Basic Algebraic Geometry
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Lecture-16
What is a Global Regular Function on a Quasi-Affine Variety?

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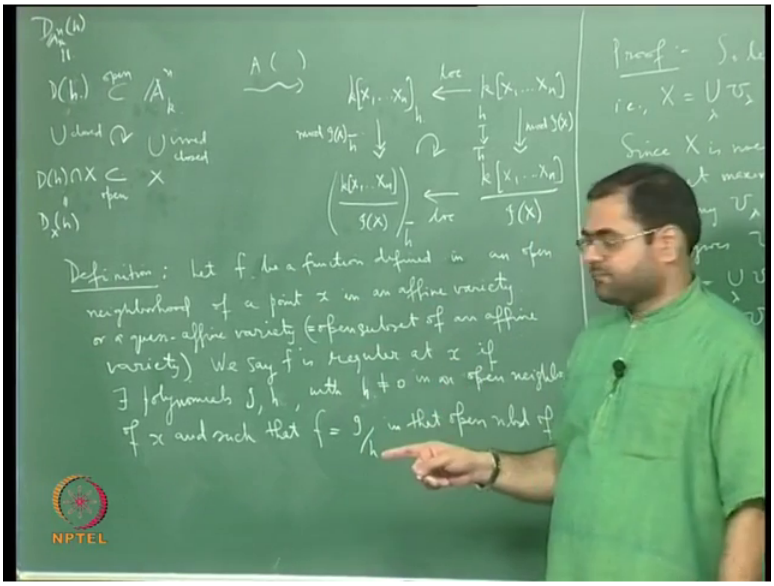

Okay so let us continue with our discussion which is, whose aim is to define what are regular function at a point is, okay. So let me again go back to this definition, the definition says that you know you have function f with values in k, the base field and it is defined in an open set containing the points small x, of course that open set could be, that open set is being considered as an open set in the affine variety or quasi-affine variety in which x exist okay.

And we say the function f is regular at that point if it is, if it can be written as a quotient of polynomials okay that is there are two polynomials whose quotient will define a function, a quotient of polynomials will define a function into the field, the base field, the scalar valued function wherever the denominator polynomial does not vanish that is actually, so this function of the form g mod h where g and h are polynomials will always define a function on D h, on this basic open set D h, which you know in its own right is actually an affine variety.

And the fact is that you will just have to find a h such that D h intersection X capital X contains small x and which is possible because it is basically because any open set is a finite union of such D h's okay for various h's, right, which is a fact we have already seen. In fact any open set here is the compliment of the closed set and that closed set is given by the zero set of an ideal and that ideal is finitely generated.

And therefore that closed set is given by the intersection of the zero sets of the generators of the ideal and therefore its compliment is given by the corresponding union of basic open sets, namely the loci, the open sets where the corresponding generators of the ideal does not vanish individually, it is the union of that, okay that is how every open set is finite union of basic open sets and therefore this is my definition.

And this definition seems to be correct for, if you look at basic open set, a good function on this is of that form okay, but the point is that whenever you make a point wise definition what you are actually doing is you are gluing okay and the problem with gluing is that it can produce new objects, new if you glue objects of a certain type we can, it can produce a completely new object okay.

So if you glue functions of locally you might get something which is very different from what an ordinary function is, for example if you glue topological spaces in a funny way you might get a topological space that you, that looks very different from the ones that you started with okay, you are going to see later that if you glue affine spaces finitely many affine spaces you can get a projective space okay.

So for example if I take two copies of the you know if I take for example the usual topology and take two copies of the unit disc on the real plane and you know if I make them look like two open two hemisphere of a sphere and then glue them like this, I will get a sphere okay and the point is that the new topological space I have got which is a sphere is very very special because it is compact which is not a property that is shared by either of the two open discs I started out to glue it.

So the problem is that gluing of objects in mathematics can produce new objects which have properties that are completely different okay, the same thing happens also when you try to glue functions and what I am saying is that whenever you make a definition point wise you are actually doing some gluing then for all you know you might ask the following thing can happen if I look at all the functions on affine space itself which are regular at every point they could be very well different from polynomials, what is the guarantee that they are no different from polynomials.

