Basic Algebraic Geometry

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Lecture-18

Translating Morphisms into Affine as k-Algebra maps and the Grand Hilbert Nullstellensatz

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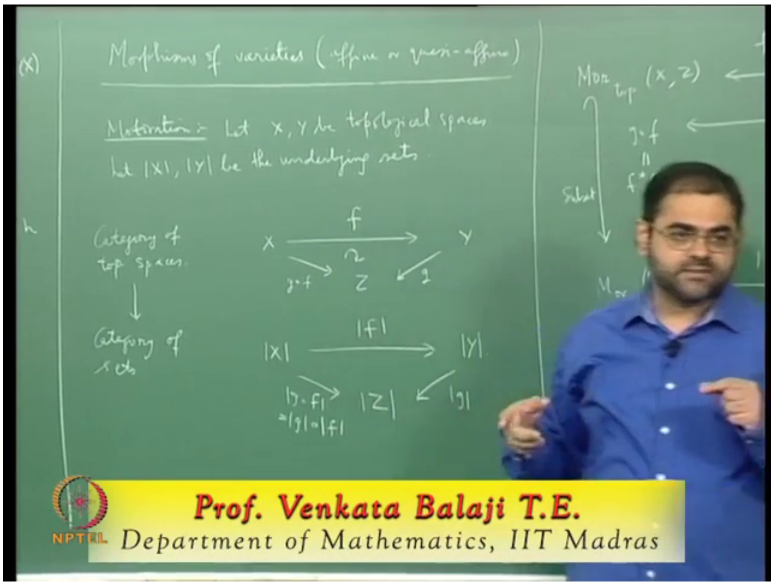

Alright so recall fr0om the previous lecture that we were trying to define what a morphism of varieties is okay so we have pair of varieties, two varieties, they could be either affine or quasi-affine varieties and then we want to say when a map between them is a morphism of varieties okay and we use the analogy that I gave you in the last lecture namely the analogy from the topological spaces okay so if X and Y are topology spaces and f is a continuous map of topological spaces of course you know that a continuous map pulls back continuous maps to continuous maps.

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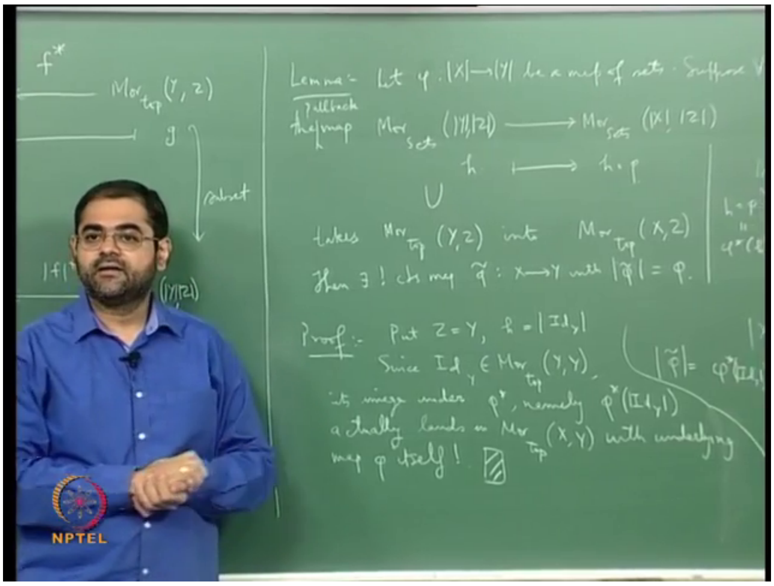

It is just a restatement of the fact that the composition of continuous maps is continuous and but the more beautiful thing is that if you have a set theoretic map that pulls back continuous map to continuous maps then it follows that that set theoretic map is itself continuous the proof is deceptively simple and we could say more or less tautological but this is the, but the importance is with the philosophy that you define a morphism as a map which pulls back good function to good functions.

