Basic Algebraic Geometry
Dr. Thiruvalloor Eesanaipaadi Venkata Balaji
Department of Mathematics
Indian Institute of Technology, Madras
Lecture-19
Morphisms into an Affine Correspond to k-Algebra Homomorphisms from its Coordinate Ring of Functions

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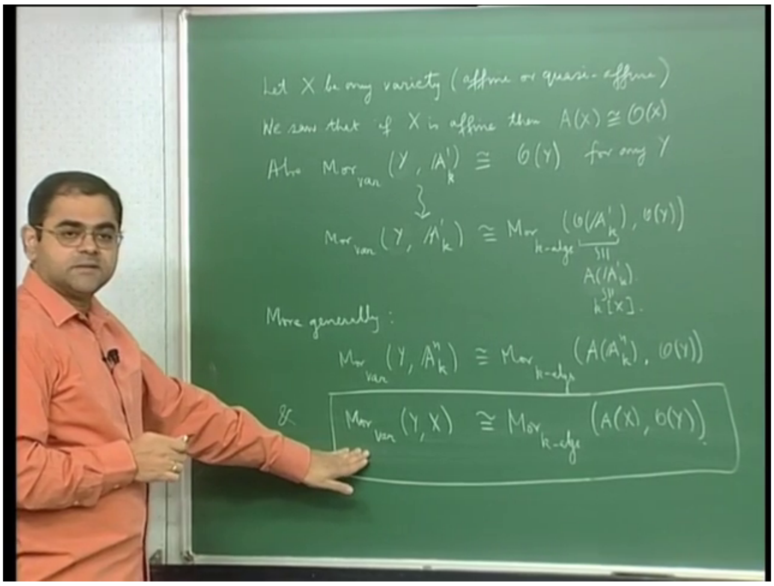

Alright so you see what we did last class was to define the notion of a morphism and the idea of a morphism is that it is a continuous map which should pull back regular functions to regular functions okay that is the definition of a morphism and then I stated a few results let me recall them, so let X be any variety in our case it means either it is an affine variety or quasi-affine variety.

An affine variety as you know is an irreducible closed subset of affine space and we are working over an algebraically closed field okay and a quasi-affine variety is a non-empty open subset of an affine variety and variety for us at the moment means either an affine variety or quasi-affine variety, which means either an irreducible closed subset of affine space or a non-empty open subset of an irreducible closed subset of affine space.

And we saw that if X is affine okay then the A X and O X are isomorphic okay so A X is the co-ordinate ring of X, the ring of polynomial functions on X which is gotten by taking the ring of all polynomial functions of the affine space in which X is sitting inside modulo the ideal of X which is a prime ideal and O X is supposed to be the ring of all regular functions on X okay.

And we saw that if X is an affine variety then there is an isomorphism like this alright and mind you this isomorphism is not just an isomorphism of rings, it is an isomorphism of k-algebras okay, it is an isomorphism of rings it is also an isomorphism of k-vector spaces okay so k-algebra isomorphism and of course O X was defined to be the ring of regular function on X and regular functions were defined to be functions which were locally given by quotients of polynomials okay.

Then I told you that we defined then the notion of a morphism of one variety into another variety and what we did was we showed the set of morphisms of varieties from X to A1 or let me put Y to A1 is isomorphic or let me again put X or let me put Y okay, Y to A1 is isomorphic, it can be identified with O Y for any variety Y, namely every regular function which is the map from Y to k okay is the same as the morphism from Y to A1 where k is thought of as A1 which means that A1 is just k with the zariski topology.

And the zariski topology there of course is you know it is a compliment finite topology it is a topology for which the open sets are complements of finite sets the closed sets are only finite sets okay so and I told you that this can be generalised that this statement can be generalised so you know this statement can be written as morphisms of varieties from X so let me go back to, so let me again write Y to A1 is isomorphic to this can also be written as morphism of k-algebras from O of A1 k to O of Y.

Because this O of A1 k is actually isomorphic to k X, O of A1 is the same as A of A1 okay and A of A1 is k of X it is a polynomial ring in one variable okay so if you want I can write, I am using the fact that you know O and A are the same for an affine variety so if you want I will write this as, this can be identified with A of A1 and that is isomorphic to k X, okay and the all possible k-algebra homomorphisms from the polynomial ring k X to any k-algebra is in bijection with the elements of that k-algebra okay.