But the answer is that you are not going to get anything new, okay and that is what, that is a surprise and it is a pleasant surprise and that is what we are going to prove okay so, so let me try to explain that, so just to stress this gluing business, let me write out, so let me make a few you know, let me make few notational conventions.

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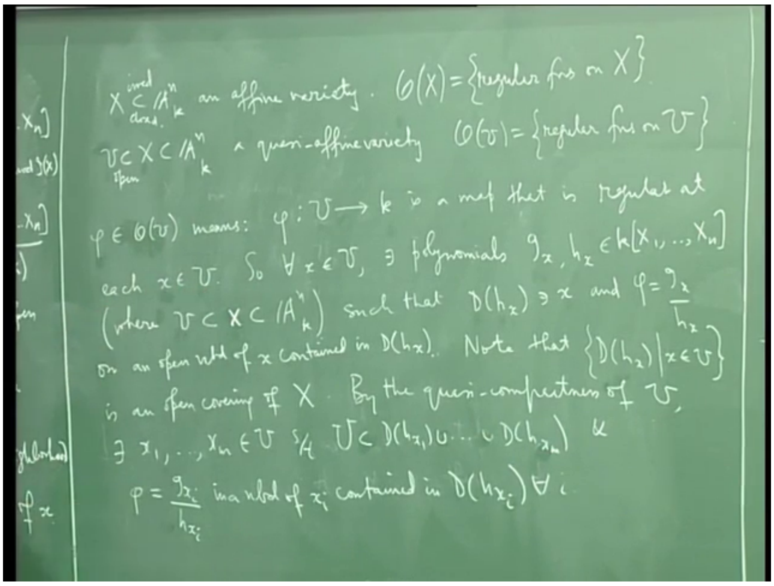

So what I am going to do is, X in An k an affine variety or a quasi-affine variety, so let me do the following thing, let me do it one by one, let X in An be an affine variety. I define O of X to be the regular functions the set of regular functions on X, okay and so you see this is the new notation I am using, this is a calligraphic O okay for regular functions.

Because as yet I do not know anything about a global regular function except that it is locally a quotient of polynomials okay and certainly I do not expect it to be anything, I cannot expect it to be anything unless I prove something about it okay, so and similarly you know if U inside X inside An so this is irreducible closed and then U is an open inside this irreducible closed and U is quasi-affine variety.

So a quasi-affine variety is supposed to be just an subset of an affine variety then again O of U is defined to be regular functions on U, okay and this is, these are global definitions okay and how do these global definitions come they come from local definitions point-wise definitions okay and so let me write down what is a definition of, so for example if you.

Suppose I write phi belongs to O U means what okay, phi is a map from U to k is a map that is regular at each small x in U okay. I am of course I am writing this for U, but you know I can as well write it for the same thing will also work for X where X is actually an affine variety okay, but I am taking a quasi-affine variety right.

So what is a regular function on a quasi-affine variety it is a function which is values in the field which is just a map but with a property that it is regular at every point that means what, so for every x in U there exist polynomials gx, hx in k of x1 etc xn where of course your U is being considered inside x which is being considered in some in An okay, your U is an open subset of X, capital X, it a quasi-affine variety and it is being considered in some affine space, it is an irreducible closed subset of some affine space.

And there are polynomials such that D of hx contains x and phi is the same as gx by hx on D hx, this is the, you see this is the definition, on D hx or rather I should say D hx may not be all of on an open neighborhood of small x contained in D of hx, this is what it means for give me a point x, I can the function phi in an open neighborhood of that small points small x is the same function that I would get if I evaluate quotient of polynomials with the denominator polynomial out-vanishing and the polynomials being taken in the appropriate number of variables in which dimensional affine space you are considering U and x okay.

So you see this definition is, you see this is the problem with this definition, the problem with this definition is that it is so arbitrary in a certain sense see I could have you know the same affine variety x could sit in so many affine spaces, see if I, if x for example is a two plane then x could be A2 or x could be 2-plane in A3 maybe the XY plane or it could be the 2 plane in some An it could be anywhere.

And depending upon on where it is I have to take polynomials in as many variables and then I will take a quotient so you see it is a there is so much arbitrary in a sense definition and that sometimes is a little scary okay but the point is that you have this for every point x so now you see note that the D hx where x is in x where x is in U is an open covering of capital X right.