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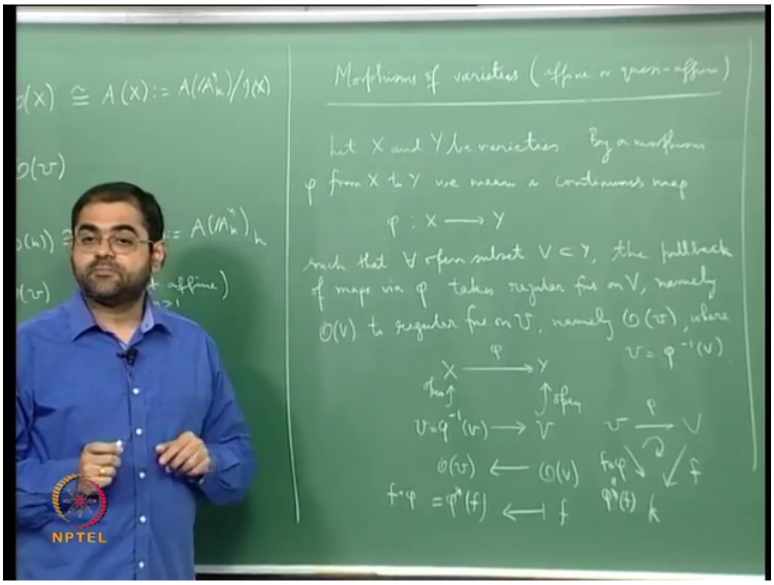

So let me make this definition of what a morphism between varieties should be, so here we go let X and Y be varieties by a morphism from X to Y, phi from X to Y, we mean a continuous map phi from X to Y such that for every open subset V in Y, the pullback map, the pull back of functions of regular functions I should say the pull back of maps via phi takes regular functions on V namely O V to regular functions on U namely O U.

So the diagram is like this here is X, here is Y, here is phi and here is the open subset V of course whenever I say open subset it is of course non-empty okay and all these situations we are not going to look at the case when the open subsets are empty because we do not want it define regular functions on an empty open sets okay, so of course I should tell you that where U is actually f inverse V okay.

So here is V and here is U which is f inverse V and this is also open that is because phi is a continuous map of course X and Y are varieties so they have zariski topology therefore you know what open sets mean and you take an open set here the inverse image of an open set is open set because and I should use not f, I have messed up the notation, it should be phi inverse here, it should be phi inverse there as well okay and I have this map from phi inverse V to V.

Now this map from phi inverse V to V will give you a map from O V to O U and what is this map from O V to O U, what is an element of O V, it is a regular function on V okay an element of O V is a regular function on V so what is a regular function on V? It is a map from V to k, a regular function on V is a map from V to k which as we had defined in the earlier lectures is a function which is locally given by quotients of polynomials where the polynomials are considered in the right number of variables in which V or Y is right number of variables is equal to the dimension of the affine space in which V or Y is sitting times i okay.

So well here is V and here is phi, this is U so U takes phi, phi takes U to V and then the pullback will be this map you pull back essentially means composition so this first apply phi then apply f and this is called as phi upper star of f so this is f going to phi upper star of f, the pull back of f which is by definition first applying phi then following it upto the f okay, so this is the requirement, the requirement is you take an open subset of the target variety, take a regular function on that, if you composite it with phi, you should get a regular function on the source okay.

This is the condition, this is just the condition that the map phi is continuous and it pulls back regular functions to regular functions okay this is the definition of what a morphism is, what you must understand is the following that if you go down from the category of varieties to the categories of topological spaces that is you forget the variety structure and just look at all this things as the underlying topological spaces with the of course with the zariski topology then what will happen is if you give me a continuous map, if you give me a morphism phi of course phi is continuous and if you give me a V and if you give me a regular function on V.

This f is of course going to be continuous because I have already proved you that regular functions are continuous okay, regular function from a variety is always a continuous map, continuous with the target k being thought of as A1 in this zariski topology this is something that we have proved last time okay so since phi is continuous and f is continuous it is very clear that f circle phi which is the pullback of f by phi is also phi upper star f, it is clear that this is continuous okay.

But what our requirement is that it is not just a continuous map from U to k of the zariski topology you should actually be a regular function from U to k for the zariski topology that is the requirement okay that is what has to be singled out okay so the moral of the story is a continuous map such that if you take a regular function on the target on an open subset of the target and take the composition okay, what you get on the source are subset of the, on the right subset of the source it is not just the continuous function it has to be more, it has to be itself a regular function.

