# Basic Algebraic Geometry

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**Lecture-15**

**Quasi-Compactness in the Zariski Topology;**
**Regularity of a Function at a Point of an Affine Variety**

Okay so let us continue with our discussion so you know we are at the stage of trying to understand the meaning of open sets in the Zariski Topology and I told you in the last class that an open set is always built up of so called basic as a union of basic open subsets and these basic open subsets are subsets which are given by the locus of non-vanishing of a single polynomial okay they are called the basic open subsets.

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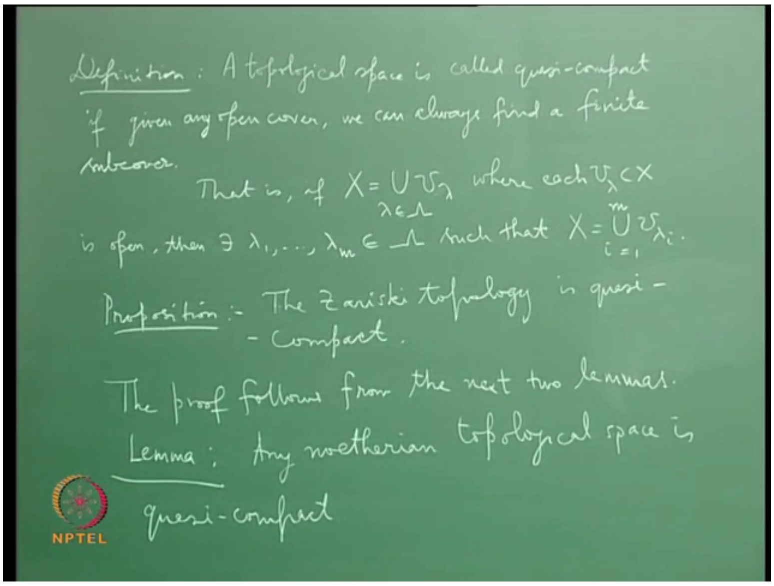


Now you see there are, there is a very important property about open sets in the zariski topology which is what stating which I ended the last lecture and that was quasi-compactness okay, so let me begin from there okay. So here is the definition, a topological space is called quasi-compact if given an open cover, we can always find a finite sub cover okay.

So this is the definition of quasi-compactness, in fact see if you general, if you study general topology this is the definition of compactness okay in general topology, normally we, this is the definition we give for compactness of the space, the compactness is if you give me a collection of open subsets whose union equals the space then from that collection you can extract a finite sub-collection whose union will also be equal to the whole space, this is what it means when we say that every open cover you can always find a finite sub cover okay.

And this should happen for any open cover, when I say an of course it means any. So if you want maybe I will say, I will modify this an to any which is what I mean so what does that mean, so if you write it in symbols you know that is, if X equal to union of U of alpha where, or let me write U lambda, lambda in capital lambda where each U lambda inside x is open then there exists lambda 1 through lambda m finitely many indices from the set lambda such that x is just the union of these corresponding U lambda.

So X is union U lambda i, i equal to 1 to m and we say that the sub collection U lambda 1 through U lambda m is a finite sub cover of the original cover which consists of the collection of all the U lambdas okay now of course each U lambda is an open subset okay, now this is the usually this is the definition of compactness in a topological space okay this is the usual definition of compactness but if you have studied topology you will always find that just compactness alone is not a very good property, usually you should also have compactness with hausdorffness.

And the most good type of spaces are spaces which is locally compact hausdorff okay these are the nice spaces on which you can do (())(06:00) good topology okay. So but in algebraic geometry we, especially in connection with the zariski topology, we do not use the word compact okay the reason is, I will tell you the reason later but the more important thing is that we use the word quasi-compact okay.

And so the first thing that I want you to notice about this definition is that this is a definition of what a compact topological spaces in general but in the zariski topology whenever you are in algebraic geometry you always use only the word quasi-compact, do not use the word compact okay and of course there should, the technical reason for that is that compactness transfers something else and even there this something else is not called compactness it is given a different name it is called properness or completeness okay.

So the word compact itself is kind of not very suitable for algebraic geometry okay and another, so you know you must remember that this quasi is especially in the case of the Zariski Topology. Now what I want to tell you is that if you now take k to be an algebraically closed field and you take affine space over k and you look at the Zariski topology then the topology itself is quasi compact.