Because I said for every point small x, I am getting this hx such that their affine open set D hx contains x okay and this affine open set basic affine open set D hx is considered in this affine space that affine space of that dimension in which is that number of variables in which, number of variables you have taken the polynomial hx, okay, and of course since I say this such a hx exist for every x in U it means that all these D hx's they cover U okay.

But then what do we know about the zariski topology? We know that it is quasi-compact therefore what it will tell you is that out of this collection of D hx's I can just be content with having only finitely many okay so by the quasi-compactness of U okay, U is afterall a subset of x which is in turn a subset of An and then you know when you take subsets and take the induced topology the noetherianess goes down, it is hereditary so U becomes a noetherian topological space and you know a noetherian topology is quasi-compact as we saw in the last lecture.

So U is quasi-compact and that means every open cover has a finite sub cover so what it means is that there exists x1, etc, xm in U such that U is D hx1 intersection, sorry union, U is contained in the union of all these D hxi hxn and well of course phi restricted to phi, phi is g xi by hxi in a neighborhood of xi contained in D of hxi for every i, this is what it means.

So you see what it means, so what it means when you say that you are having a regular function what it means is it you are actually taking quotients of polynomials and finitely many such quotients such that these quotients they agree on the intersections you see if you take the intersection of D hxi with D hxj then in that intersection phi will locally be both gxi by hxi and it will also be equal to gxj by hxj.

So what you are saying is, you are saying that a general regular function is simply gotten by taking finitely many quotients of polynomials with the property that these quotients of polynomials agree the functions that they define upon evaluation agree on these intersections okay and this is exactly what gluing is, gluing is you take functions locally and then so many functions with a certain property glue to give a bigger function if they all give on the intersections the functions should agree.

So what you are saying is just take finitely many quotients of polynomials such that the locus of the non-vanishing of the denominator polynomials covers your space that is U in this case and such that these quotients wherever the loci where the denominator polynomials do not vanish intersect the quotients evaluate to the same functions okay such a, this is what a regular function on U is, okay.

It is got by locally, so what you, so global regular function on open set is actually gotten by gluing finitely many quotients of polynomials so it is a gluing process and now so this is how it is for U it is a same definition instead of U, I can put capital X also same definition works for any subset in fact of affine space but particular in fact instead of U, I could have put any subset of affine space okay.

And I could have said regular functions on that set but we are interested only in regular functions on either open sets or on closed sets either they should be regular functions on irreducible open sets they are open subsets that is either they should be, we are interested in functions on either irreducible closed subsets that is functions on affine sub varieties or we are interested in functions on open subsets of such affine sub varieties okay.

So U is an open subset of x and of course whenever I am considering a subset I am certainly not looking at empty set so U is a non-empty open subset and mind you a non-empty open subset is both irreducible and dense okay because capital X is both irreducible and dense, I mean because capital X is irreducible okay. So you see what we have defined as regular functions is something that is very strange it is, these are gotten by gluing finitely many quotients of polynomials.

So now I can ask what will I get if I put O of, if apply this O to An itself what will I get, if I apply O to D h what will I get, if I apply O to x what will I get, if I apply O to D h intersect x what will I get? The answer is very beautiful, the answer is you will get exactly what you will get if you apply A okay that is the beautiful thing okay and the fact is that is the sophisticated way of saying that is that describes that is why all the four are actually affine varieties okay.

So in more general algebraic geometry you can define A the A for objects which are affine objects okay and then you can define the O for any general object and then the theorem is that the general object is an affine object if and only if the O is the same as the A okay and that is exactly what is happening here alright. So I mean that is saying it in very loose terms okay to understand the exact import of that statement you should study scheme theory which should be a second course in algebraic geometry okay.

