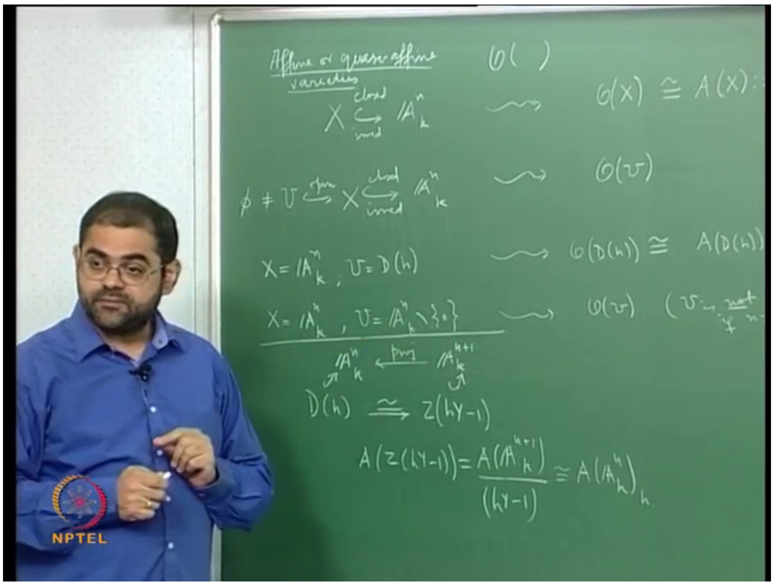
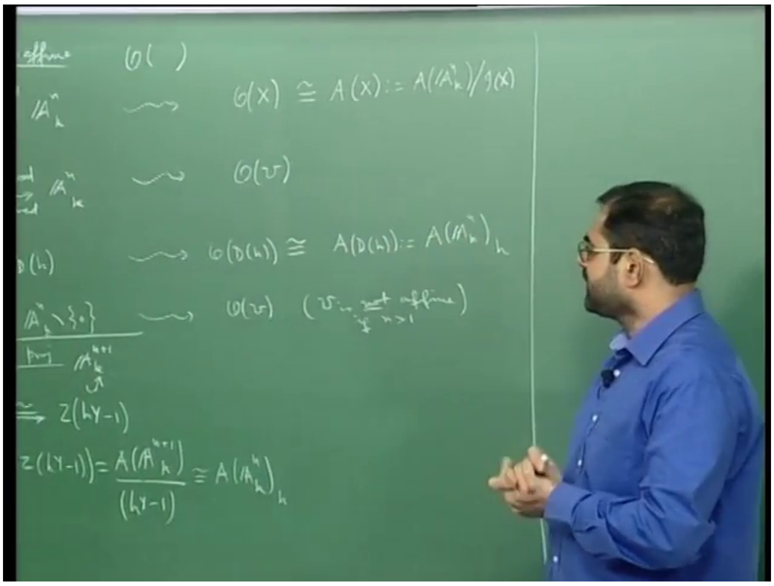
Basic Algebraic Geometry
Dr. Thiruvalloor Eesanaipaadi Venkata Balaji
Department of Mathematics
Indian Institute of Technology, Madras
Lecture-17
Characterizing Affine Varieties;
Defining Morphisms between Affine or Quasi-Affine Varieties

(Refer Slide Time: 02:04)





Okay so see what we are trying to study now at this stage of the course is the so called good functions that you can define on a variety okay and actually these good functions are called as regular functions okay and you know we have defined, so for any variety or a quasi-affine variety in affine space, so if I call that as a X, X is either closed irreducible subset of some affine space or well or I can take an open subset of course non-empty.

Open subset of such an irreducible closed sub variety that is an irreducible closed subset of some affine space then we have defined this operation O, this operation O gives you O of X in this case and in this case it gives you O of U and what is O of X, O of X is supposed to be the ring of regular functions on X okay, O of U is similarly the ring of regular functions on U so here what I am having on this side is affine or quasi-affine varieties.

These are the objects on this side okay so an irreducible closed subset of affine space is called an affine variety if you recall and an open subset of non-empty open subset of such an affine variety is called quasi-affine variety, okay, and for an affine variety or a quasi-affine variety I define the ring of regular functions to be functions which can be locally written as quotients of polynomials okay.

So what it means is that see your whether it is x or whether it is u, they are all sitting inside an affine space and on the affine space I have natural functions giving by polynomials. Polynomials in the right number of variables okay so if you give me, so in this case if it is An I am taking polynomials in n variables and you give me a polynomial in n variables that gives you a function from affine space into the base field k okay which of course to just to remind you is always an algebraically closed field and so any polynomial here gives you a map from An to k okay.