Namely it has to be locally given by quotients of polynomials that is the requirement okay so it is a philosophy that it is a map of varieties is a map which is continuous and which pulls back good functions to good functions and the good functions here are regular functions okay, now so here is the, so the point I want to make is the following.

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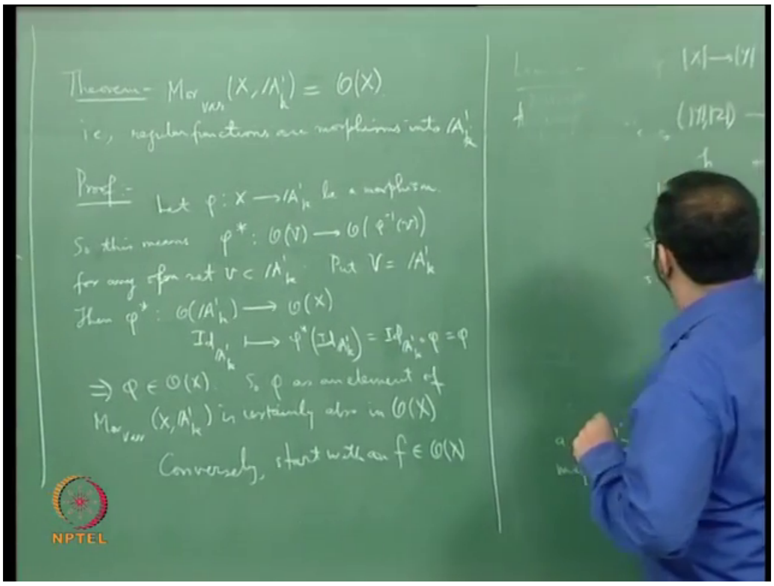

So here is a nice little theorem, the theorem is if you look at if you take a variety X and look at all the possible morphisms into A1 you will get exactly all the regular function in this okay so morphisms of varieties from X to A1 is naturally isomorphic to O X okay so the regular functions are actually morphisms into A1 there is no difference okay, what are elements of O X? They are maps into A1 which is just k with the zariski topology alright and which are locally given as quotient of polynomials that is what a regular function on X makes okay.

And what is a morphism of varieties from X to A1? It is also a map from X to A1 which is a map from X to k but then the condition is that it has to be continuous and it should pull back regular functions to regular functions okay and the beautiful thing is that there is no difference between regular function and a morphism into A, so regular functions are exactly the same as morphisms into A1, so let me write that that is regular functions are, so in fact I should not even put isomorphic, I should put equal to, are morphisms into A1.

So of course it is A1 k, where k is the (())(12:05) algebraically closed field over which we are working okay so it is a very nice theorem, it tells you that you have morphisms into A1 are the same as regular functions into A1 okay, so let us try to prove this, so you know, so what you do is so what is a map, of course this map is, it is a identity map okay, it is the identity map but only thing is that when you take a regular function on X the target is taken as k and the target k you do not care about the topology of the target k.

But if you put a topology on the target k and call it A1 then you know regular function is continuous so it certainly continuous map from X to A1 but the fact is it is more it is actually a morphism from X (())(13:00) that is what this theorem is, so let us prove this. Let phi from X to A1 be a morphism okay, what do I have to show? I have to show that phi is a regular function alright now so this means phi upper star from O of V to O of phi inverse of V for any open set V inside A1 okay.

So if you, what is a morphism? A morphism is a map which is continuous and it pulls back regular functions so if you give me an open set in the target which is an open subset of A1 then and you give me a regular function on that open set on A1 which is an element of O V then applying phi upper star the pullback should give me a regular function on phi inverse okay now what I am going to do is I am, this way it is actually easy you put V equal to A1 itself.

Put V equal to A1 okay then you get phi upper star will go from O of A1 to O of X okay because phi inverse of A1 is X the inverse image of the target space under any map is the whole source space okay I will get this okay and then what you and then we will use the fact that you know on O of A1 see I have the identity map, see the identity map on A1 is a regular function on A1, see the identity map on A1 is a map that sends every point of even to it to its every point of A1 namely every point of K to itself.

And what is it you know O of A1 if you want is A of A1, the ring of functions on A1 which is k X okay and what is the identity map correspond to, it correspond to the polynomial X identity map corresponds to the polynomial X, the polynomial X evaluate at lambda gives you lambda okay so O of A1 is a same as A of A1 okay that is something that we have already seen O and A coincides for affine varieties alright so O of A1 is same as A of A1 and identity map in O of A1 is actually that corresponds to the function define by the polynomial X okay.