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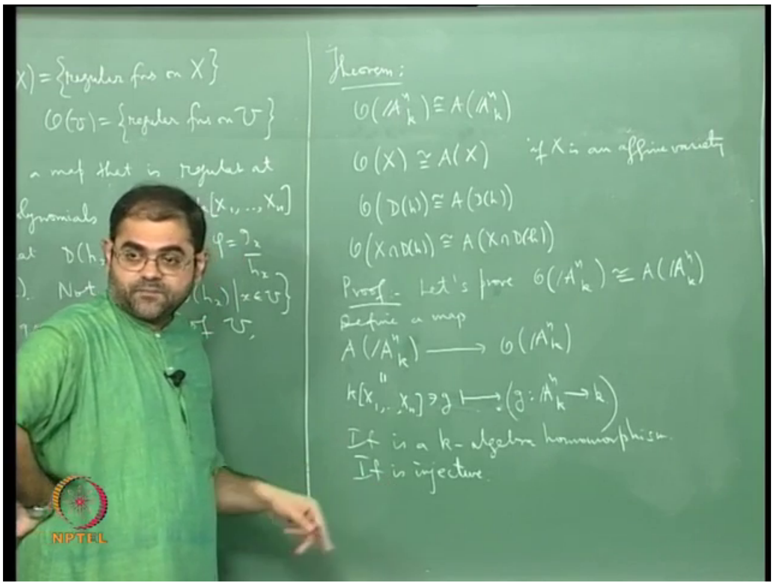

But nevertheless does not do any harm in while stating it here so if at all you go ahead to study scheme theory you can come back and try to remember this statement okay so let me make that statement, let me make this remarkable statement. So here is theorem O of An is equal to A of is isomorphic I should say okay let me put isomorphic.

Okay so here is the theorem, the theorem is that what you define as regular functions are going to give back exactly these functions which are given for, which are given when you apply A to these objects here affine space or affine varieties or basic open subsets of affine space or basic open subsets of affine varieties okay this is the statement right, so the technique of proof is literally the same it is a basically it is a technique in commutative algebra.

So what I will do is I will, let me first prove the first one and in fact you can see that you can deduce the remaining if you little careful and just apply the same philosophy as proof of the first okay. So what I will do is let me prove let us prove O of An is isomorphic to A of An let us prove this alright let us prove this, so how does one do it, so what I will do is I will do the following thing.

So I will define a map, define a map A of An to O of An by very very simple map take the A of An is just the polynomial ring in n variables over K and simply send it, send a polynomial g to the function g from An to k so it is a very very simple map so what you do is take an element here what is it? It is a polynomial in n variables over k and a polynomial in n variables over k is a function.

It is a function from An to k by evaluation, you can evaluate the polynomial at every point and that is in certainly a regular function because a regular function is something that is locally given by quotients of polynomials and this is globally given by a single polynomial and the single polynomial g can be written as g by 1 if you want, you can think of g as g by 1 and the locus where 1 does not vanishes everything okay.

And therefore it is also a regular function so all I am just trying to say is that every polynomial is certainly a regular function there is no doubt about it, a regular function is something that locally look like a quotient of polynomials but something that is globally a polynomial is also a regular function because it is a polynomial divided by 1 if you want and 1 is a constant polynomial which always makes sense okay.

Now the fact is that you see just like all the polynomials form a ring you can add two polynomials, you can multiply them and then they form a vector space over k and then, the ring of polynomials is a k-algebra okay, it is a finitely generated k-algebra in fact it is a free polynomial algebra in so many variables, in the same way the ring of regular functions is also, the set of regular function is also a ring that is the first thing you have to realize.

Because you see you take some of two regular functions okay the sum will also be regular because if you take local, regular function locally given by quotient of two polynomials and if you take two such regular functions and add in a suitable neighborhood the corresponding quotients of polynomials the sum of quotients of two polynomials is again a quotient of two polynomials okay in the correct neighborhood where the denominator does not vanish and the product of two quotients of polynomials is again a quotient of polynomials right.

So the moral of the story is that when I define this O of something whatever it is, the regular functions on that thing that is a ring, in fact that is a k-algebra, it has addition, it has multiplication, it is a vector space over k you can easily see that if you take a regular function multiplied by a scalar the result is again a regular function because locally you are just multiplying the numerator polynomial by that scalar okay in the expression locally as a quotient of polynomials for that function okay.

And it is also clear that that is a k-algebra and so on. So these isomorphism that I have written here, they are not just isomorphism of rings and they are isomorphism of k-algebras, okay they are isomorphism, they are ring isomorphism, they are isomorphism of vector spaces also okay that means scalars go to scalars, scalar lambda in k thought of as a constant polynomial lambda goes to the same constant function lambda thought of as a regular function okay.