And that map can be restricted to any subset, so such a map can be restricted to X, such a map can be restricted to U, okay and what we are saying is that we are saying that if you take two such polynomials okay and suppose I take the quotient of these two polynomials and suppose I look at, suppose I restrict this quotient to subset where the denominator polynomial does not vanish then that will also give me a function from that subset into k okay and functions like this which I am calling as regular functions.

So more generally a regular function is something that locally looks like a quotient of polynomial okay so the right way to say is that regular function is something that, it is a function which is got by, which is locally a quotient of polynomial and it is gotten by gluing such quotients okay and of course because of the quasi-compactness of the zariski topology you know that that this gluing can be done on just one of finite number of basic open sets okay that is something that we have seen last time.

And in fact what we prove was we proved that you know that whenever the variety is, whenever X is isomorphic to an affine variety okay so I am again making a statement which is kind of retrospective kind of statement, it is a futuristic statement in the sense that I have not defined what is meant by isomorphism of varieties so it is very strictly not correct or pedagogically sound for me to tell you give a statement like let X be isomorphic to an affine variety but assume it for the moment okay.

So the fact is that whenever a variety or a, so if X is already a closed sub, irreducible closed subset of affine space it is already an affine variety by definition but if you take an open subset of such an affine variety the beautiful thing is that it may or may not be an affine variety what I mean by that is that it may or may not be isomorphic as a variety to another variety which is affine.

So the both cases happen, so there are special cases when you take U when you take U equal to, you take X equal to, if you take X equal to An okay and if you take U to be the basic open subset defined by the locus of points where a polynomial hits in the right number of variables n variables does not vanish then you know that this is actually an affine variety in the sense that the, this can be identified as a closed subset of an affine space of dimension 1 more you have seen this.

And in this case what happens is that even though U is only a quasi-affine variety in An, it is actually isomorphic to a variety in An plus 1, so in that sense U is not only a quasi-affine variety but it is also an affine variety it is not an affine variety in An but it is an affine variety in An plus 1, okay and on the other hand you know I give you, if you take this statement like this if you take An, I mean if you take a situation like this take X equal to An and take U to be just the compliment of the origin okay.

So you take An minus topology okay now this is also an subset of affine variety and the fact is that of course the fact is that if n is greater than 1, this is not, this can never be isomorphic to an affine variety it is a fact, that we will see later okay this cannot be isomorphic to an affine variety okay. So there are open subsets of affine varieties namely there are quasi-affine varieties which can be isomorphic to affine varieties.

And there are quasi-affine varieties which cannot be isomorphic to affine varieties and the point is that whenever a variety or a quasi-affine variety especially whenever you take a, so let me begin like this if you take an affine variety okay namely an, it is already an irreducible closed subset of An, if you calculate the ring of regular functions okay then I prove last time that this is isomorphic as k-algebras to the ring of polynomial functions, the ring of co-ordinate functions, so called affine co-ordinate ring of X which you know is defined to be just the affine co-ordinate ring of the full affine space in which x is sitting modulo the ideal of X which is a prime ideal.

So this is a finitely generated k-algebra and this is how I define the ring of functions on a irreducible closed subset okay and the fact is that if X is an irreducible closed subset of An the ring of regular functions is naturally isomorphic to the ring of, to this ring, the affine co-ordinate ring of functions that come out of polynomials okay so what you are saying is you are saying the following, so what does it mean? What it means is if x is an irreducible closed sub variety of, irreducible closed subset of some An then a function on X which is gotten by locally gluing quotients of polynomials is actually represented globally by a single polynomial function mind you, what are the elements of A X?

The elements of A X are just the co-sets of, I mean the elements of A X are just co-sets here this is a quotient ring this is a co-set, so an element of A X is written as F bar where F is an element of the affine co-ordinate ring of affine space namely F is just a polynomial in n variables and F bar denotes the co-set F plus IX okay and the fact is that this F plus IX also defines a function on X what is that function? It is just F restricted to X.

You have if you take an F here, it is a polynomial in n variables so it is a function on affine space and X is after all a subset of affine space so I can restrict that polynomial function to X and the resulting function on X is not F itself, I mean it is a restriction of F but it can also be represented by a g such that the difference F minus g is in the ideal of X, so writing a function here as F bar means that you are writing it upto the ideal of X.