So this is, so identity map is certainly a regular function, identity map on A1 is certainly a regular function and what does it go to? It will go to phi upper star of the identity map on A1 but phi upper star of the identity map on A1 is just phi, phi upper star of the identity map on A1 is identity map on A1 composition with phi which is just phi and but then you are saying that this is here so you are just saying that phi is inverse.

So this will tell you it belongs to O X, so it is very clear that a morphism from X to A1 is certainly an element of O X okay so phi as an element of morphism, phi as a morphism of varieties from X to A1 is certainly also in O X right, so this is very trivial alright (())(17:34) things the other way around, I will have to say that I have start the regular function and I like to start with the regular function on X and say that that is a morphism okay.

So let me do the other way conversely start with an f inverse okay start with an f inverse alright I want to show that f is actually a morphim into A1, so you know my f in O X means that f is a regular function defined on X its target is k and we have already seen that if this target k is given the zariski topology and is thought of as A1 then this f is of course continuous okay.

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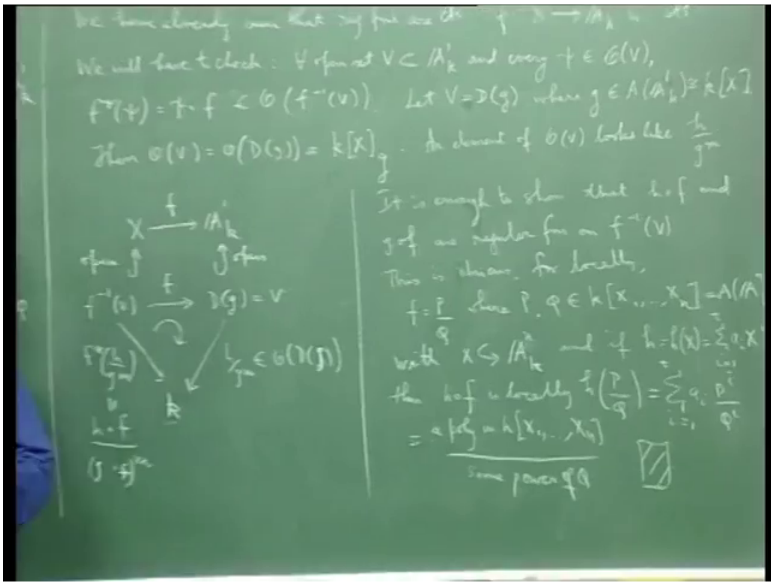

We have already seen that a regular functions are continuous so that means that f from X to A1 is continuous so what are the two defining conditions for a morphism, it should be a continuous map and then it should pull back regular functions to regular functions so we will have to check for every open set V in A1 and every psi in O V, f upper star psi which is just f followed by psi is in O of f inverse V, okay, this is what you will have to check.

You have to check that it pulls back regular functions to regular functions alright so now you see what you must understand is that V is an open subset of A1 of course I am certainly not looking at empty set alright so you know a non-empty open subset of A1 is actually a basic open set okay because you see what you must understand is the open sets in A1 are compliments of finitely many points okay and therefore they are compliments of finitely many points and there is always a polynomial with roots exactly at those points.

And it is the zero set of that polynomial which is the compliment of this open set and therefore this open set is defined as the basic affine open set given by the non-vanishing of that polynomial okay that polynomial which vanishes at those points in the compliment of the open sets, you must understand that the important fact we are using is that even as the only closed sets in A1 are finitely many points and this is basically commutative algebraic reflection of the, it is a reflection geometric reflection of the commutative algebraic fact that every ideal in the polynomial ring in one variable or the field is generated by a single element okay.

It is a PID alright so the fact is that so let V be D of g okay where g is an element of, if you want A of A1 which is identified with k X, g is the polynomial in one variable right and V is D of g so it is a basic affine open defined by g right, then what is O of V, then O of V will be O of D g and what is O of D g, our definition as we have checked it is just k X localised at g okay, the regular functions on D g are the same as the A of D g and that defined to be the localization by g okay.