So this is overall k-algebra homomorphism okay, so this map that I have written here is actually it is very easy to see that it is obvious to see that it is a ring homomorphism because g you know f g1 plus g2 what g goes to and what I mean this this association will preserve addition, multiplication it will, it is k linear and all that so this is a ring homomorphism it is a k-algebra homomorphism okay.

And then the fact I want to make is that I will have to prove two things, I will have to prove that it is injective, I have to prove it is surjective, if I prove that then I get that this is an isomorphism of k-algebras which is what I want okay so it is, so let me write that it is a k-algebra homomorphism, that is obvious, no doubt about that, then how do you show it is injective? It is injective, why is it injective? Well if you take two polynomials g1 and g2 which as functions are different.

It means that there is a point in An where the values of g1 and g2 are different that means the difference polynomial g1 minus g2 does not vanish at that point okay and that should tell you that g1 and g2 cannot be the same okay, so it is very clear that it is injective, the injectivity is just very severe okay, in other words you now I am, another way is to say it is that I am saying that if you take polynomial and if I evaluate it as a function and suppose as a function it is a zero map as a function then the polynomial has to be the zero polynomial okay.

So this is something that is obvious probably it just requires the fact that the field k is infinite okay which is true because filed is an algebraically closed field and an algebraically closed field is infinite okay, of course the problem is over finite fields you can always find polynomials which are non-zero polynomials but which evaluated as functions end up being the zero function okay that can happen for finite fields, but small k is not a finite field, it is an algebraically closed field and an algebraically closed field is always infinite.

And for an infinite field if you have polynomial which if it is, if upon evaluation it is zero map then the polynomial has to be zero polynomial okay that is the statement that it is injective okay but what is really crucial is the fact that it is surjective, so that requires a little bit of proof so let me do that, so that is why a little bit of commutative algebra will come in and you will recognize what is going on if you have done a course in, an earlier course in commutative algebra and where you have proved that you know the, you proved the quasi compactness of the zariski topology when you take the prime spectrum of a ring okay.

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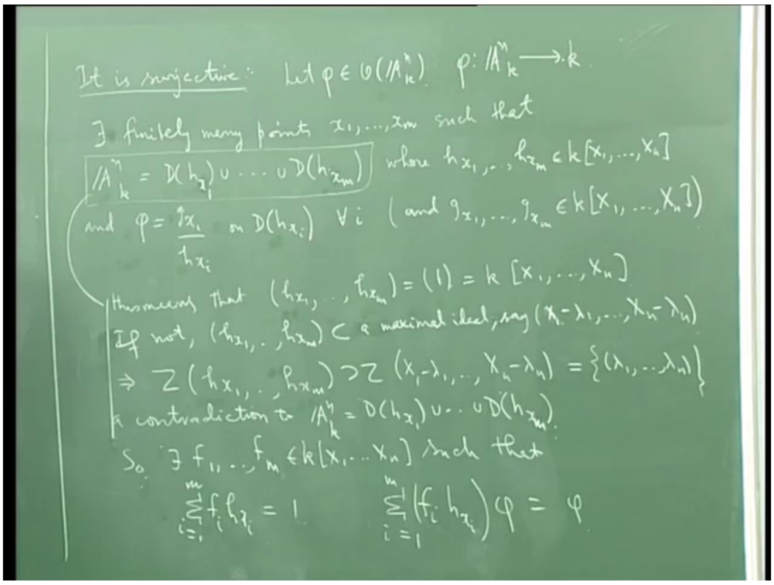

So let me do this, it is surjective, so what I will do is I will start with, so let phi be a regular function okay take a regular function right so phi is map from An to k and by our definition of what a regular function is there exist finitely many points okay and such that the union is a whole affine space and such that phi is quotient of polynomials in this sense okay so let me write that down.

There exists points same such that An is D h1, h sub x1, D sub hxm okay where hx1 through hxm are polynomials in n variables and phi is equal to gxi by hxi on D hxi for every i and of course gx1 through gxm they are also polynomials in n variables okay this is from the definition of what are regular functions okay, now you see the first thing is that this has a meaning in terms of commutative algebra, the meaning that it has in terms of commutative algebra is that the ideal generated by the hxi is the unit ideal okay.