Because if you have a function on X and you add to it a function that is identically 0 on X okay then the resulting thing is also going to give you the same function on X okay so function on X is of course polynomial function on X is of course a polynomial function on An upto an element of the ideal of S because element, the polynomial functions in the ideal of X when your restrict them to X they are going to give you the 0 function okay.

And what is this isomorphism signifies? This isomorphism signifies that you give me, if you take an irreducible closed subset of affine space then a function on that with values in k if it is obtained by locally gluing quotients of polynomials then it can be represented globally by a polynomial as the restriction of a single polynomial function that is what it says and in particular you know if you take X equal to An itself, what this will tell you is that if you take a function on all of affine space which is locally the quotient of polynomial X then globally it has to be polynomial and that polynomial has to be unique.

Because in that case you will get an isomorphism of O X namely O An with A An itself because I An will be the zero ideal okay so that is the significance when you get an isomorphism when you apply the O and A okay mind you, you can apply O and A has been defined only for some certain special objects in this side okay so you know if you remember if you take basic open set like this okay then you know I have defined A of that basic open set, the functions on that basic open set to be just the localization of the functions, the ring of functions on the ambient space at h.

So and this is, so this is very simple when you take a ring and you put a subscript saying you are localizing at that element it means you are just inverting that element and that makes sense because, basically because on D h, h does not vanish so 1 by h and all powers of 1 by h are also valid functions on D h and therefore a general valid function on D h will be some polynomial by a power of h and that is exactly the kind of elements that you have in this localization so we made this definition okay.

And in fact I justified that this definition is correct in, at least I gave three partial justifications as to why this definition is correct of course you know one justification was that it is correct to invert h because h is non-zero on D h which is a locus where h is not zero therefore a general function on this should be some polynomial divided by a power of h okay which makes sense and when you collect all these things together and identify them properly you will get the localization at H that is one justification.

What is the other justification? The other justification is that D h is also as I told you an affine variety because it can be identified with a closed subset of An plus 1 so you know this D h if you recall this D of h which sits inside An okay can be identified with the zero set of hY minus 1 in an affine space of dimension 1 more okay where Y is the extra co-ordinate which you are adding okay and this identification comes because of the projection.

The projection map from An plus 1 to An if you are restricted to this closed subset you will get this and what is so you know if you belief I asked you to check that it is a matter of topology to check that this identification is not only bijective but it is not a bijective map of sets but it is also a homeomorphism of the zariski topology which I hope you check okay but the fact is you can go one step further in fact this is an isomorphism of even varieties okay.

And that is again statement that you will have to, that we will have to fix up later on when we come to the notion of isomorphism but if you believe that then it tells you that since this is an irreducible closed subset of an affine space, this is an affine variety and this is something that is isomorphic to an affine variety and therefore it is correct to define A of this to be the same of A of this and the A of this is actually isomorphic to because of properties of localization in commutative algebra A of this is precisely this okay.

Because A of Z of hY minus 1 is just A of An plus 1 divided by hY minus 1 which is the ideal of Z of hY minus 1 because hY minus 1 is an irreducible polynomial and this is actually isomorphic to A of An localized at h okay because this is k x1 through xn, Y by hY minus 1 and that is isomorphic to kx1 through xn polynomial ring localized at h okay this is the property from commutative algebra and I was just trying to tell you that this isomorphism between these two rings is actually commutative algebraic translation of this isomorphism of variety okay.

And therefore I told you that is the justification as to why you can define A of D of h to be this okay that was the second justification okay and the third, there was yet another justification that I gave and that justification was the third justification was that if I take this A so you know it is the, third justification is that whenever for example I start with An okay I go to A of An if I take max spec I get back An okay.

So which means so I am just saying max spec of A of An is the same as An that is just the nullstellensatz if you really but it is not just nullstellensatz set theoretic bijection, it is actually I told you to check that this is even and in fact we check I think we actually check that this identification of max spec of A of An k which is the set of maximal ideal is in A of An with the zariski topology for the maximal spectrum which is induced by the zariski topology and the prime spectrum that, with that topology max spec of A of An and An itself are homeomorphic and again let me go one step further we will see it is a fact that this homeomorphism, it is just not even a homeomorphism, it is an isomorphism of varieties in the most general sense okay.

So the fact is that, similarly if I take an irreducible closed sub X then I will check that if you take max spec of Ax then that is isomorphic to x itself at least you could have, you would have, you must have check that it is homeomorphic to X topologically but the fact is it is even isomorphic to X okay, so moral of the story is that somehow if something is affine if you, then you define the A of that okay and then if you apply the max spec you should get back that thing okay.