So O of V is just this okay and so what is an element of O of V of the form, it is of the form some h by g power m where m is the positive non-negative integer okay and element of O V looks like h by g power m alright, this is how an element in localisation looks like and you are thinking of this h by g power m as a function on, as a regular function on V and every regular function on V looks like that right.

And now if you, so let me draw this diagram so I have X, I have A1 and here is f and here is D of h okay which is my V and then I have f inverse V this is open of course f is a regular function it is continuous so the inverse image of an open set is open so f inverse V is open and then I have on D of h, I have a regular function okay and that regular function is given by h by g power m, this is an element of O of D of h and then I compose it with f to get the pullback this is a regular function into k okay, sorry this is D of g it should be D of g, I am sorry it should be D of g.

I think I have messed it, did I messed it somewhere there no I did not okay, I messed up here alright it should be D of g so yeah so I have h by g to the m which is the regular function on D g and then I compose this to get the pullback function which is f upper star of h by g power m which is by definition equal to first apply h and then apply, first apply f and then apply h by g power m and so it is going to be well.

So if you calculate it is actually f h circle f by h circle g to the m this is what it is, so because this is what it will be when you evaluate if you take an X here then it will go to f x and when you evaluate h by g power m on f x you will get h of f x by g to the m f x so the effectively this is h circle f by h, h circle g to the whole to the m alright and here of course and now the point is that what do I have to check? I have to check that this is, I have to check that this is a regular function on f inverse V.

I have to check that this is a regular function f inverse V okay so it is something wrong oh sorry this is g, sorry you are right, it is g circle okay, it is g circle ops yes g circle f to the m but yes yeah please check that it is f followed by h divided by followed by g whole to the m yeah thank you, so I will have to check that this is a regular function I mean if you think about it for a moment you will not hesitate to realize that this is a regular function okay.

But then if you want the way to look at it is the following the way to look at it is that you see the locus see f is a, see so let us look at what f is, so f is a regular function on X okay so you know I want to say the following thing I just want to say if I tell you that h circle f is regular function on f inverse V and g circle f to the m is also a regular function on f inverse V that does not vanish on f inverse V, then the quotient will also be a regular function on f inverse V.

I am just using the following thing namely that suppose you have a regular function that does not vanish then it is reciprocal will also be a regular function because the only condition for a regular function is that it is locally given by quotient of polynomials and if that quotient of polynomials does not vanish it mean that the numerator polynomial does not vanish if they are in the, if there are no common factors and with the numerator polynomial does not vanish then the reciprocal of the quotient is also a valid quotient of polynomials and that tells you that the reciprocal of a regular function that does not vanishes is also a regular function alright.

So therefore if I claim, therefore it is very clear that it is enough to show that h circle f is a regular function, it is enough to show that h circle f is a regular function on f inverse V, g circle f is also regular function on f inverse V and g circle f does not vanish on f inverse V okay it is very clear that g circle f cannot be vanish on f inverse V because if g circle f vanish at a point in f inverse V it will mean that at the image of that point under f, g will vanish but then the image of that point is supposed to lie in D g where g cannot vanish so it is very clear that g circle f cannot vanish on f inverse V.

So the only thing therefore I have to prove is that h circle f or g circle f are actually regular functions okay but so let me write that down it is enough to show that h circle f and g circle f are regular functions on f inverse V this is all I will have to show okay now this is again something that is very very easy to see because you see, see f therefore it started with first regular function on X, so f is given by a quotient of polynomials in the right number of variables, the number of variables being the dimension of the affine space in which X sits alright.

So f is locally a quotient of polynomials okay so h circle f will also be a quotient of polynomials the same number of variables because h is just a polynomial in 1 variable when I take a polynomial in one variable and substitute for that variable another polynomial in some m variables the resulting thing will again be a polynomial in m variables it is obvious therefore the moral of the story is that it is very clear that since f is locally a quotient of polynomials h circle f is locally a quotient of polynomial on X okay then h circle f is also locally a quotient of polynomials on X okay.

So let me write that down, this is obvious for locally f is equal to p by q where p, q are polynomials in the right number of variables which is A of An with X siting inside An okay and you know and if h is equal to h of X is equal to sigma Ai X to the i, i equal to 1 to say some t then h circle f is locally h of t by q which is sigma i equal to 1 to t Ai p power i by q power i okay which is a polynomial in k of X1 etc Xn divided by some power of q, so you are done.