So this means commutative algebraically, so you know it is a translation, it is a translation of all these geometric facts into commutative algebra okay and the extracts of information from commutative algebra so the fact that A is union of all this means that the ideal generated by hx1 etc hxm is 1 is ideal generated by 1, namely it is a whole polynomial okay, this is the first thing that needs to be noted.

And why is this true? Why this is true is because well if this ideal is not the unit ideal then it is a proper ideal and then we know that every proper ideal is contained in a maximal ideal okay and that you know the maximal ideal is corresponds to a point okay so what will happen is that if you, that point will have some co-ordinates okay but that point will be here, it is a point of An so it has to be in one of these so that means that at least one of the hxi's does not vanish at that point but that will contradict the fact that this is contained in the ideal of that point okay.

So this is exactly what I am going to prove if not hx1 ideal generated by hx1 etc hxm is contained in a maximal ideal which is of the form say x1 minus lambda 1 etc x, n minus lambda n okay this will tell you that you see it will tell you that, so you know if I apply the Z you know if i1 is contained in i2 then Z of i1 contains Z of i2 where z is the operation that associates to an ideal zero set.

So this will tell you that Z of hx1 etc hxm contains Z of this maximal ideal which is actually the point which is actually simply the point singleton point lambda 1 through lambda n okay alright that means what this tells you is that this point is you know it tell you that this point is a common, this point is a common zero of all these h's okay and if this point is a common zero of all these h's that contradicts the statement because this point is here okay and by definition every point you has to be contained in some locus where a certain h does not vanish.

But then you are saying that but I have been able to find a point which is not in any of these okay it is a contradiction, to An is equal to D h1, D hx1 union D hxm okay right. So the moral of the story is that indeed this condition, the fact that the affine space is a finite union of certain basic affine open sets means that the equations corresponding to those basic affine open sets, those polynomials actually generate the unit ideal okay so what all these will tell you is that these generates the unit ideal so what it will tell you is there exist you know, let me give me some other names okay.

So let me use f so there are f1 etc fm polynomials in n variables such that sigma fi hxi is equal to 1, I get this okay, in other words the 1 is in ideal generated by the hi's so 1 should be generated by a ring linear combination of the hi's and that is what I have written here, these fi's are L ring elements, the co-efficient from the ring okay so of course I is equal to 1 to m okay.

Now you see now the point is that I will have to show that let us go back to what I started with and what I want to show I started with regular function on An okay and I am just trying to prove surjectivity of this map, so I am starting with something here, regular function An and I am trying to find that, show that comes from here so I will have to cook up a polynomial which is equal to that function okay.

So I will have to use this information to cook up a polynomial g such that if I consider g as a map I get this phi that is what I will have to do okay and that is pretty easy to see, so the trick is you multiply both sides by phi okay, multiply both side by phi so what I will, so let me do that so sigma i equal to 1 to m fi hxi into phi is equal to phi okay I get this just by multiplying both sides by phi and then I use the fact that you know you see phi is chi xi by hxi on D hxi but if I cross multiply I will get phi times hxi equal to gxi not only on D xi I will get it everywhere okay.

So I am using the following fact if two polynomials are equal on an open set if two polynomials treated as functions they coincide on a non-empty open set if the two functions define by two polynomials if they coincide as a function on a non-empty open set then the polynomials are equal okay so what this equation tells me is that on this non-empty open set D hxi the function phi times hxi and gxi they are the same okay and I want to say from that that they are the same phi times hxi and gxi are the same if you want as regular functions on the whole space.

So I am using, so in fact let me restate it more correctly I am saying that if you have two regular functions which are, which coincides on an open set then they have to coincide on the whole space if two regular functions coincide on a non-empty open set then they coincide on the whole space the reason is topological, the reason is because regular functions are continuous and open subsets are dense, it is very simple, the reason is topological okay.

So if two, so all I am saying is that if I have two regular functions, okay and if they, the fact that they coincide on an open set means that they the difference regular function is zero on an open set okay alright but the set of points where a function is zero is a closed subset because the function is continuous therefore this open subset is contained in a closed set but the open subset is dense which will tell you that the open subset that, it will tell you that these two regular functions are the same everywhere okay.