So the fact is that if I take D h if I apply A to that I get A of D h which is this okay if I take if this is a correct definition if I take max spec of this I should get back D h and that is again a fact I asked you to check at least topologically that you get it back okay so these are three justification as to why this is the correct definition but then here is yet another definition, the for D h I can define O of D h okay and then it is a fact that O of D h turns out to be isomorphic to this okay.

So this is part of this philosophy that I was trying to tell you that you know whenever something is affine if you apply O to it and you apply A to it. A is always defined whenever something is affine okay and O is defined whether something is affine or not and the fact is that the characterization that something is affine is given by the fact that when you apply O to it and apply A to it you should get the same thing if you get the same thing then and only then is that thing affine okay, that is then and only then is that object can realizable as an irreducible closed subset of some affine space okay.

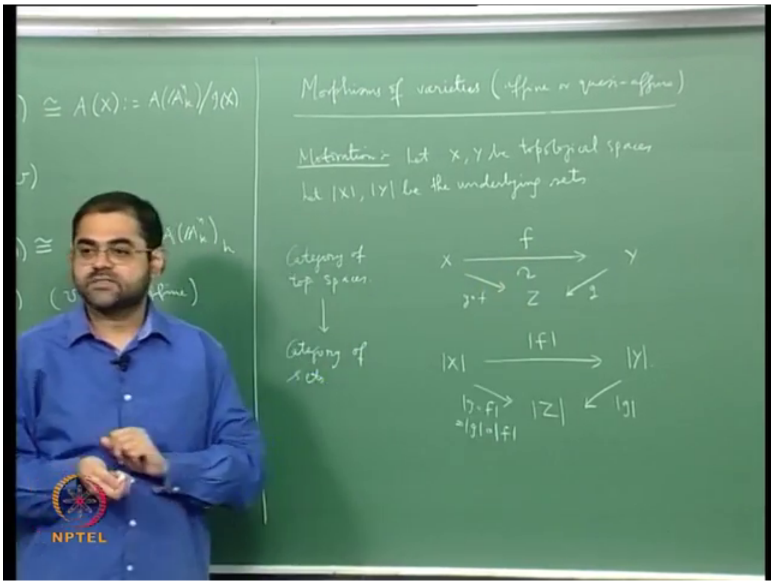
So the fact that an object is, can be realizable as an irreducibly closed subset of affine space okay is captured in checking whether O of that and A of that are isomorphic okay and I told you that well, the so you know if I take O of U okay I should not say O of U is not, so in this case if you take affine space minus the origin I should not say O of U is not isomorphic to A of U, the fact is that I do not define A of U because I know it is not affine okay so A is somehow defined only in cases where you know things are affine okay.

And the fact is that for all those things that are affine if you apply O also you will get the same result as you would get when you apply A okay. Now yeah so this is the story so far alright and so I was trying to just tell you that so here I should say here U is not affine U is not affine if n is greater than 1, by that I mean U is not isomorphic to any affine variety provided n is greater than 1, of course n is 1 then you know then you are going to get A1 minus the origin and A1 minus the origin is affine because it is just D of x what is D of x? D of x is the set of points where x does not vanish and the set of points where x does not vanish on A1 is precisely A1 minus 0.

So when n equal to 1 it is actually affine because it is a D of x and all I am trying to say so when n is 1 it is actually even basic open affine okay but when n is greater than 1 this is very very far away it is not even affine okay it is very far away from being affine. So alright so you know now somehow you know I am, now let me go back and try to tell you that in all this things you know I have been using the statement that some variety is isomorphic to some other variety okay so I have to define what an isomorphism of variety is.

And you know in general philosophy, the general method to define an isomorphism is to define a morphism which has an inverse which is also a morphism okay this is how you define an isomorphism alright, so basically isomorphisms can be defined if I can define morphisms and so what I am going to do next is how do I define morphisms okay so let me make that statement here.

(Refer Slide Time: 24:56)



So let me put that as a title here morphisms of varieties so by varieties I mean either affine varieties or quasi-affine varieties how do I define what are morphisms right, now to understand this let me give you a, let me go back to something more basic let me go back to topology okay so see let me take so motivation is you know let X, Y be topological spaces okay and let mod X, mod Y be the underlying sets okay.

So you see what I am trying to do, do not confuse mod X with the cardinality of X which is usual notation okay so here for me mod X is not the cardinality of X and mod Y is not the cardinality of X when I write mod x I mean throw away the topological space structure on X and think of X only as a set so mod X is just X is a set mod Y is just Y as a set okay now what I am going to do is there are two categories.