I mean the zariski topology is quasi-compact for free, it is God given so it is nothing special okay but you can remember that when we study general topology, compactness is a very special thing okay so you know in Euclidian space a subset is compact if and only if it is closed and bounded okay so if you take subset of Euclidian space which does not have a boundary point then it cannot be compact, if it is not closed then it cannot be compact so compactness means so many things when we study the Euclidian topology okay that is Rn okay n dimensional real space with the usual topology.

But as far as Zariski Topology is concerned this compactness in the sense of every open cover having a finite sub cover that comes for free okay so that is why we reserve the word quasi-compact for that so this is the last statement I made last time so I should say if you want proposition the zariski topology is quasi-compact okay, it is quasi-compact, so let me write it okay.

Now why is this happening so the reason is actually it is because of the noetherianness okay, it is actually because of the noetherianness which I will explain as follows so what I am going to do is first is I am going to say that this follows this the proof of the proposition follows from a couple of lemmas okay.

So here is, so let me write that the proof follows from the following the next two lemmas okay, so the first one so here is the lemma and the lemma is any noetherian topological space is quasi-compact okay this is the first thing which says that you take a topological that is noetherian then automatically it is quasi compact alright and in the zariski topology is you know is noetherian in this sense that if you take any affine variety of all that matter you take any closed subset, any algebraic set in affine space then you know that the, it can be broken down, it has noetherian decomposition and affine space is of course noetherian okay.

Affine space, if you take affine n space over k okay which is just Kn with the zariski topology, Kn algebraically closed field okay then you know the affine space is noetherian topological space for the zariski topology that is just, because that just translates to the noetherianness of the ring of functions of affine space which is the noetherianness of the polynomial ring in n variables which you know is true because of Hilbert's Basis Theory okay.

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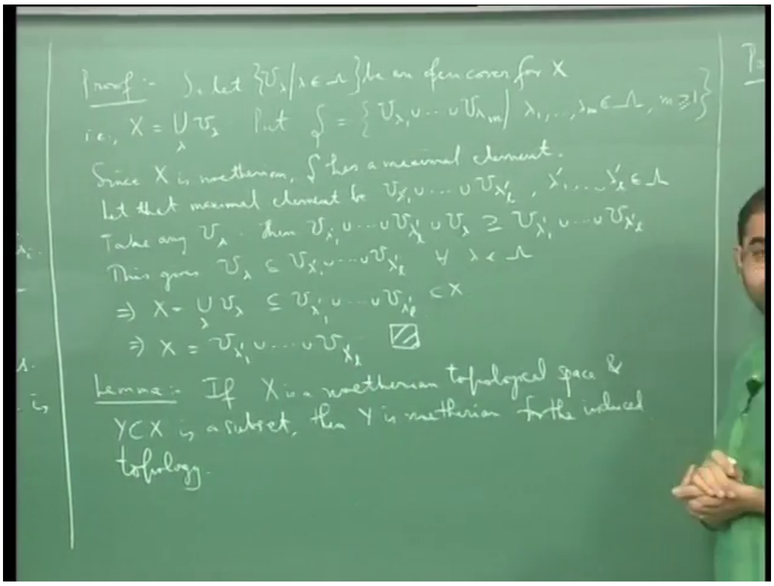


So I am saying that the quasi compactness just is a result of the noetherian property so how does one prove this? The proof is pretty easy, a proof is well what do I have to show, I will have to show I have a topological space which is noetherian then I have to show it is quasi-compact, so I assume that the topological space is called X, I assume that U lambda is an open cover for X, I have to cook up for U a finite sub cover I have to find finitely many indices lambdas such that the corresponding U lambdas, finitely U lambdas their unions is also equal to X okay that is what I have to do.

So what I will do is, so let U lambda, small lambda and capital lambda be an open cover for X, that is X is union over lambda U lambda okay I will have to show that there is a finite sub cover from this collection for X okay, so what do I do? I do the following thing I use, I try to somehow make use of the hypothesis, my hypothesis is that it is noetherian topological space now what is the definition of noetherian topological space is if you recall, there are several definitions, equivalent definitions.