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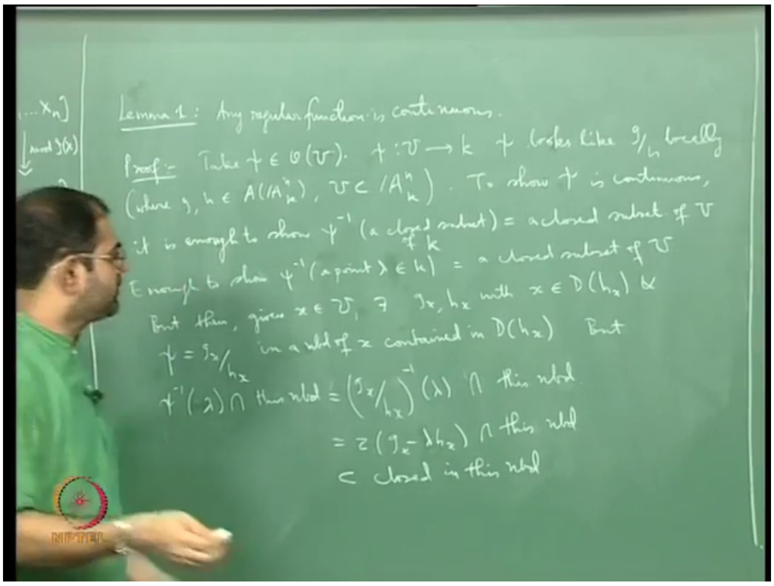

So let me write that down so that you know, so let me rub this side I still want this side of the board, so let me write that down so that it becomes clearer to you so what I do is here is the lemma, lemma 1 any regular function is continuous so here is the lemma, any regular function is continuous so you see, so what is the proof? So the proof is take psi in O of U if you want okay or let me take O of something, okay so let take O of U right.

Then so psi is actually a map from U to k psi looks like g mod h locally okay where you know g and h are polynomials, they are polynomials in the right number of variables and you are considering U inside that affine space okay and then you see to show psi is continuous, it is enough to show psi inverse of a closed subset is equal to a closed subset okay alright so psi inverse of a closed subset of here this is a closed subset of k and is a closed subset of U, this is what I will have to show alright.

But then what is the close? You know the zariski topology on k which is A1, the zariski topology is just the compliment finite topology namely the only closed set of finitely many points so to, so psi inverse of a closed set is psi inverse of finitely many points and to show that that is and psi inverse behaves well with respect to unions psi inverse of a union of sets is the union of psi inverse of those sets, psi inverse behave well with respect to the operation of taking unions.

So to show that psi inverse of a finite set of points is closed it is enough to show that psi inverse of a single point is closed okay so enough to show psi inverse of a point lambda in k is a closed set, a closed subset of U it is what I will have to show okay so but you see, but then given x in u there exists gx hx with x belonging to D of hx and psi is equal to gx by hx okay so what this will tell you is that you know it will tell you that, so in a neighborhood of x contained in D hx, this is the definition, local definition, alright.

But then psi inverse of lambda intersection this neighborhood is just gx mod hx inverse lambda in intersection this neighborhood and this is just equal to Z of, so gx mod hx has to be lambda so gx has to be lambda hx so gx minus lambda hx has to be 0 so it translates to Z of gx minus lambda hx intersection this neighborhood which is of course closed in this neighborhood and you are done.

Because you know, to check that a subset of a topological space is closed it is enough to check it on an open cover okay so what I have done is I have for every neighborhood, every point I have shown that psi inverse lambda intersection that neighborhood is a closed subset in that neighborhood okay and if I vary x I get an open cover and for each of the sets in the open cover I verified that the inverse image intersected with that set of open cover is a closed subset okay.