Because this is the universal property of the polynomial ring in one variable okay namely any k-algebra homomorphism from k X to O Y is completely controlled by what, by the image of X okay, it is controlled by the element of O of Y that X goes to and therefore you will have as many k-algebra homomorphisms as there are elements in O Y so this set is the same as O Y okay so I am just translating this statement like this using the universal property of the polynomial ring in one variable.

And then I told you that more generally you can also write it as what I have written for A1 you can write it for An and not only for An you can write it also for any affine variety okay so the statement is morphism of varieties from Y to An can be identified naturally with morphisms of k-algebras from A of An to O Y and what I can do for A1 and I have done for An I can do it also for an irreducible closed subset of An namely an affine variety so I can also write it as morphism of varieties Y, X is isomorphic to, is in a natural bijective correspondence with morphisms of k-algebras from A of X to O Y okay where of course X is an affine variety okay.

So this is the most general statement, okay. So in this general statement if you put X equal to An, you get this statement, if you put X equal to A1, you get this statement which actually, so if you put X equal to A1, it boils down to this statement that regular functions are nothing but morphisms into A1 okay regular functions are no different from morphisms into A1 right so what I am going to do is, this is a very very important statement okay.

Because as I told you at the end of the previous lecture that this is the statement that gives an equivalence of categories between, on the one side the category of affine varieties with morphisms being morphisms of affine varieties and on the other side the category being the category of finitely generated k-algebras that are integral domains with morphisms being k-algebra homomorphisms. So what I am going to do is am really going to focus on proving this fact okay.

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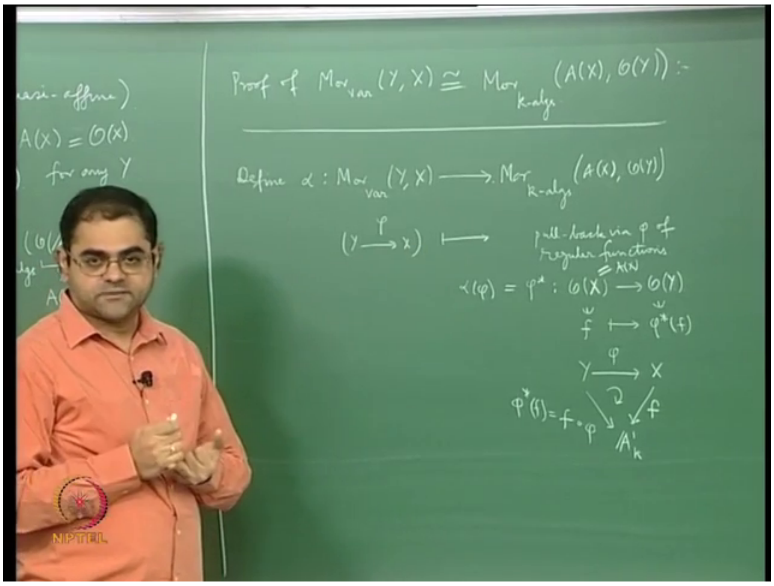

So that is what I am going to do now let us say that is very important factor, so proof of the statement morphisms, let me rewrite it again varieties from Y to X isomorphic, here of course this means there is a bijection of sets okay, this what you have there is a set but the point is that at this side represents the geometric side, the algebraic geometric side, this side represents the commutative algebra side okay and as you shall whole point about algebraic geometry is that you go from the, you know from the geometric side to the commutative algebra side this translation is very important to algebraic geometry okay.

So what I am trying to do is I am trying to prove this, this is the algebraic geometric side morphism of varieties from Y to X and here is the commutative algebra side which is morphisms as k-algebras of Ax to O Y so this is what I am going to prove okay so I am going to prove that there is a bijection like this there is a natural bijection like this now how do I do that? So define alpha from morphism, again let me write this down I have defined a map like this first.

I define a map like this which is if you give me a morphism phi of varieties from Y to X this is an element on this side, I have to send it to something here and you know what I am going to send it to here is a very natural thing it is a thing that characterises what a morphism is, a morphism is a continuous map which is supposed to be as considered as a pullback map something that pulls back regular functions to regular functions, so what I am going to do is I am going to just take the pullback via phi of regular functions and what does that means, it means, so the notation for that is phi upper star which is a map from regular functions on Y to regular functions X okay.

So this is my map, so what is it that is happening if you give me a regular function f on Y, I get the pullback of that regular function to X is a regular function on X this is the defining property of a morphism apart from its continuity and what does this mean, this is just composition, so you know you have Y to X you have this map, you have this morphism phi and f is a regular function on Y, I have yeah I have messed up X, I have mixed up X and Y this should be X and this should be Y.