One definition is perhaps the basic definition or the usual definition is that there is DCC for closed sets that is if you have a sequence of closed subsets one becoming smaller and smaller that is every next one contain the previous one then this sequence has to at some point it has to stabilize, that means if it is a strict sequence then it has to stop and if you do not demand its a strict sequence then all the terms in the sequence becomes the same beyond a certain finite stage okay.

This is the DCC that is the descending chain condition for closed sets okay and you know when you put this condition for affine space you know affine space in affine space closed sets correspond to radical ideals in the polynomial ring and therefore the descending chain condition for closed sets will correspond to the ascending chain condition for the corresponding ideals in the polynomial ring which is true because the polynomial ring is noetherian by Hilbert's Basis Theorem and this is what gives you the fact that affine space is with the zariski topology is a noetherian topological space.

So that is one definition that there is DCC for closed sets, there are other definitions, the other definitions is that since open sets are the compliments of closed sets okay you can say that there is ACC for open sets okay that is one equivalent definition and then there is another definition of, there is yet another equivalent definition that is given any non-empty collection of closed sets, there is always a minimal element with respect to inclusion.

This is one more definition, equivalent definition of noetherianess of the topological space which in the case of n space actually translates to the polynomial ring having the property that you give me any you know if you give me a collection of ideals there is always a maximal element if you give me a non-empty collection of ideals there is always a maximal element that is for example the ring theoretic definitions of one of the equivalent definition of noetherian ring okay.

And then there is yet another, yeah so probably I use, I will try to use that so I will try to use the fact that in this since the topological space is noetherian if you give me a non-empty collection of open sets there is always a maximal element, so what I will do is so I will have to apply to a collection, what is the collection? I will take the collection of consisting of unions, consisting of finitely many members from here okay.

I will take all finite subsets of lambda capital lambda, I will take the corresponding unions and take that collection okay, that is a collection of non-empty open sets and that should have a maximal element and my claim is that maximal element which will be a finite union anyway by definition will be all of X and then I am done okay.

So what I am going to do is put script S to be set of all U lambda 1 union U lambda n such that lambda 1 through lambda m or elements of lambda and m is an integer, m greater than (())(16:41) take this collection, what I am doing is I am just taking finite unions from this direction okay and I am putting all these thing together and hitting this family of subsets then this of course this family of subsets is non-empty okay because of course my X is a non-empty topological space and there is at least one open set and that one open set will occur here okay.

This m could be 1, small m could be 1 in which case even the single terms are there okay. Now since X is noetherian S has a maximal element okay, let that maximal element be well U lambda 1 union let me call it, let me give some special names lambda 1 prime to U lambda L prime where lambda 1 prime through lambda L prime are all elements of lambda okay.

So there is some finite collection of lambdas which I want to call lambda 1 prime through lambda L prime and the corresponding open sets in there that union, that union that finite union is a member in this collection and that is the maximal element now take any U lambda okay then you see U lambda 1 prime union U lambda L prime union if I put together is U, this U lambda also that will contain this element U this maximal element lambda 1 prime union U lambda L prime okay this is obvious because I have just added, I have taken union to bigger set with another set.

So this is contained in this but mind you this is in script S because this is also a finite union okay but this is supposed to be maximal so what it will tell you is that this is equal what is the property of maximal element the property of maximal element is that if it is, if there is an element bigger than that then it has to be equal to that okay. So this element is bigger than that so it is to be equal to that.

So this gives that U lambda is actually contained in U lambda 1 union U lambda prime 1 to lambda prime L. See if this is equal to this then this extra set I have taken union with has to be already contained in this only then I will get equal to okay. So but this U lambda I took was arbitrary, the small lambda subscript I took was arbitrary so what this tells you this is true for all small lambda and capital lambda.

So this tells you that X which is a union of all the U lambdas because all these U lambdas are a cover for X that also contained in this U lambda 1 prime union U lambda prime L which is of course contained in X okay and this will tell you that X is actually this maximal element okay and I am done that is the end of the proof. I have proved that there is a finite sub cover okay.

So this is how I get very easily that noetherian topological space is always quasi-compact okay. Now I will give you another lemma, here is another lemma and what this lemma actually tells you is that the property of a space being noetherian is a hereditary property namely if a topological space is noetherian then any subset of that topological space given the induce topology also automatically becomes noetherian okay.