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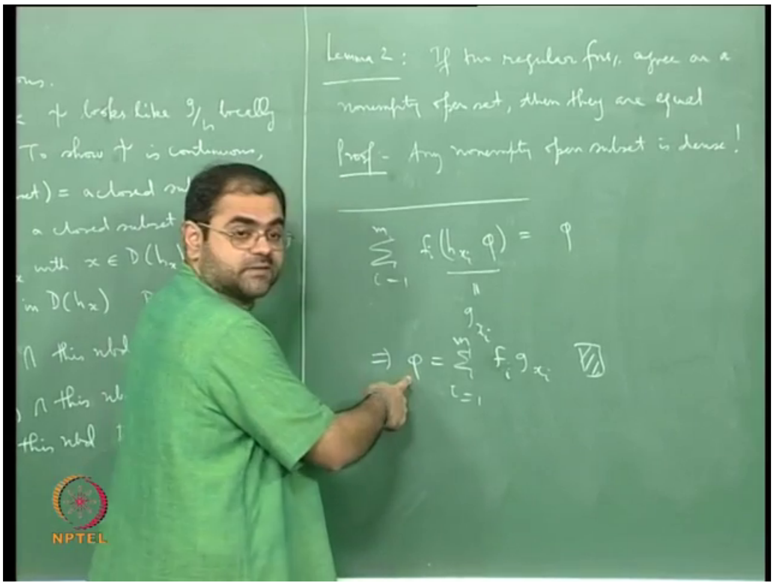

So this proves the lemma, now let me prove another lemma, which is the lemma that I want to apply there so of course you know I am here I am looking at regular function on either on a affine variety or quasi-affine variety mind you okay so here is lemma 2, in fact yeah if two regular functions agree on a non-empty open subset then they are equal, if you have two regular function they are equal or non-empty open subset then they are equal everywhere proof is two lines, proof is trivial, two functions agree on a non-empty open subset means the difference function is zero on a non-empty open subset.

But any non-empty open subset is dense okay for the zariski topology, so you have function that is, you have a condense function that is zero on a dense open set so it has to be zero everywhere so it is obvious, so proof is any non-empty open subset is dense and it says and a function which is zero on a dense, continuous function which is zero on a dense open subset has to be zero everywhere simple topology.

So if you apply these two lemma I mean I need this lemma now you look at now you look at this equation I have written here, in this equation you know if I calculate if I look at the product hxi times phi, the product hxi times phi is gxi on D hxi okay but hxi times phi is also a regular function so product of two regular functions and gxi is also a regular function because you already seen every polynomial is a regular function.

So you are saying the regular function phi times hxi is equal to the regular function gxi on the non-empty open subset D hxi therefore by this lambda x the same everywhere, therefore for hxi times phi I can put gxi okay so what I will get is I will get sigma i equal to 1 to m fi gxi equal to phi and that tells you that phi is actually that polynomial given by the (())(52:29) that is the polynomial that I wanted okay.

So now I will let me continue, so let me rewrite that sigma i equal to 1 to m fi hxi phi is equal to phi this is equality as regular functions okay but hxi times phi is the same as gxi because hxi times phi is equal to gxi on D hxi and therefore on whole affine space because of this lemma, therefore I can replace hxi times phi as gxi everywhere so what this will tell you is that it will tell you that phi is actually sigma i equal to 1 to m fi gxi and that is the end of the proof. I have proved that the regular function is actually the function that you get by evaluating a polynomial what is that polynomial? g is the polynomial okay, that is the proof.

So you see there is a tricky bit of commutative algebra coming inside the proof okay, so the moral of the story is there is no difference between the ring of regular functions on affine space and polynomials in affine space okay. So every regular function on affine space which is define locally by gluing polynomials, by gluing quotients of polynomials if you take a regular function on the whole affine space which have gotten locally by gluing quotients of polynomials, the resulting regular function is actually a polynomial, in other, what that means to say is that it is the function that is gotten by evaluation of single polynomial.

And that this is the that fact written in the form of an equation okay so you do not get anything new okay now you can use the same technique of proof to prove the other statements okay, all the time you will use this fact that whenever the corresponding space is a union, is contained in the union of finitely many basic open affines then the corresponding equations, polynomials that occur in those, that define those basic opens the ideal generated by that is (())(54:45) that is the key.

And then from that by applying this lemma you can get this, you can get the proof for all the other cases okay so in the case of, so in all these cases your regular functions agree with the ring of functions that we defined, the co-ordinate ring of functions that we defined okay so I will stop here.