So you have a regular function on X, f is the regular function on the target so it is a, if you want f is a morphism into A1 or it is a regular function into k okay and then what is a pullback, it is this map which is a composition this is first apply phi then apply f okay this is otherwise called as phi upper star of f this is the pullback and the fact is that if you have a regular function and the target by composing with the morphism should get a regular function on the source.

The target is X so give me a regular function on X and by composing with phi the pullback will be a regular function on Y, this is the definition of a morphism okay, this is part of the definition of morphism and mind you that this O X can be this O X can be canonically naturally identified with A X, in fact I should even put equality okay O X is the same as A X alright, so here also when I say that if X is affine then A X and O X are isomorphic in fact you can say that they are equal okay if you think as them as functions into K into A1 which is k with the zariski topology they are equal actually this is actually an equality here right.

And so it is very clear that I have a very natural map, given any morphism look at the pullback which takes regular functions to regular functions and it is very easy to see that this pull back map is of course a k-algebra homomorphisms because you know phi upper star of f, f1 plus f2 will be phi upper star of f1 plus phi upper star of f2 and phi upper star of f1 into f2 will be phi upper star of f1 into phi upper star of f2 you can easily verify that this what you get here is not just a map from A X to O Y but that is a k-algebra homomorphism.

Because functions by the definition of functions everything is done point wise sum of two functions is defined point wise product of two functions is defined point wise multiplying a function by a scalar (())(17:14) function is also defined point wise, so you can easily check that this is certainly a k-algebra homomorphism okay, what we need to prove is that this map so I have written alpha but well so this is alpha of phi but the better notation is phi upper star okay.

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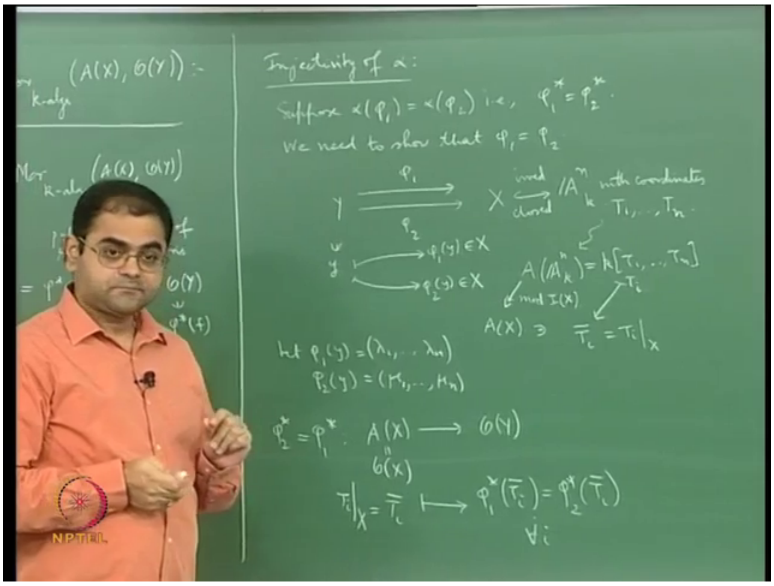

But since I have written alpha here I will also write it here alright and what will I have to prove, I will have to prove that this map is injetive, I have to prove this map is surjective that is all I have to prove alright, so let us try to prove the injectivity let us try to prove the surjectivity so I guess the injectivity is very easy so what is the situation suppose there are, suppose I have two morphisms which give rise to the same k-algebra homomorphism as pull backs of regular functions then I have to say that these two are the same okay.

So suppose alpha of phi 1 is equal to alpha of phi 2 which is the same as saying that you know phi 1 upper star is same as phi 2 upper star okay I will have to show phi 1 is the same as phi 2 alright we need to show that phi is the same as phi 2 so you have to show that these two morphisms are one and the same and what do you mean by equality of morphisms, equality of morphisms means equality of the underlying maps as sets okay, equality is just a set theoretic map, so it is a set theoretic checking that you have to do okay.

So how do I check two maps are equal, I check two maps are equal by evaluating them at every point and showing them that they are equal so what I do is well so here is my situation so I have Y so there are these two maps P1 and P2 which are morphisms into X and mind you X is an affine variety which means that X is sitting inside, X is an irreducible closed subset of a suitable An so let us consider that it is sitting inside, it is an irreducible closed inside some An okay and let us put co-ordinates on this An okay.

Let me say with co-ordinates T1 etc Tn let these be the co-ordinates on An alright so when I say A1 with co-ordinates T1 etc Tn what it means is that I am just saying that if you take the if you take the ring of polynomial functions on Am I am identifying it with K polynomial ring T1 etc Tn okay this is what it means, the choice of variables in the ring of polynomials in An those variables, the choice of variables will also give you the labels for the co-ordinate functions okay.

