 0 set of ideals so on this side you have ideals, okay here you have subsets of here you have the closed subsets of affine space and there you have the ideals in the polynomial ring so you already have the picture on both side becoming clearer on one side you are studying ideal ideals of the polynomial ring in n variables on the other side you are studying the closed subsets of affine space, okay.

So this is how ideal theory enters into the picture, okay but it does not enter without its share of apprehensions and they are as follows see the fact is that you know the worry comes from a very simple observation, see even if my set S was a single element suppose I was just S was a single element that means S is only one polynomial, okay then Z of S is just 0 locus of that polynomial there is a set of points in that polynomial vanishes, okay but if take the ideal generated by that polynomial it is a principle ideal it consists of all multiples of that polynomial and that and the number of polynomials in that ideal is huge, okay you have so many polynomials, okay. And this set is a huge set, okay if k is infinite I mean for that matter even if k is finite but the fact is that since k is assume to be an algebraically closed field some filed theory will tell you that it has always been infinite set, okay an algebraically closed field cannot be a finite set, okay.

So the fact us that this even if S is a single polynomial this is an infinite set so it seems to you as if you are looking at zeros of infinitely many polynomials, okay. Now intuition will tell you that if you put too many equations if you have too many equations to solve then you may not get any solutions I mean for example in linear algebra if you have if you are solving n so many equations in say n variables if there are more equations it is very likely that you do not get any solutions, okay and so you can imagine that I am if I change from 1 polynomial to say infinitely many polynomials I might not get any solutions.

So therefore this set may turn out to be empty but that is not the case the case is the fact is that because of the Hilbert Nullstellensatz you know that because you know these two are the same and since Nullstellensatz says that this 0 set is of a single non constant polynomial is non-empty such a thing is not going to happen but there is something more that is happening and that is the following fact the fact is for that matter if you take not just a singleton set if you take any set and take the ideal generated by that that ideal will be generated by only finitely many elements.

So what it means is that you are actually always looking at the common 0's of only finitely many equations even if you start with infinitely many equations if at all you get a 0, okay then which you will, okay so long as the ideal generated by those by that subset is not the whole ring then you are actually studying only the common 0's of finitely many polynomials. So this is so you know this somehow removes apprehension that you know even though you may be trying to solve a large set of equations I mean you are trying to look trying to get common 0's of in fine set of polynomials actually in principle you are actually solving only finitely in many polynomials I mean you are looking at 0's of only finitely many polynomials.

And this fact follows from the so called Hilbert basis theorem or Emmy Noether's theorem we just one of the fundamental theorems that you study when you study Noetherian rings in commutative algebra.

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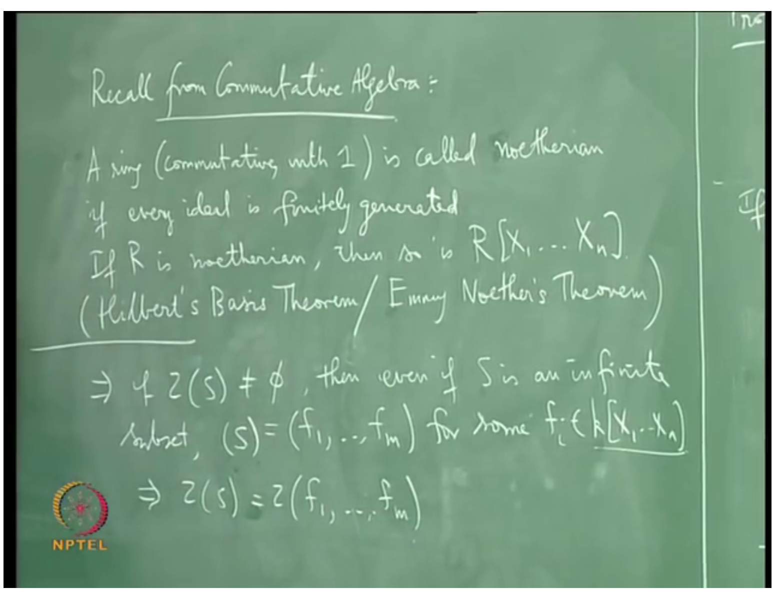

So let me recall that and write that down, so recall from commutative algebra a ring commutative with 1 is called Noetherian if every ideal is finitely generated, okay. A Noetherian ring is a ring in which every ideal is finitely generated, if R is Noetherian then so is R X1 etcetera Xn in a polynomial ring in finitely many variables over a Noetherian ring is again a Noetherian ring and this is so called Hilbert Basis theorem or is also called I mean more generally is called Emmy Noether's theorem, okay.