So let me write that down if x is a topological space is noetherian topological space and Y in X is a subset okay then Y is noetherian for the induced topology okay. Any soft space over noetherian topological space is also noetherian that is a property of topological space being noetherian is a hereditary property okay. Now how does one show this? So that is the proof of that is also but easy.

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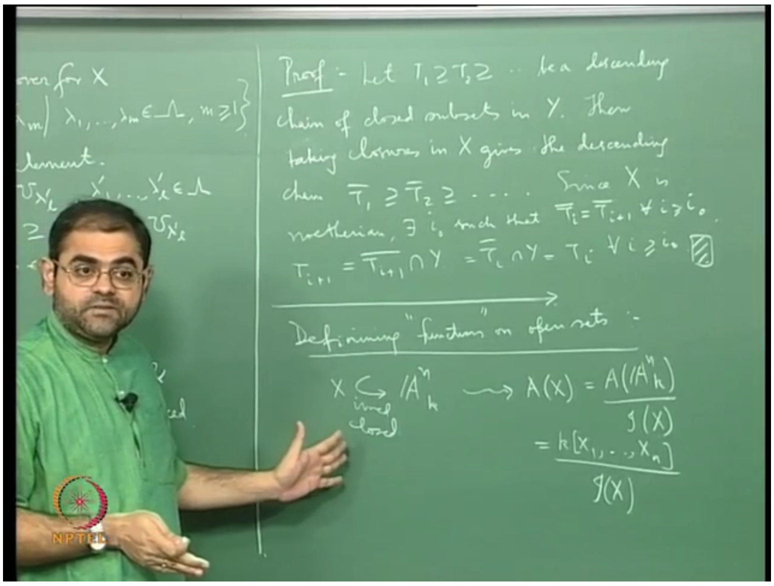


So proof, so you know I will have to show that Y, so Y is subset of X I have to show that I know that capital X is noetherian I have to show that capital Y is noetherian so what I will have to do is for Y I will have to verify DCC for closed sets okay. So what do I do is I take a descending chain of closed sets in Y okay, let T1 containing T2 and so on be descending chain of closed sets in Y, okay take a descending chain of closed sets in Y.

Now what you do is you take closures of these sets in the bigger space X okay then their closures, then taking closures in X gives the descending chain T1 bar containing T2 bar and so on, okay so I am just I have a, I am taking closure in the bigger space X okay, so what you must understand is that you see T1 closure is the closure of T1 in capital X, T2 closure similarly is the closure of T2 and capital X and see T1 closure contains T1 which contains T2 so T1 closure is a closed set in capital X which contains T2 and therefore it has to contain T2 closure because T2 closure is the smallest closed subset of capital X which contains T2 that is how this descending chain gives rise to this descending chain okay.

But then what you know about capital X you know about capital X that it is noetherian and therefore this descending chain has to stabilize at some point, so since capital X is noetherian there exists an i, i not such that Ti is equal to Ti plus one for all i greater than or equal to i not I have this, this is a simply the definition of the noetherian property that you have if you have a descending chain then it has to stabilize.

Of course you know if I am assuming that the original chain is not a strict chain if I had assumed it is a strict chain then what I will, then I will have to show that the strict chain is only it terminates, it is only finite okay but I am not assuming that, what I am assuming is that I am not assuming that these containments are all strict okay so these containment also need not be strict okay and then I am just trying to use the noetherian condition to say that stabilizes at some point, now what I want to tell you is that this holds with the bars and it also hold without the bars okay.

Why is that? So because you see if you calculate if you calculate Ti plus 1, Ti plus 1 is actually Ti plus 1 bar intersection with Y okay so you take a closed subset of Y okay it is not closed in the bigger space X then you take its closure in X and then you intersect it with Y you will get back the closed subset Y because when you take closure in X you are adding limits, you are adding the boundary even in X which is in the bigger space.

And then if you intersect it back with Y you are only looking at those boundary points in X which are already in Y but then the original set is closed therefore it contains all its boundary points in which are already lying inside the sub space therefore Ti plus 1 is Ti plus 1 bar intersection but then Ti plus 1 bar is actually Ti bar intersection Y and that is equal to Ti, this is true for all i greater than equal to i not and I am done.