So the moral of the story is that if you really think about it, it is very clear that a regular function is also a morphism, so there is no difference between regular function on a variety and morphisms into A1 there is absolutely no difference okay so the beautiful thing is that to define a morphism we started with regular functions and then found that regular functions are themselves special cases of morphisms okay.

In that sense it should not be very surprising but you must understand that you see right from the beginning this is again you know amplifying more and more the philosophy of Felix Klein that you know the geometry of the space is completely by the functions you choose on it so you know we started with the affine space, we started the n-dimensional space kn, we choose the functions to be polynomials using the polynomials we define the zariski topology on kn okay and then from the zariski topology we started translating back into commutative algebra and we came down to defining.

We first came down to guessing what should be a regular functions okay for example on basic open sets and then generally what should be a regular function, what the regular function should be on the general open set okay and then after we define regular functions once we define regular functions we found that if you take affine varieties the regular functions are just polynomials okay and we also found that using this regular functions we could define morphisms.

And then it is not surprising that finally regular functions themselves turn out to be morphism alright so that is a very nice thing, it is a very nice result but now what I am going to do is that I am going to say that this is, this statement here is only, it is still a special case of a very very general statement okay so let me let me make that statement.

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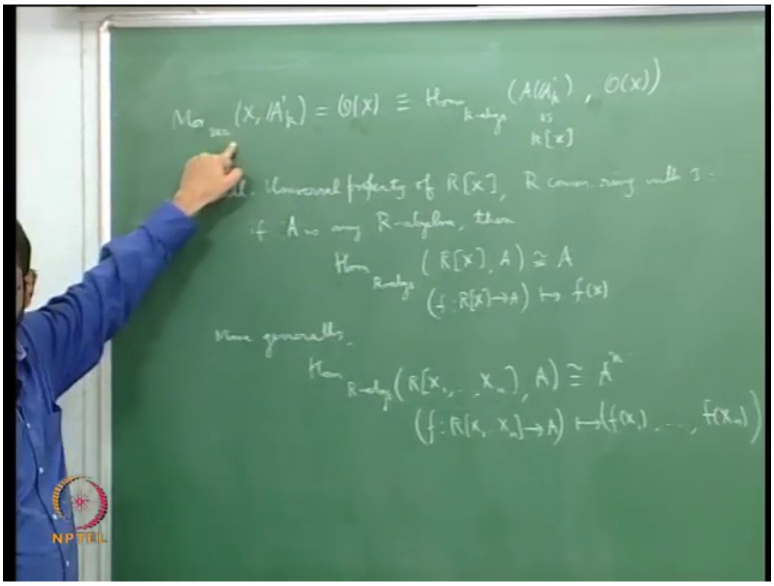

So again let us look at that statement, morphism of varieties so this is morphism of the category of varieties from X to A1 is identified with O X okay and you know and I just want to say that this is isomorphic to homomorphism of k-algebras from A of A1 which can be identified as k X to O X okay so you see now what I am trying to do is I am now trying to translate things, see on this side the morphisms I have is in the category of varieties now I want to go, this is geometric side, I want to go to the commutative algebra side and I want to translate morphisms of varieties into morphisms of rings, in this case morphisms of k-algebras.

And this is the this is the way I do it what I do is I think of O X as the I intervaled as being bijected the set of all k-algebra homomorphisms from the A of A1 which is k X to O X, see you know there is this universal property of the polynomial ring in one variable it says that a k-algebra homomorphism from a polynomial ring in one variable is completely dictated by the image of the variable X okay

So any k-algebra homomorphism from k X to O X, from k X to any ring is simply controlled by what you are sending X to, so more generally so let me recall universal property of R X okay where of course R is the commutative ring with 1, what is the universal property, the universal property is that if A is any R algebra then Hom r-algebras from R X, A can be identified with A by simply sending a map f from R X to A to f x to the element f x because it is a substitution alright.

Sending a map from R X to something means that you are substituting X for something in every polynomial in X with quotients in R so it completely is controlled by what you are substituting X if you substitute X with some lambda then you have to substitute every polynomial in X with lambda in place of X so it is controlled by that, this is the universal property and in fact this universal property is also valid in several variables more generally homomorphism as r-algebras from R X1 through Xn to A is isomorphic to A to the n where you simply send a map f from R X1 through Xn to A to the tuple f x1 dot dot dot f x to the n topology.