And what happens is that you know this is an irreducible closed subset therefore this if I go mod the ideal of X which is going to be prime ideal, I am going to get A of X okay so it is going to be so this is the, this is the polynomial functions on Am restricted to X okay to get the ring of functions on X, I will have to go modulo the ideal of X which means I will have to identify two functions of X if and only if they differ by a function that is 0 on X and that identification corresponds to doing modulo I of X okay.

And so you know under this map you know all the every Xi, any Ti will go to some Ti bar okay if you want thought of as an element here okay and mind you Ti bar is geometrically Ti bar is just the co-ordinate function Ti restricted to X so this Ti bar is just Ti restricted to X okay actually what is this quotient map give any polynomial consider it as a function on the affine space and restrict it as a function on X that is this quotient, this restrict, this quotient map is actually restriction geometrically it is a restriction of polynomials to a closed subset this closed subset X okay.

And how do I check these two maps are equal I check these two maps are equal by looking at the image of a point so let me take a point small Y okay the point small y will go to under phi 1 it will go to phi 1 of y and which is going to be a point of X and under phi 2 it is going to go to phi 2 of y which is a point of X again okay and both phi 1 of y and phi 2 of y are going to be points in affine n space, so they are going to have co-ordinates.

So let phi 1 of y be well lambda 1 etc lambda n because it is a after all phi 1 of y lies in X and after all it is a point in m space it is given by n co-ordinates okay and phi 2 of y is say given by mew 1 etc mew n these are the co-ordinates of this point okay and what I will have to prove is that if phi 1 star, phi 1 upper star equal to phi 2 upper star then I will have to show that these lambdas are the same as the mews which will say that the point phi of y and phi 2 of y, one and the same point.

And if I, and since y is arbitrary it would tell you that phi 1 and phi 2 are one and the same map okay. So the point that one has to understand is that what is the meaning of phi 1 upper star equal to phi 2 upper star so phi 1 upper star is the pullback map from the ring of functions on the target, the regular functions on the target to the regular function on the source, so it is from A X to O Y of course A X is the same as O X because X is affine okay.

So this is the pullback okay and what is the pullback map do given a regular function on X, I compose it with this to get a regular function on their source that is how the pullback map works take a regular function on the target, compose it with the map you get a regular function on the source, so in particular you know the Ti's okay restricted to X they are also regular functions on the target mind you Ax is the same as O X because X is affine.

So these Ti bars they are regular functions on X, the co-ordinate functions, they are just the co-ordinate functions on the affine space restricted to the irreducible closed subset S okay so if I take each of this Ti bars which is the same as Ti restricted to X they will go to, it will go to phi 1 upper star of Ti okay and what is given to me is phi 1 upper star is the same as phi 2 upper star, so what it means is that phi 1 upper star equal to phi 2 upper star means that this is the same as phi 2 upper star of Ti bar for every I, this is what it means, this is what is given to me okay.

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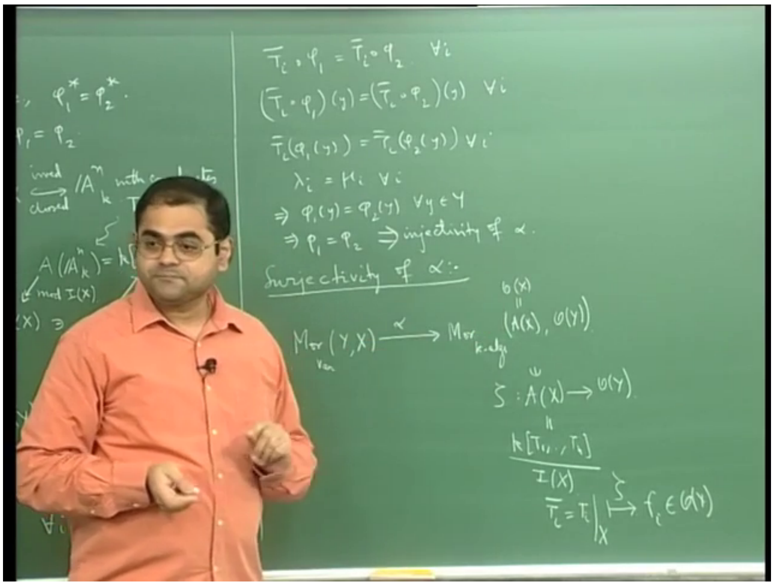

Now just think for a moment what is the meaning of phi 1 upper star of Ti bar, phi 1 upper star of Ti bar is the regular function on Y which is gotten by pulling back the regular function Ti bar on X and what does pullback do? Pullback is this composition with phi so what this means is, it means that phi upper star of f as you see phi upper star of f is f circle phi okay it just composition so if you translate that you will get Ti bar composition phi 1 is equal to Ti bar composition phi 2, this is true for every i, this is what it means.