So what I proved is that the fact that the T bars descending chain of T bars stabilizes implies that the descending chain T stabilizes okay, so that is the end of the proof of this lemma which says that noetherianness is a hereditary property okay. Now apply both of these lemmas to the zariski topology okay, first of all notice that if I take affine space any An is a noetherian topological space that we have already seen, that is just as I told you reflection of the fact that the polynomial ring in n variables is a noetherian ring.

So any affine space is a noetherian topological space, now since it is a noetherian topological space any subset of An given the induced topology is also a noetherian topological space because of this lemma, the second lemma okay and if you now apply the first lemma that subset given the induced topology is noetherian implies that subset is quasi-compact.

So what this altogether will tell you is that, it will give you this proposition that you give me any subset of affine space, any subset of affine space it will be quasi-compact in the induced topology okay and that is what is meant by the statement, the zariski topology is quasi-compact okay.

So you see for any subset of affine space quasi-compactness is just comes for free okay it is nothing, it is not, it does not have a speciality that compactness, the usual compactness as per Euclidean Spaces okay and that is the reason in a way that you know it calling this kind of compactness, the compactness here is quasi-compactness kind of justified okay.

So when you say quasi it means that this something is left out okay and in this case something serious is left out okay, so how serious that is a something that you will understand when we define what is meant by completeness or properness which is a correct analogue of compactness in algebraic geometry okay but for the movement that is the proof of this statement okay.

So now fine so this is, so this kind of ends the discussion about open sets in zariski topology, so let me summarize, the summary is that any open set in the zariski topology is a finite union of basic open sets okay and these basic open sets are very special in the fact that these basic open sets are actually themselves affine varieties okay, they are isomorphic affine varieties okay.

And you also have this property that the open set in zariski topology have the quasi-compactness property namely you give me any subset of affine space, if it can be covered by a collection of open sets then I can extract from that finite sub cover okay, so this is about the open sets in the zariski topology okay. Now what I am going to do is I am going to shift, focus to something else which also was there in this discussion and that is trying to define the so called functions on an open set of an affine variety okay so you know.

So the next part is defining functions on open sets okay, so of course when I say functions I put it in ports because what kind of functions we want is something that we have to decide upon and so you know let me recall, so what we did was if you take affine space and you take X inside affine space of course small k is an algebraically closed field and this is the affine n space of the zariski topology.

And I am taking X closed sub variety there okay that is an irreducible closed subset so this is irreducible closed and if you remember we defined the ring of functions on X so called affine co-ordinate ring to be nothing but the affine co-ordinate ring of the bigger space namely the set of polynomials in n variables modulo the ideal of X okay which is a prime ideal.

So you know this is an integral domain and it is a finitely generated k-algebra and then I told you that in this way we actually have very deep correspondence which can be made sense of as bijective correspondence or even as an equivalence of categories okay which goes on one side from affine varieties to the other side being finitely generated k-algebras which are integral domain okay.

So but the point is for every irreducible closed subset this is the ring of functions we defined and this is a very legitimate definition because see what you are doing is the ring of functions on the affine space is defined to be just the polynomial ring in that many variables because every polynomial in n variables can be thought of as a function on the affine space because you can evaluate at each point of the affine space, it gives you as scalar, okay.

So these polynomials in n variables are certainly bona fide algebraic functions on the affine space and then when you want the affine, when you want the functions on a closed subset, an irreducible closed subset then you have to go modulo the ideal of that closed subset and that is because two functions on the full space on the full affine space, two polynomials on the full affine space will define the same function on the subset, closed subset x if and only if their difference is zero on that closed subset and that is the same as saying that the difference lies in this ideal and so you have to identify functions modulo the ideal and that is the reason why you are taking this quotient and therefore you know this definition absolutely fine.

This definition is intuitively correct and it is also technically correct okay, so this is fine so long as you are trying to define the algebraic functions on an irreducible closed subset, remember the question is what are you going to do if you want to define algebraic functions on an open subsets okay and more seriously in the spirit of analysis how you are going to define a function to be algebraic at a point okay see what is it that we do normally in analysis when we want to define continuity you can define continuity at a point okay.

If you want to define differentiability you can define differentiability at a point similarly if you want to define analyticity or holomorphicity you can define at a point. So all these properties you can always define at a point alright whether a function has that property at that point or not okay so in the same way you can also ask an algebraic geometry, give me a function on some subset okay.