That is a map, an r-algebra homomorphism from a polynomial ring in n variable is dictated by the image of those variables this is, these statements are restatement of universal property of the polynomial ring over a given ring okay so and I am just applying instead of R, I have put the field k and instead of A, I have put O X so the homomorphism from the k-algebra from k X to O X is just O X and that is what this is okay.

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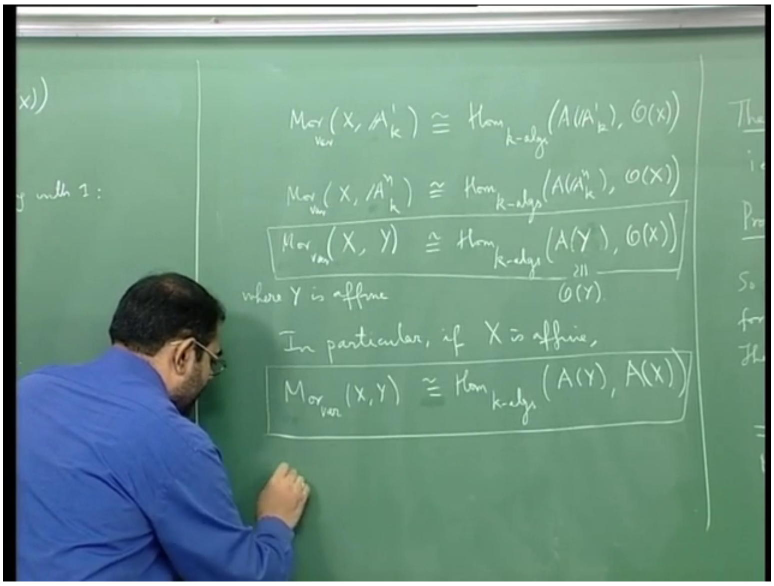

And now let me tell you how this statement generalises the statement generalises like this okay it is very beautiful so it generalises like this morphisms from X to A1 can be identified with homomorphisms of k-algebras from A of A1 to O X okay and morphisms from X to An can be identified with homomorphisms k-algebras from A of An to O X okay and morphisms from X to Y okay where Y is affine okay so instead of taking the target to be A1 you can take the target to be An instead of taking the target to be An I can take the target to be an affine variety.

So the morphisms from any variety into an affine variety can be identified with homomorphisms of k-algebras from the affine co-ordinate ring of that affine variety to O X sorry this should be A Y and mind you this A Y is the same as O Y there is no difference because Y is affine and the, in particular you see, in particular if X is affine the last statement says that the morphisms, so here of course this is all morphisms is varieties on this side it is a geometric side there are morphisms between geometric objects and this side is the algebra side, the commutative algebra side.

So the morphism of varieites from X to Y can be identified with homomorphisms of k-algebras from A Y to A X and mind you I can replace O X by A X because O X and A X are the same because X is an affine variety okay so this is the, these are the two important statements this one and this one and why are these statements important? The statements, we will prove this statements okay and they are just you know generalised versions of this first statement which is just the simple statement that the regular functions on variety are the same as morphisms into AY okay it is just a generalisation, the grand generalisation of that.

But how grand it is, is that it actually gives you an equivalence between the category of affine varieties and the category of finitely generated k-algebras that is the reason why this is very important statement, so you have category, so category, so let me do that in the next board so that I have little bit more space to write what I want to write down, so what I do is now I can really complete the picture that I gave several lectures ago and then I was just you know throwing statements at you.