Now this is an equality of regular functions on Y so apply to the points small y so what you will get is Ti bar of phi 1 of small y is equal to Ti bar of phi 2 of small y so this will tell you that Ti bar of phi 1 y is equal to Ti bar of phi 2 y okay but phi 1 y has these lambdas as co-ordinates, phi 2 has mews as co-ordinates and if you take a point like this the action of Ti bar on that point is to give you the i th co-ordinate okay, it is just the function Ti on affine space picks out the i th co-ordinate of the point when you evaluate it at a point.

So if you do that what I am going to get is that I am going to get lambda i is equal to mew i and this is going to be true for all i and this implies that phi 1 of y is equal to phi 2 for all y involved and that tells you that phi 1 equal to phi 2 okay so if alpha phi 1 equal to alpha phi 2 then you get phi 1 equal to phi 2 and that is injectivity okay the slightly more involved part of the proof is the surjectivity namely to show that if you give me a k-algebra homomorphism from A X to O Y it arises from a morphism okay.

And of course when once you prove it arises from a morphism we are already saying that that morphism is unique because we already proved uniqueness that is the injectivity okay so surjectivity is what we have to see next, so how does one do the surjectivity, so what I am going to do is I am going to start with, so here is my situation, I have this morphisms of varieties from Y to X, I have this map alpha and on this side I have morphisms of k-algebras from A X which is same as O X to O Y okay and I am trying to prove surjectivity of this map.

I have already proved injectivity, so what I do is I start with a k-algebra homomorphism zeta, so zeta, I start with the element here zeta is a k-algebra homomorphism from A X to OY, I start with the zeta and I am trying to show that there exists a phi such that alpha of phi namely phi upper star is the same as the zeta this is what I have to prove okay, so but what is A X, A X is if you look at it carefully A X is just k T1 Tn mod I X okay and so this A X is just k T1 etc Tn mod I X okay, this is what it is, in fact so it is k T1 bar etc Tn bar, it is generated by the images of the Ti's okay.

It consists of just polynomials in the Ti bars that is okay and so in particular you know you have this elements Ti bars which as you know are just the polynomial functions Ti restricted to X okay and what you do is, you take their images under zeta they will go to some you know let me look at it for a moment they will go to some fi's and these fi's will lie in O Y okay, see after all the Ti bars are regular functions on X okay.

And zeta is a ring homomorphism, it is a k-algebra homomorphism so it is going to take an element here to an element here so if I take Ti bar it is going to give me an element there and what is an element there, it is a regular function on Y so fi is the image under zeta of the Ti bar for every i okay, so the moral of the story is very simple the moral of the story is the moment you give me a k-algebra homomorphism from k A X to O Y, the n co-ordinate functions on A X which are actually the n co-ordinate functions in the affine space in which X is sitting restricted to that X they automatically give you the bunch of n functions on O Y okay.

What does it mean to give you n regular functions on O Y, it means that you are giving n maps into k, n scalar maps into k on Y but that means what you are trying to do is if you think about it for very point of Y if you evaluate this n maps you are getting an n-tuples of points okay, so you can see that this should give you a map from Y to Am and that will be the map, that will be the morphism that you are looking for okay that is how you get this surjectivity.

So we will have to use these fi's to define a map from Y to An okay by simply evaluating a point of Y at these fi's and putting them as co-ordinates okay so you will get a map Y to An and then we will easily see that the map will actually factor through X and we will prove that the map is a morphism and we will prove that the alpha of that map is zeta okay therefore we will get the surjectivity of the zeta okay.

So let us do that so what I am going to do is let me rub this side off, so I mean what you should remember is that the moment you are given n regular functions on a variety you must always remember this, whenever you are given n regular functions on a variety you are actually given a map of that variety into An, this is the idea okay.