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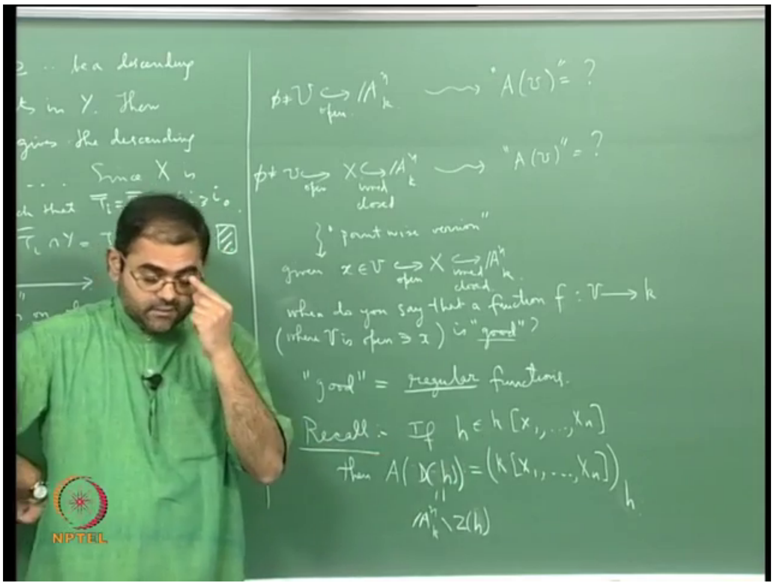


When will you define it to be a good function an algebraic function at a point okay, so this is the, in the spirit of analysis how do you do this okay so the clue is, so basically we want to study functions on open sets okay so you know, so the question is if you, so here is a question U inside An open subset okay or you know I will, let me say that after the statement so take an open subset of An and the question is what is A of U, what is this?

How do you define functions of U? Alright and then more generally I can do the following thing, take x inside An to be an affine variety so this is an irreducibly closed subset so this s an affine variety in An it is closed sub-variety of An and take U an open subset here okay of course in all these cases I am assuming U is non-empty because nobody want to work with empty set.

Take a non-empty open subset not an affine space but take a non-empty open subset of an affine variety and the question is how you are going to define the good functions on U? How you are going to do this? How to define functions? So the key to this is, the key to this is the following is this is trying to look at try to define what are the functions on the whole open set but now I can make it point wise and say give me a point of an open set.

How do you define a function define in a neighbourhood of the point to be a good function okay, so given so here is a point wise version, the point wise version is given small x in U which is an open subset of X which is an irreducible closed sub variety An okay, when do you say that a function v, let me call this is as f from v to k where v is an open containing x is good.

By good I mean algebraic function okay, a bona fide function, a function that comes in the algebraic sense okay how do you define function to be algebraic at a point okay, so the answer to all this is that, first of all I cannot keep always saying good good all the time and so we need a notation for that, we need a terminology for that and the terminology is regular so the our, we define good functions to be the so called regular functions, okay.

We define the good functions to be regular functions, so my aim is give me here as a, give me an open, give me a point on an open subset in an open subset of a closed sub variety and give me a function defined in the neighbourhood of that point okay, when do I say that it is regular at that point okay that is my question and how do you answer this question? The answer to this question is already there we only have to dig it out, it is already there in the discussion that we have had so far, in fact see if you go back we already proved that any open set is a union of affine open subsets okay basic affine open subsets, okay.

And therefore you can say that give me a point x and an open neighbourhood to the point this open neighbourhood can be is a union of basic affine opens and therefore one of the basic affine opens will contain this point and therefore you can restrict the function to that basic affine open and then I have to define what is a regular function what is a good function on a basic affine open but that I have already done, I have already defined what is meant by a regular function, a good function on a basic affine open set namely it is just localization it is the ambient ring of function localized at that that function, okay.

So recall if f is polynomial then A of D of f is defined to be equal to the polynomial ring localized at f okay this is our definition okay and what is D of f, D of f is supposed to be, it is the compliment, it is An affine space minus the zero set of f, it is a open set given by the compliment of the hyper surface defined by f of course you know if f is irreducible then you really get a hyper surface but if f is not irreducible then Z of f will be a union of hyper surfaces.