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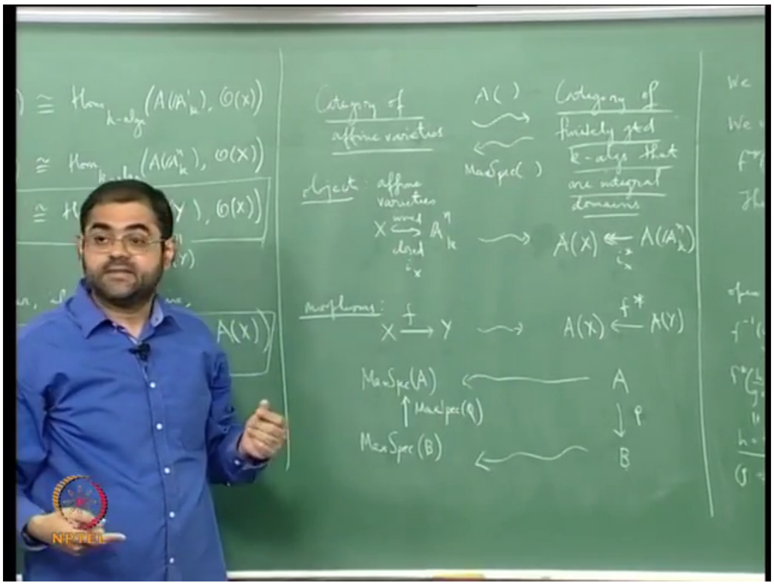

So here is, on this side I take the category of affine varieties okay and here the objects are affine varieties namely irreducible closed subset of Am for some n positive these are the objects in the category and I am going from there to there category of finitely generated k-algebras that are integral domains sometimes instead of writing so much people just say the category of affine co-ordinate, category of affine rings over k, the affine rings means they are actually rings of functions on affine varieties okay.

So and abstractly they defined as finitely k-algebras that are integral domain which means that you are just taking the finite, you are just taking a polynomial ring in finitely many variables over k and going modulo prime so that we get an integral domain okay and you know I have this, I have a functor that takes objects to objects here so what it does is that if you give me an X it takes this X to well A of X okay so it tends, it takes X to A of X and it takes An to A of An and it takes this closed immersion to a quotient okay.

And you know what that map is if you call this closed map if you call this closed inclusion as i x you know what this map is this is just i x star you can check that this closed, this map the inclusion map is actually a morphism of varieties and you can check that the pullback i x star that induces it from the O this to the O this which is the same as the A of this to the A of this is simply the restriction we can check that okay.

So this i x gives you this i x star okay and what are the morphisms, the morphisms here are well if you have X and Y two affine varieties I get you, you have a morphism f then that will rise to a map on this side that will go the other direction so it will go from A Y to A X and you know what that is that is just f star is a pull back, so the fact is that if you give me a morphism of varieties the pullback map is actually a ring homomorphism okay that is something that you can very easily check it is obvious and in fact it is a k –algebra homomorphism.

So the morphisms on this side are k-algebra homomorphism, they are ring homomorphisms which take one to one which respect the vector space structure they are k-linear, so they are k-algebra homomorphism and the fact is that this is the functor, now A is a functor it is a functor because it goes not only does it associate an object to an object but it associates a morphism to a morphism okay, to every morphism variety as you get a associate in k-algebra homomorphism so this is a functor.

And the fact is that this is an equivalence of categories because there is an inverse functor which has I told you is given by max spec, the max spec functor starting with here, starting with finitely generated k-algebra A, I can give you, I can look at max spec A that is a variety okay that can be identified with a variety and if you give me a k-algebra homomorphism g, phi from A to B then what you will get is you will get a from the max spec to the max spec you will get a homomorphism in the reverse direction namely max spec phi.

And this is just pull back of given a prime ideal in B, I mean given a prime ideal in B the inverse image of a prime ideal will be a prime ideal in A and the fact is you can check that because it is a finitely generated k-algebras which are integral domains in k is an algebraically closed field if you take a maximal ideal in B and you pull it back you will get a maximal ideal A and so it will take max spec to max spec and this is the inverse functor.

And these two functors are inverses of each other in the sense that you start with an object you go there and come back what you get is an isomorphic object okay and then you similarly if you start there you go here and come back you will get something that is you will get an isomorphic k-algebras so if you start from here and go and come back you will get an isomorphic variety, if you start from there with an algebra go and come back you will get an isomorphic k-algebras okay.

And that is why these two functors are called inverse functors and we say that each of them defines an equivalence between the geometric side which is the category of affine varieties and the commutative algebraic side which is the category of finitely generated k-algebras which are integral domains and you know it is easy but and certainly not an exaggeration to say that this is the full blown form of the Hilbert Nullstellensatz okay with all the other definitions in the right place this is the grandest form of Hilbert Nullstellensatz, so with that I will stop this lecture.