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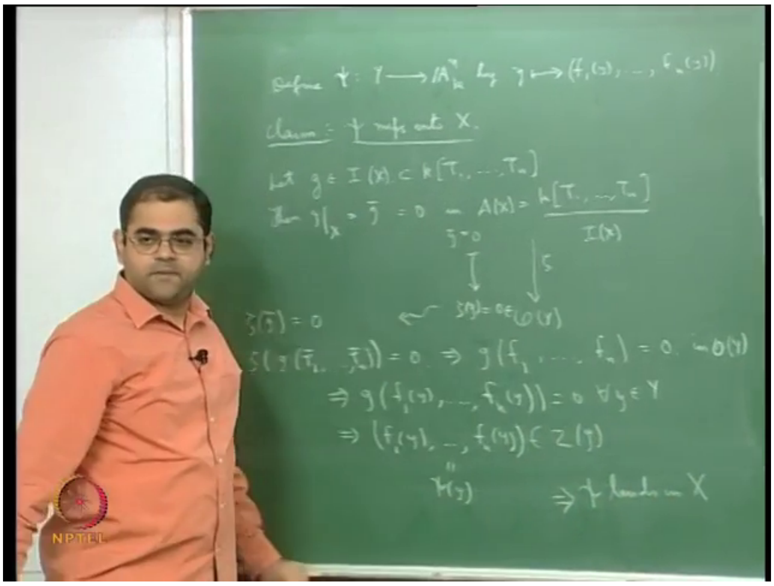

So let me write this down, define phi, let me use psi from Y to An by Y go into f1 y fn y okay this is the well-defined map because the fi's are all regular functions of Y if you evaluate a point of Y at any of the fi so we are going to get a scalar you are going to get an element of k and this is n-tuple of elements of k so it is a point in An which is just kn with the zariski topology and here is my map alright.

Now the first claim is that psi maps into X, this map is not just going into An but it is going to go into the irreducible closed subset X of An, it is going to go into this irreducible closed subset that is because it came from zeta as you will see, so how do you show psi maps into X? So it is very simple, so how do you, so what does it mean to, how does one verify this? You have to show that given a point of small y of capital Y the corresponding point of psi of small y is a point in of An which actually lies in X, this is what you will have to show.

But how do you show a point of An affine space lies in irreducible closed subset, you will show that by showing that that point is in the common zero locus of the ideal of X how do you show a point of An lies in closed subset you will just have to show that it is a common zero of the ideal of that closed subset, so all I have to do is I would take a point small y in capital Y, I will take its image under psi which is this okay and then I have to verify that this point, the point with these co-ordinate satisfies every equation, every polynomial in the ideal of x okay.

So that is what I will have to do, so let g belong to ideal of X okay let g be the ideal of X mind you g is the polynomial in n variables T1 Tm, g is a polynomial in so many variables okay then g restricted to X is actually g bar is zero in A X which is just k T1 etc Tn by I X obviously an element in the ideal if you take its image in the quotient it is 0, but then you see from Ax, I have this map zeta, I have started with the map zeta with the k-algebra homomorphism from A X to O Y okay.

Zeta is what I started with and zeta is a k-algebra homomorphism in particular it is a ring homomorphism and ring homomorphism will take 0 to 0 okay so moral of the story is that if I take zero which is g bar under if I apply zeta I will get zeta of g bar with 0 in O Y, this is what it means okay so if I write that down what I will get is I will get zeta of g bar is 0 which means that, but so this is if you write it carefully it is and what is g bar it is just g of, so you know now I am going to use the fact that zeta is a k-algebra homomorphism okay.

Zeta of g bar is same as g of T1 bar etc Tn bar so I will have to say something so let me write that again zeta of g bar of, well if you want okay g T1 bar etc Tn bar is 0 okay mind you g is a polynomial in the T's, g bar is a polynomial in the T bars okay, g bar is a polynomial in the T bars and zeta is supposed to take this 0 but what, how is zeta, what does zeta do to the T bars the zeta maps the T bars to the f's so this implies that g bar of f1 etc fn is 0.

This is what it means, you see zeta takes T1 bar to f1 okay and it takes Ti bar to fi then it will take a polynomial in the Ti bars in the corresponding polynomials in the fx okay maybe I should just remove this, maybe I should just remove this bar because the bar is already inside I should just remove g bar is g of this, just the image of g okay I will remove this bar that is (())(39:56) okay so what this translates to is that g of f1 etc fn is 0 and what is this, see this is what is g of f1 etc fn, it is another regular function on Y see notice that fi's are all regular functions on Y okay.

And what is g of f1 etc fn, it is some polynomial in expression in the fi's with co-efficient in k okay and you know if you write a polynomial in bunch of regular functions the resulting thing is also a regular function because the product of regular function is regular, a sum of regular functions is again regular and a regular function multiplied by a scalar is also a regular function because scalar functions are regular functions which are constant okay, constant functions are always regular okay.

