Basic Algebraic Geometry
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Lecture-14
The Ring of Functions on a Basic Open Set in the Zariski Topology

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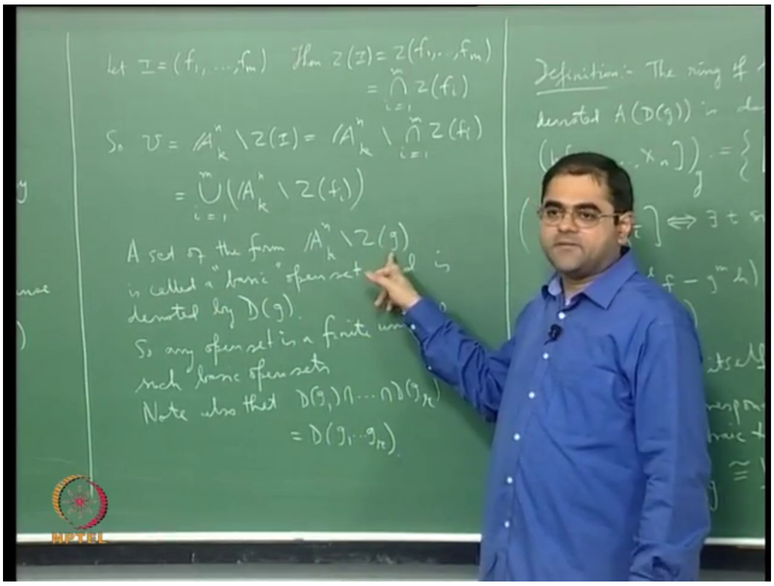

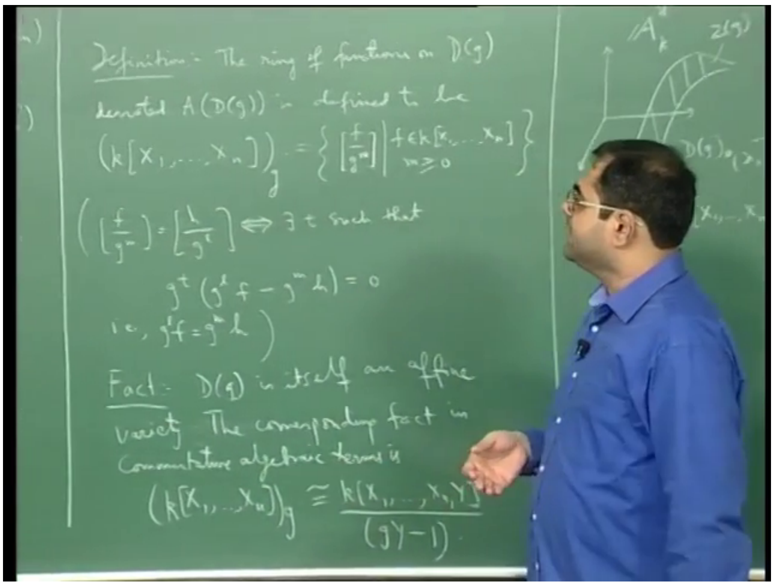

Okay so let us continue with our discussion of Basic Open Sets of the Zariski Topology. So let me again recall, you start with polynomial g in n variables and you look at the compliment of the zero locus of g and call it D g, it is called the basic open set defined by g and any open set can be written as finite union of such basic open sets and then we define the ring of functions on this basic open set to be functions of this form.

Namely these are the polynomial functions multiplied by you know inverting powers of g okay which makes sense because g does not vanish on that locus okay but then the fact is that this basic open set D g is actually itself isomorphic to an affine variety and I have not defined what an isomorphism of affine varieties is but I am trying, at least I will try to give you the isomorphism at least at the level of topological spaces.

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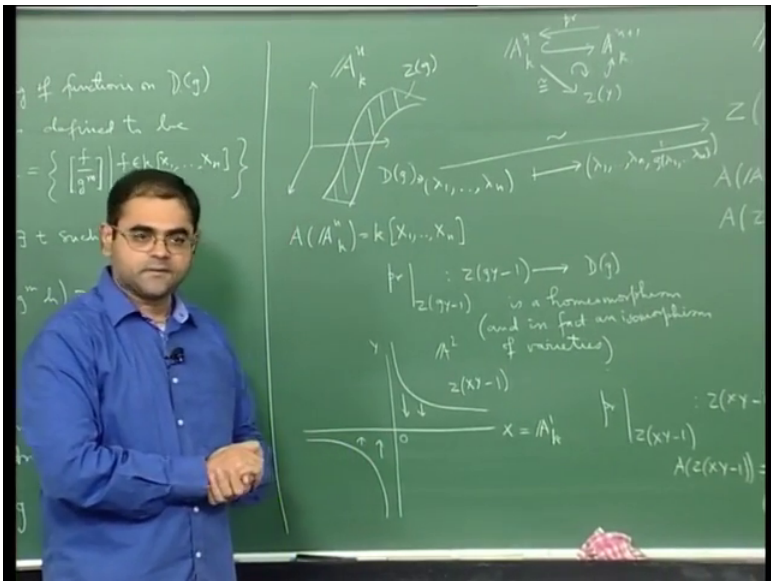