There is a category of topological spaces and there is a category of sets okay so in the category of sets the object is sets and the maps, so just to recall a category very naively basically consists of two pieces of data one piece of data specifies the so called objects of the category and the second piece of data is the maps between these objects, maps with certain properties and that is why the word morphisms is also used in subsets maps okay.

So every category is specified by defining what the objects are and what the maps are okay so when I say category of sets, the objects are sets and the maps are just maps of set okay just functions from one set to another without any other properties when I say category of topological spaces the objects are topological spaces and what are the maps? The maps are not just maps of sets but they have to be continuous maps okay.

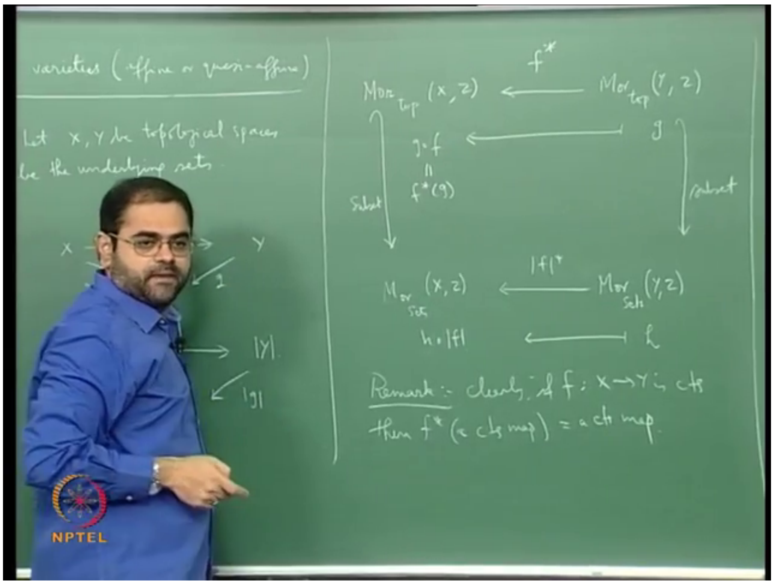
And you can go on like this for example if you take the category of rings, the objects then are rings and the maps are not just set theoretic maps of rings they are ring homeomorphisms okay and so on and so forth but I am only interested in these two categories with respect to these two guys so you know if you take the object x okay and if you take a topological space X then I have the corresponding I have the corresponding underlying set mod X okay.

And if you give me a topological space Y I have the underlying set mod Y okay and then well if you give me another topological space Z okay then I will again get the set mod Z okay and now well you see I start with so I do the following thing if you give me a map f from X to Y of topological spaces okay then I will get this map mod f which is a map of from mod X to mod Y and the point about mod f is that I forgotten the continuity of f.

Here f is a map is a morphism from X to Y is a morphism in this category so it has to be continuous map where as if I take mod f, I am looking at the map as a set theoretic map okay and then well if you give me any map g any continuous map g from Y to Z then I get the composite map which is first apply f then apply g and you know composition of continuous maps is continuous therefore if g is continuous then f followed by g is also continuous.

And the corresponding diagram here will be, here I will get a mod g and here of course I will get mod of g circle f which is of course mod g circle mod f I am simply forgetting the continuity alright and see the fact is that, that is a functor like this and this functor is called the forgetful functor it is a functor that forgets the topological space structure okay it is called the forgetful functor because you are forgetting everything connected to the topology given a topological space you are attaching to your, you are just associating to it just the underlying set.

(Refer Slide Time: 31:09)



And given a continuous map of topological spaces you are just associating to it the underlying set theoretic map okay, now what we get from this diagram is the following you get so you know let me let me give some symbols okay so let me do that here so you see what I am trying to say is you have a home okay or let me not write home I think home alright so let me use mor, so this mor is supposed to be abbreviation for morphisms so when X and Y are objects of a category mor XY is a set of morphisms from X to Y okay in that category.

So you see if you have morphisms from if you take morphisms as topological spaces from X to well, so let me do the following things I will take morphisms as topological spaces from Y to Z okay and if you give me a morphism of topological spaces from Y to Z namely g, I get a morphism of topological spaces from X to Z which is g circle f and this map which is going in the reverse direction, one writes this as f star okay.

This is called the pull back of morphisms okay, it is very simple you have two objects in the category okay you have a morphism between them and what you are doing is given a morphism on the target you compose it with this to get a morphism on the source so a morphism, a map, a morphism from one object to the other object pulls a morphism on the target to a morphism on the source you are pulling back morphisms, it is called the pullback induced by a morphism.