Those that correspond to the irreducible factors of f okay and we have already defined A of D of f to be this of this form okay and just so that I do not mess up notation let me do something here, let me call this is as h okay because I have already used f there let me call this as h, put h everywhere.

And now you know what I am going to do I am going to say that give me a point give me okay, so I need to still make one more statement this is for an affine open set in the, it is a basic affine open subset in the whole of affine space okay but I can also look at the basic of affine open set intersected with a closed subset okay and it is not very difficult to see what the ring of functions on that will be.

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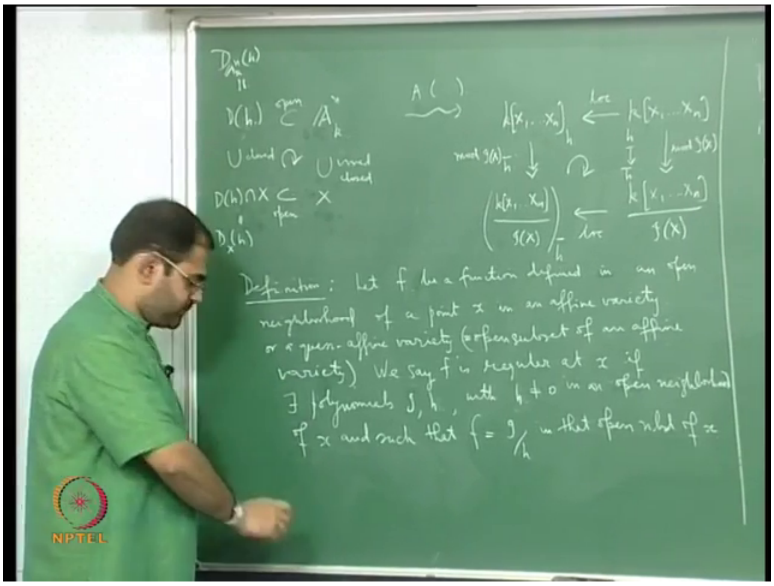

That is something that can be written out so let me go to this side and do that, so it is again given by the appropriate localization so you know, see if you take, so here is my An and here is my locus D h okay and the as far as the co-ordinate, at the co-ordinate ring level what happens is that if I apply this A, this A function to this side what I will get is I will get A of An which is k of the polynomial ring in n variables and here I will get A of D of h which is the localization of this A of x1 through xn localized at h, this is what I will get.

And of course you know this inclusion as an open subset shows up here as a the localization map, it is a localization map, this is a natural map from a ring to its localization okay and well now if I take a subset x irreducible closed, if I take an irreducible closed subset x then I can also look at D h intersection X which is inside this.

So you see this is closed, this is open and this is open okay this is how the diagram looks, this is closed inside this because this is the, this is intersection of a closed subset there and this is open inside this therefore this is an open inside this okay by the definition of the induced topology okay so the point is sometimes one can also write this as Dx h, one can write it like this if you want it is and this D h will be D h in An.

So this is D An k h that is what this is okay and you know what you are going to get here what I am going to get here is, see this irreducible closed subset this corresponds to this quotient going to the affine co-ordinate ring, the ring of good functions, regular functions, the ring of functions on x which is just the polynomial ring divided by the ideal of x okay and you know what this is going to be this is just going to be the localization of this ring and the image of h here which is h bar.

So this is going to be simply k x1 xn modulo I of x localized at h bar, where h bar is simply the image of h here, h is here it goes to the element h bar okay and if you actually look at it this way it is a quotient by I and this way also it is a quotient by I this is a quotient by the I x localized at h bar.

So this is mod I x and this one is mod I x localized at h bar this is localization of an ideal at an element, so these two are quotients that corresponds to these two closed subsets okay and these two are localizations, this is also a localization, they correspond to these two open inclusions this diagram commutes, this diagram commutes that means this followed by this is this followed by this, this followed by this is this followed by this okay.

So this is what happens when you apply the A functor to this side so what I want to tell you is that you already know what are the good functions on an affine open subset intersected with a closed subset okay so you already have definitions for what are good functions in affine space namely the polynomial ring, what are the good functions on a irreducible closed subset namely the polynomial ring modulo the ideal of the closed subset is a prime ideal.