So this is certainly a regular function and it is zero in O Y so that means if I evaluated it every point of capital Y, I am going to get 0, so this will tell you that g of f1 of y etc fn of y is 0 for all y capital small y in capital Y, this is what it means, a regular function is 0 means if you evaluated every point has to be 0 but what does this mean, this is the same as saying that the point f1 y with co-ordinates f1 y through fn y this point is actually in the zero set of g okay.

So what I have proved is but what is this point? This is just psi of y, I proved that for every small y and capital Y the corresponding points psi Y is a 0 of every function g which is in ideal of X and that tells you that psi lands in X okay so therefore this map that you got from Y to An actually factors through the closed subset X so you actually have a map into X okay, now the next part of the proof is to show that this map is a morphism and I will have to show that this map is a morphism and I have to show that the pullback map that it induces is actually your zeta and then I would be done alright.

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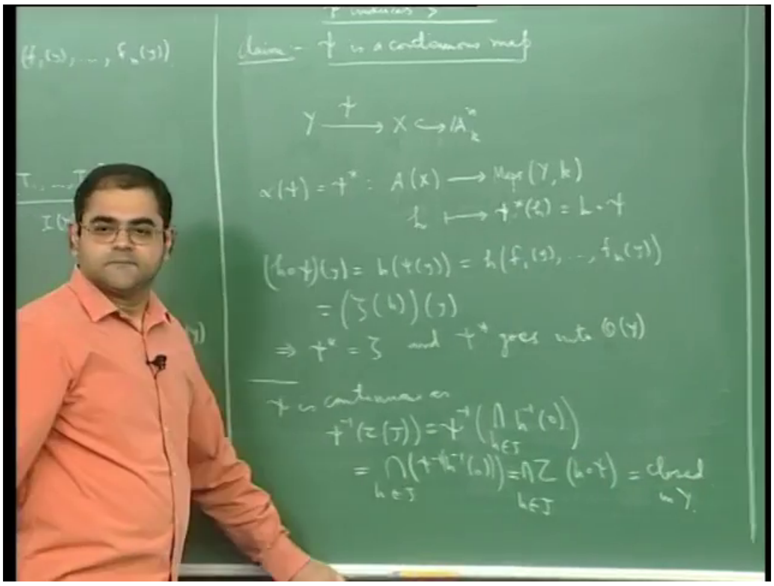

So let us see how to do that that is also pretty easy to do, okay so here is an next claim, the next claim is psi is a continuous map okay so the situation is like this I have Y, I have psi, it goes into X which is sitting inside An okay and I want to show that this is continuous okay and mind you if I show that it is continuous I am actually more or less done showing that psi is a morphism because the moment psi becomes continuous okay then the only thing I will have to check, to check that psi is a morphism that it pulls back regular functions to regular functions but the way we have defined it will follow very trivially that zeta is actually psi star, that is alpha of star, that will be very simple to see okay.

So actually this is the crucial thing that needs to verify that it is continuous okay, so the, well the, to see that that is also pretty easy if we think for a moment yeah so let me, so it probably involves showing that the pullback of, the pullback under psi is actually zeta I need that so let me let me do that, so let me write that here, psi induces zeta and psi is a continuous map okay, how do I explain that psi induces zeta?

It is very very simple, so you see alpha of psi which is psi star is going to be a map from A X to O Y okay, in fact I should not say O Y you know strictly I should say I can define for any map from psi is any set theoretic map from Y to X then given any set theoretic function on X, I can pull it back to your set theoretic function on Y so you know what I will do is I will just write maps from Y to k okay and what is this, this the usual pullback, pullback map exists for any set theoretic map even at the set theoretic level okay.

So you see if you give me any polynomial h, if you give me any element h which is a polynomial restricted to X actually then h goes to psi upper star of h and what is psi upper star of h, it is just first apply psi and then apply h this is the map okay and what I want you to understand is that, what is this map, if you take a point of Y, h circles psi of y is h of psi y okay, this is by definition psi y has been given a definition which is just h of f1 y and so on fn y because that is what psi y is okay but this is if you look at it h of f1 y, fn y if you look at this calculation it is actually zeta of h operating on y.

Because you see zeta of g is g of f1 etc fn, so similarly h of f1 etc fn is zeta of h by the same argument, so moral of the story is, so this will tell you that psi star, it will tell you that and mind you so this will tell you that psi star is actually zeta and it will also tell you that psi star goes into, goes just not into maps from Y to k goes into regular functions on Y, O of Y is a subset of, this is a set of all possible maps from Y to k okay just all possible functions they need not even be continuous okay.