So f star is the pullback induced by f okay so it is simply g going to g circle f okay so this is the pullback functor, this is a pullback induced by a map alright and you have similar map here so you have mod f star which is the same as mod of f star it is not, probably I should not say mod of f star, so this mod f upper star, it will go from morphism as sets from Y to Z to morphism as sets from X to Z okay and this is give me any h it will send h to well g circle h, h circle mod f this is what I will get okay this is, if you give me, so mind you h is just the morphism of sets it is just a set theoretic map from Y to Z.

You give me a set theoretic map from Y to Z since I have a map f since I have a map mod f from X to Y, I composite with this set theoretic map h from Y to Z and I get this composition which is this okay and the fact is that as you can see that this, this is a subset of this, of course maps of topological spaces are certainly maps of sets but they are not just maps of sets they are continuous maps okay.

And similarly this is subset of this here also this set is the set of all possible maps functions from Y to Z but these is the subset which consists of only the continuous functions okay so and you know two maps of topological spaces are equal if and only if they are equal as maps of sets because whenever you see equality of maps you only check at the set theoretic level okay so this is the subset of this, this is a subset of this okay.

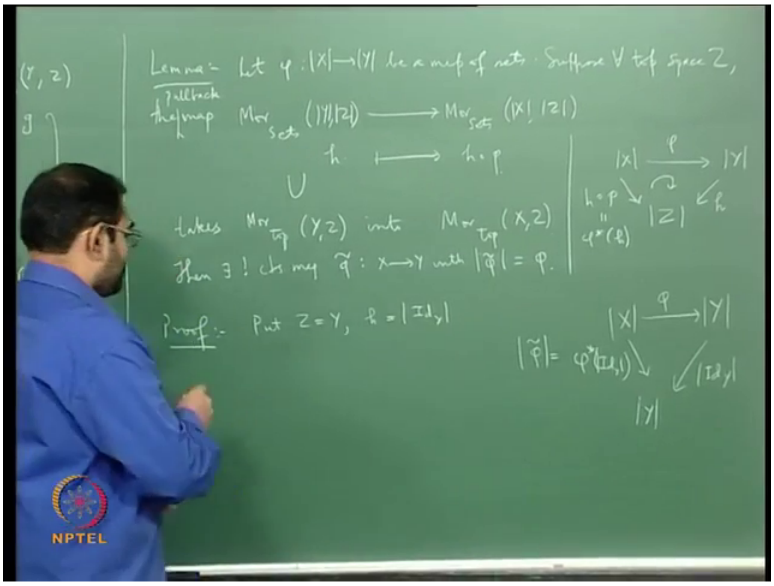
And now what I am going to do is I am going to go in the other direction so I am going to say suppose I have a suppose I have two topological spaces, so let me begin by saying the following thing if f is a continuous map from X to Y okay then the pull back of f induces a map from, it induces a map which takes continuous functions to continuous functions okay so g is a continuous function from Y to Z, the pull back of g this if you want I will call this is as f upper star g this is also a continuous function from X to Z okay.

So let me sum it up like this you start with the continuous function from X to Y then the pullback induced by that takes continuous function to continuous functions that is what it says okay now this is the model for defining what are morphisms okay so the rule is, this is a very general philosophy, you want to define a certain map as a morphism the rule is you specify that the pullback induced by that map must take good functions to the good functions okay.

So whenever you are in a situation where you know what good functions means so for example in our, in the case of topological spaces good functions are continuous functions if you are working with for example in our case of algebraic geometry good functions are regular functions okay so you can define a morphisms to be a map with the property that its pullback has the property of taking good functions to good functions, the pullback of good, so a map is a morphism if it is pullback takes good functions to good functions, this is the general philosophy.

And this works, it not only works in algebraic geometry, it works everywhere it works in analysis works in manifold theory, works universally okay, so keeping that in mind let us do the following thing let us make a, let us test this, so here is a remark, so let me make that remark here, clearly if f from X to Y is continuous so I am abbreviating continuous to cts then f upper star of a continuous map is equal to a continuous map okay that is what this say f upper star of a continuous map is again a continuous map.

(Refer Slide Time: 38:23)



And this is just if you want in this case it is just a result of the fact that you know a composition of continuous map is again continuous it is nothing more than that okay but the beautiful thing is that the converse to that statement holds and that is the power of the statement so here is a lemma, let g, okay let me write something else let me use phi from X to Y from mod X to mod Y be a map of sets okay.