So this is the fact from commutative algebra, right? So what this tells you is that this implies that if Z of S is non-empty, okay then even if S is an infinite subset the ideal generated by S is f1 etcetera up to fm for some fi in k X1 etcetera Xn. So you know what I must remind you at this point is that if I take R equal to a field k then a field k is of course Noetherian because a field has you know the only ideals in a field are the 0 ideal and the full ideal, okay and the 0 ideal is obviously finitely generated by the elements 0 and the whole field as an ideal is generated by 1 so both there are only 2 ideals they are finitely generated therefore a field is always Noetherian and therefore a polynomial ring over a field is also Noetherian by this fact therefore what it means is that this polynomial ring in n variables over a field has every ideal finitely generated.

So if I start with subset S of that n matter that subset is probably even an infinite subset if I pass to the idea generated by that subset because it is ideal in this Noetherian ring it is finitely generated which means that it is the ideal generated by finitely many polynomials, okay and so this implies that Z of S which is Z of becomes just Z of f1 etcetera fm. So the moral of the story is you know when we started defining algebraic sets we were looking at common 0's of bunch of polynomials which could be even infinite but in principle finally we are only studying common zeros of finitely many polynomials and that is very heartening, okay.

So why is it why it is heartening is because if there are finitely many polynomials there is something with you can compute you can do some computation and you can expect that you know this is not going to be a computation that will never end and this is one of the reasons that lot of algebraic geometry is nowadays done on the computer, okay so you have some special software programs like Macaulay and Coco and things like that which allow you to do ring theoretic computations on the computer and you can do this computations so you will be doing computations on the commutative algebra side and from those computations you can get consequences which means something geometrically.

So you can try to prove geometrical statements by doing computations on this side and the fact that you can do the reason why you can do the computation is one of the reason is this you are never dealing with an infinite bunch of equations you are always dealing with only finitely many equations and in fact you may ask when I write here for some fi it seems as though that this is fi are very abstract that you cannot catch them but that is not true the in fact there is a if you look at the proof of the Hilbert Basis theorem then the proof also can be refined to a constructive procedure where you can get all these finitely many polynomials you can really get them, so you can do this on a computer, okay I can really do this on a computer and that is what helps me to do computations in commutative algebra which when translated to algebraic geometry give nice results, okay.

So the moral of the story is because of Hilbert's Basis theorem this the close sets you are looking at in affine space and Zariski topology are just 0's of finitely common zeros of finitely many polynomials so you are simply studying finitely many problems, okay. So the moral of the story is that when you do algebraic geometry you are actually trying to look at the geometry of the set of common zeros of a bunch of polynomials and usually the bunch of polynomials is having coefficients in a certain commutative ring, okay even if you go to a general commutative ring but make sure that the commutative ring is Noetherian then the Hilbert's Basis theorem will tell you that even algebraic geometry in that very broad sense you are only looking at finitely many equations.

So mind you Hilbert Basis theorem will work if R is any Noetherian ring, okay so this tells you that you will always be looking at only finitely many equations if you are working with coefficients from a Noetherian ring, okay that is the importance of this that is a geometric significance, okay.

And the other important thing is of course you must not forget that we are looking at the case when the ring is an algebraic closed because of Hilbert's Nullstellensatz which assures that you really get 0's that is it ensures that the 0 set is not the empty provided this ideal generated by this collection is not the whole ideal that means it does not contain a unit, okay. So we will see more about that in the next lecture, okay.