What are the good functions on a basic open set in affine space, it is just localization at that element which defines the basic open set and what are the good functions on the this, the open set that you get by intersecting a basic open set in affine space with a closed irreducible closed subset which is simply given by the localization of that corresponding I mean it is just given by either the correct localization or the quotient in whichever way you want to set okay.

So I am saying that we already have the value of this A for four kinds of objects, we have it for affine space, for affine space we know what are the good functions, for x we know what are the good functions for an irreducible closed subset, we know what are the good functions on a basic open subset of the affine space and then we also know what are the good functions on the intersection of the basic open subset of affine space with an irreducible closed subset, now from these four we have to cook up the correct definition of a regular function okay and the definition of obvious but the surprise is the following.

With this new definition if you look at, so what you have done is you have defined regular functions at a point and once you define it at a point, you can define regular functions on any subsets okay the moment you define it for a point you can define regular functions on any subsets, so the question is if I start looking at regular function of a whole affine space what will I get okay will I get back my polynomials or will I get more, the answer is you would not get any more that is the beautiful thing.

Beautiful thing is on the affine space the regular function will still be only (())(48:33) on any irreducible closed subset, the regular functions will still only be this quotients okay you would not get anything more and this tells you that your definition of regular functions is correct. That your definition of regular function gives you the right thing when you for these known objects okay, so I will make that definition now.

Definition let f be a function defined in an open neighbourhood of a point x in an affine variety okay or a quasi-affine variety okay. A quasi-affine variety is just open subset of an affine variety. Okay we say f is regular at the point x if there exists polynomials f, g with g not equal to 0 in an open neighbourhood of x and such that f is equal to I am sorry I think I should use not f and g, I should use g and h which with h not equal to 0 such that f is equal to g mod h in that open neighbourhood of x okay.

So look at this definition this definition is, this is what tells you when a function defined in an open neighbourhood of a point is a good function it is a regular function at that point it is very simple all you are saying is that to say that the function, see the function is when I say a function define a neighbourhood of a point, it is a function of the values in k which is the field, scalar valued function okay.

And all I am saying is that you can deem the function to be a regular at a point if the function in a neighbourhood of the point is a quotient of two polynomials that is all, you can, you are able to find two polynomials g and h such that the function you get by evaluating this quotient of polynomials is the same as your original function f in a neighbourhood of the point and that neighbourhood of the point obviously should be in the locus where h does not vanish otherwise it cannot evaluate if h vanishes at a point then I cannot evaluate g by h at a point because I will be dividing by 0 okay.

So it is a very very simple definition, it is a lot to write down but the idea is very simple you are saying a set theoretic function is good, is regular at a point if it can be written if it is the same function as you get when you evaluate a quotient of polynomials in a neighbourhood of that point okay that is all and what you must understand is that if your point is lying in a basic open set then you know that already the functions, the good functions you are defined on a basic open set are just localizations, functions here.

And what are the functions here, they are of the form g by h, in fact they are of the form g by h power m where you know you also allow a power of you do not only invert h but you also invert powers of h because inverting h automatically will also invert powers of h, so in general element here will be look in the form, will be of the form g by h power m okay.

But any case it is one polynomial divided by another polynomial with the bottom polynomial not vanishing that is how the functions look like and you are saying that that is the kind of inspiration to define a general function to be regular and why is that inspiration correct? It is correct because every open set is always broken down into a finite union of basic open sets like this in fact if you take any open subset in An it is a union of finitely many D h for finitely many h.

And if you take any open subset of x, a closed sub variety that will also be a finite union of such D h's intersected with x, all this just follows because of the fact of that you know that these are all noetherian topological spaces and therefore the subsets are all quasi-compact so any open cover that means a finite sub cover okay, it just follows from that right.

Now I will have to justify that after with this new definition of what a regular function is I will have to show which is amazing thing, the amazing thing is if you take affine space and look at all the regular functions in affine space, so I am looking at all functions on affine space that locally look like quotients of polynomials okay that looks a little more complicated than just looking at all the polynomials in n variables.

But the fact is they are all the same that is the surprise and that is what I am going to prove okay I am just going to show that this definition is correct if you take affine space or if you take a closed subset, irreducible closed subset of affine space okay that kind of tells you that this definition is in the right direction okay. So I will stop here and continue in the next lecture.