Let phi from mod X to mod Y be a map of sets suppose for every topological space Z, the map morphism as sets from Y to Z so I should here maybe be very strict when I say morphism are sets I should probably put this parallel bars to tell you that I am just looking at morphisms of sets okay but you should have understood that even if I did not put it, I am anyway taking morphism in the category of sets so I am only worried about the underlying sets okay.

The map from, the pullback map, see so this is a pullback map from the morphisms of sets from Y to Z to morphisms of sets from X to Z which is given by h going to, you first apply so it is this h circle phi okay so the diagram is like this so here is, so here is mod the underlying set of X, the underlying set of the topological space X, this is the underlying set of topological space Y and then I have Z and this is underlying set of topological space Z.

And I have this map if you give me a h from Y to Z, I have a phi from mod Y to mod Z if you have a h set theoretic map, if you give me a set theoretic map phi from mod X to mod Y, I have this composition which is just first apply phi then apply h and its composition okay, this is just phi upper star of h, this is a pullback okay, see this pullback map okay that is the following has the following property I have here as subset, I have here the subset which is morphism as topological spaces from Y to Z okay mind you the way I am considering this as subset here is by associating a morphism topological spaces from Y to Z as a set theoretic map from Y to Z by forgetting the topological structure and forgetting the continuity okay.

That is how I am identifying this with the subset of that okay that is what this inclusions mean when I say this is a subset of this and this is a subset of this I am actually identifying a map with its mod right, so this morphism takes, the map takes this into morphism of topological spaces X, Z you see it is a property then there exists a unique continuous map phi tilde from X to Y with the associated map phi tilde set theoretic level to be equal to phi.

So this gives you a criterion as to when a map is continuous, suppose you give me two topological spaces X and Y and you give me a set theoretic map from X to Y that means you take give me a set theoretic map from the underlying topological space of X to the underlying topological space Y. How do you check that this set theoretic map is actually a continuous map? The power of statement is whenever it pulls back continuous functions to continuous functions then it is automatically a continuous function okay.

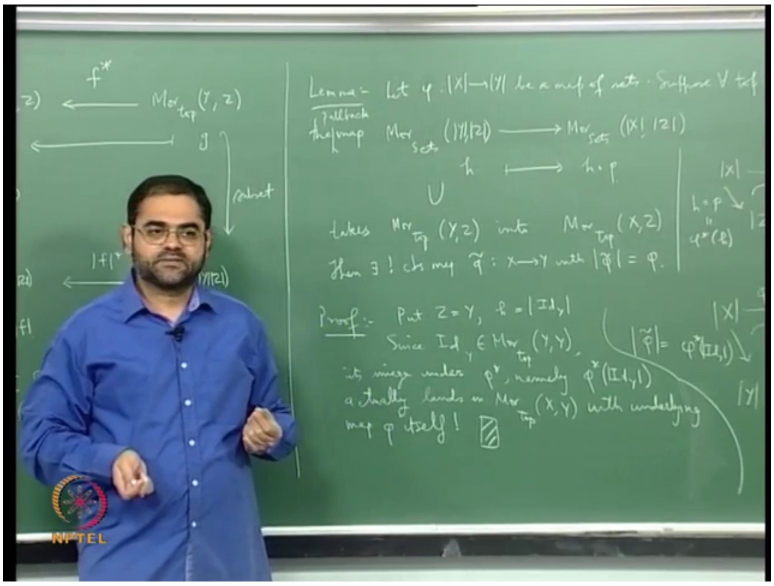
So this tells you that this philosophy of defining a morphism to be the, to be a map which pulls back good functions to good functions is the right definition it works even to define the continuity of a map between two topological spaces okay so this is the, it is a very simple observation but it is a beautiful philosophy to work with and it is exactly this kind of philosophy that we are going to use to define regular morphism of variety okay.

We are going to define morphism of varieties exactly in this field and well so what is the proof? See the proof is, the proof is very very simple, the proof is well you see put Z equal to Y okay and you put h is equal to the set theoretic map connected associated with the identity map on Y okay so you see the diagram is that the diagram is as follows I have X, I have Y okay this is underlying set of X, this is underlying set of Y okay and here is my phi set theoretic map and then well the Z, I am going to put is just equal to Y so this Z is just Y okay.

So the underlying set of Y is same as underlying set of Z and then the map h that I am going to take from the underlying set of Y to the underlying set of Z is the identity map on Y okay and the claim is that when I do this what I will get here is phi upper star of identity map on Y okay and what is the claim? The claim is that, notice the identity of Y, the identity map on Y is of course a continuous map, as a topological, as a map of topological spaces from Y to Y the identity map on a topological space is always a continuous map because after all the inverse image of and a set that is open is a set itself and that it is anyway open.