Whereas O Y is very special, these are all morphisms of Y into k they are the regular functions of Y, they are very special okay and the fact that psi star is zeta which you have verified point wise will tell you that and since zeta takes values in O Y it tells you that psi star also lands inside not just in this, but in this subset okay so this proves that psi induces zeta okay.

So the only thing to show that the zeta is started with comes from a psi here is to show that actually psi is here namely I should show that psi is actually a morphism okay and what is the defining property of a morphism, the defining property of a morphism there are two conditions, the first condition is that it has to be continuous, the second thing is it should pull back regular functions to regular functions.

The second condition is already satisfied, the pullback map induced by psi is already zeta and zeta is pulling back regular functions to regular functions, its taking every regular function on X to a regular function on Y okay so it is already pulling back regular functions to regular functions therefore the only thing that one has to check is actually that it is a continuous map okay, so that is the reason that I said in the beginning that the essential claim is that it is a continuous map and once you do that you have proved everything, you have proved the theorem alright.

So how do I check something is the continuous map? I will have to just check that you know inverse image of open sets are open or I have to check inverse image of closed set are closed but that is a pretty easy thing because you see how do you define the zariski topology on X what is the zariski topology on X? How do you define zariski topology, you for example you specify closed subsets given as zeros of polynomials okay.

So what is the closed subset of X, it is given as a common zeroes of a bunch of polynomials okay and if you compose those polynomials with psi okay those polynomials are regular functions okay therefore if you compose them with psi, the resulting things will be regular functions on Y and the inverse image will be precisely the locus of those zeroes of those regular functions on Y and what you must understand is the zariski topology given by, the zariski topology on Y coincides with the topology that you get by taking the closed subsets to be common zeroes of regular functions.

You see after all either you can, when we started defining the zariski topology we always took set of common zeroes of polynomials but then when we went to a general variety which could even be an open subset of a closed irreducible subset then we had do define regular functions and these regular functions were locally given by quotients of polynomials okay and the zero set will be therefore locally the zero set of the numerator polynomial which is anyway closed okay it has got nothing to do with the denominator polynomial okay the zero set of a quotient of polynomial is essentially the zero set of the numerator.

The denominator polynomial is anyway assumed to be non-zero okay otherwise we would not put it in the denominator, so what you must understand is if you take a variety okay and if you take a bunch of regular functions on that variety and you take the common zero locus of a bunch of regular functions on that variety the results is again a close subset of that variety just like closed subsets of affine space are given by common zeroes of a bunch of polynomials, closed subsets of any variety are simply given by set of common zeroes of a bunch of regular functions okay.

And so if you start with a closed subset in here that will be the common zero locus of a bunch of polynomials okay and it is inverse image here will be the common zero locus of the bunch of regular functions which are pull backs of these polynomials and that is again closed so what this argument tells you is that psi pulls back closed sets to closed sets so it is continuous and we are done okay so let me write that down psi is continuous as psi inverse of Z of some J is going to be intersection of okay so if you want I can write the full blown set theoretic thing that you, that you might, so Z of J is h inverse of 0 intersection where h belongs to J right.

What is this zero set of an ideal, zero set is you take this for each element in the ideal, each polynomial in the ideal, look at it zeroes which is h inverse of 0 and you take the intersection, this is the common zeroes and you know psi inverse taking inverses behaves well with respect to intersection, so this is going to be intersection h belongs to J, psi inverse of h inverse of 0 okay and if you see that this is just the zero set in, this is just the zero set of first apply, this is a zero set of h circle psi, intersection of the zero set of h circle psi where h belongs to J okay and this is closed in Y.

So this implies psi is continuous and since psi is continuous it is already the map that it induces pulls back regular functions to regular functions so it is a morphism and so I found, I have found psi here, I have been able to found a psi which goes to zee that gives me the surjectivity and I am done with the proof okay, so the proof ends here.

So the crucial point you will have to understand is the following which you can think of as you know as a separate exercise if you want, I mean it is actually if you understand it if you have understood it well it is just a tautology but take any variety take a bunch of regular functions okay then the common set of common zeroes of those regular functions is a closed subset of the variety okay, it is a closed subset.

Because regular functions are locally quotients of polynomials and taking the common zeroes of such regular functions actually amounts you taking common zeroes of a bunch of polynomials, namely the numerator polynomials local okay and that is a fact that you will have to understand okay so I will stop here.