So inverse image of open sets are open therefore the identity map on a topological space is always continuous okay therefore this Id Y identity map on Y is a continuous map of topological spaces and so you know and what is the, so what is a conclusion that the pullback by phi takes continuous maps to continuous maps so it tells you that phi star of identity Y is will give you a map from the underlying space of X to underlying space of Y and that map is actually a continuous map.

(Refer Slide Time: 46:28)



But what is this map? This map is a map which set theoretically is the same as phi so you know if I set this as if I call this as phi tilde okay if I, so all I am trying to say is that phi tilde is actually this map okay so then so the proof is very very clear since identity map on Y belongs to it is a topological map, continuous map from Y to Y, its image under phi upper star namely phi upper star of Id Y actually lands in the set of continuous maps from X to Y with underlying map phi itself and that is the proof.

So the moral of the story is that a set theoretic map between two topological spaces is continuous if and only if it pulls back continuous functions to continuous functions okay a set theoretic map phi of topological spaces is a continuous map if and only if it pulls back continuous functions to continuous functions okay, so more generally you this is the method define morphism, a morphism is a map such that under pull back it takes good functions to good functions and now you know what I am going to do how am I going to define a morphism between two varieties.

Two varieties can be affine quasi-affine does not matter, so what am I going to do, I am going to do the following I am just going to say a map is a morphism, a morphism between two varieties is a set theoretic map which pulls back good functions to good functions and what are good functions of varieties? They are regular functions, so I am going to just say a morphism between varieties is a map of varieties, a set theoretic map of varieties that pulls back a regular functions to regular functions alright.

And the beautiful thing is that there is a small glitch in this proof, in this definition the glitch is that I have to specify for avoiding certain pathologies that the map that I start with is not just a map of sets I should already specify that the map is already continuous okay so you see here what we did was the base structure, the base category was the category of sets and the topological and the category with more structure was the topological category.

So here you had, these were just sets and these were sets with topological structure okay but if you go to varieties it is even more, it is one step even more because as far as varieties are concerned they not only have the topology namely the zariski topology but it is not just a topology that describes all the geometry of varieties, the geometry of varieties far more higher than just the topology of varieties okay.

To study the geometry of varieties the first step is to study the topology of varieties so when you look at a variety you should look at it at three levels, the variety as a set is the base level you are looking at it at the category of sets, then the next level is you look at the variety as a topological space in which case you are looking at it at the zariski topology okay, then there is a third level which is the variety as a variety itself okay the variety as a variety itself is something more it is not just a topological space okay.

And therefore if you will, so the point is that in that situation the philosophy, this philosophy will work you will have to replace this forgetful functor from topological spaces to sets to the next higher level which is from varieties topological spaces so what you must assume at the base is not just a map of sets which you already assume something that is a map at the level of topological spaces.

So what is the correct definition of a morphism of varieties, it is a morphism of the underlying topological spaces namely it is a continuous map which under pull back takes regular functions to regular functions, that is the definition okay so with that definition you can define what a morphism of variety is and all this, in all the previous lectures whenever I said isomorphism of varieties I meant a morphism like this okay namely a continuous map that pulls back regular function to regular functions with an inverse which also is a morphism namely which also pulls back regular function.

So that is the very easy definition of what an isomorphism varieties has to be, it has to be a bijective map and both the forward map and the reverse map should pull back regular functions to regular functions okay that is what it means to say that map is an isomorphism of varieties okay and I should tell you with a word of caution that there are many categories in which usual categories in which a bijective morphism is also an isomorphism that is if you have morphism which is bijective in the inverse map is also a morphism okay but it is not true with varieties unfortunately okay.

So for example if you have a bijective linear map then the inverse is automatically a linear map so it is an isomorphism, if you have a bijective homomorphism of rings the inverse map is automatically a homomorphism of rings so it is an isomorphism of rings okay but not all bijective morphisms are isomorphisms for example as I told you I think one of these, I do not know that I told you but if you take the map from the real line, if you look at maps from real, the real line to the real line which are with the property that they are not just continuous but with the property that they are differentiable then you know if you take a bijective differential map need not be a differentiable isomorphism it is inversely need not be differential.

For example if you take X going to X cube then that map is bijective differentiable map but the inverse map is X to the 1 by 3 is not derivable at the origin okay so bijective morphism never it is not necessary that it is an isomorphism namely the inverse map need not be a morphism okay and it is also true with varieties that a bijective morphism need not be an isomorphism okay so that is the word of caution alright, so I will stop here and continue next in the next lecture.