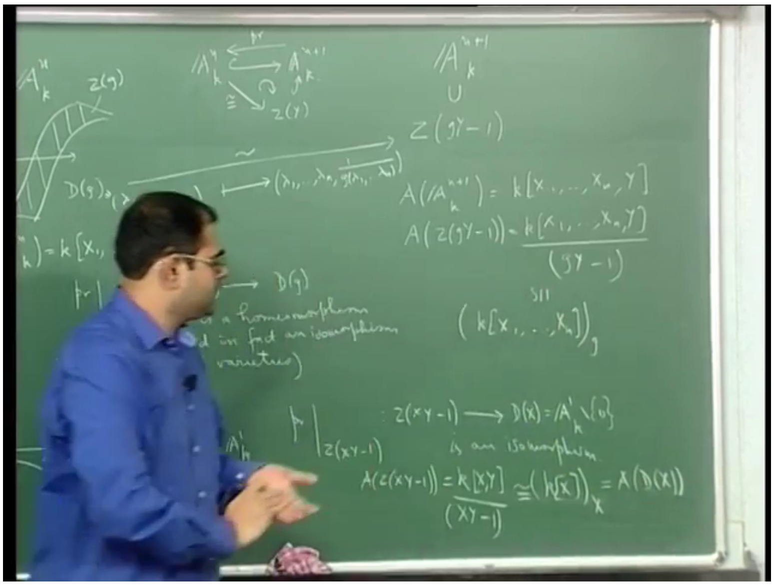

And this is how we do it, we take An the affine n space over k of the usual zariski topology and look at these zero locus of g, this is a hyper surface okay if of course if g is irreducible it is a hyper surface if g is not irreducible then it is a union of hyper surface okay and it will be a union of hyper surfaces which will be the irreducible components, they will be hyper surfaces corresponding to the irreducible components of g which occur in the factorization of g okay.

And then the compliment of this hyper surface is the affine open set is the basic open set D g okay and this D g can be thought of as a closed subset as a close sub variety of an irreducible closed subset of a larger affine space namely an affine space of dimension 1 more and that is done in the following way, you look at the zero set of gy minus 1 where y is a extra variable that you add to get the ring of functions on this affine space of 1 dimension more.

And since this polynomial gy minus 1 is irreducible, the ideal generated by gy minus 1 is prime and this zero set therefore, the zero locus of that polynomial gy minus 1 is hyper surface okay and it is a closed subset of this n plus 1 dimensional affine space and it is an irreducible close subset and its ring of functions is defined to be the ring of polynomials on the ambient affine space divided by the ideal of functions that vanish on that hyper surface which is the ideal generated by gy minus 1.

And the fact is that we have a bijective map from this basic open set here in An and this reducible close subset, this hyper surface in An plus 1 so what is happening is that a basic open set in affine n space is being identified with a hyper surface in affine n plus 1 space okay so it may look, at first it may look a little confusing because here it is open and there it is closed okay but you must remember that the affine spaces are of different dimensions.

And you must also remember that the ambient affine spaces are of different dimensions but the dimensions on these two spaces of course they match okay, see the dimension of a hyper surface is always 1 less than the dimension of the affine space. So here it is n plus one dimensional affine space and it dimension is 1 less so its dimension n okay and this is a something that I have not told you about but there is something that I will try to explain to you, the dimension of an open set, a non-empty open set is essentially the same as a dimension of, I mean it can be, the dimension of a non-empty open set can be defined as the topological dimension okay.

And the fact is that the dimension of this will also, the dimension of this will also be n okay and that coincides with the dimension of this, in fact what will happen is that you know dimension can be defined for any topological space okay, it is defined to be the maximal length of you know a chain of irreducible close subsets properly contained in one another, each one properly contained in the next and provided you start indexing with 0 and then you take the length of maximal such chain okay.

Essentially you should take the length of the maximal such chain and take away 1 from that okay, so you can define the dimension of a topological space in that sense you can define the dimension of any subset of any topological space and it will turn out that the dimension of subset is same as dimension of the closure of that subset, by going to the closure the dimension is not going to change okay.

And therefore you know if you believe that statement that dimension of D of g will be the same as the dimension of the closure of D of g but then the closure of D of g will be the whole affine space because D of g is a non-empty open subset of the affine space and you know any non-empty open subset is irreducible and dense, in particular it is dense so its closure will be the whole affine space and by going to the closure you do not change the dimension. Therefore the dimension of this is same as the dimension of the affine space and the dimension of the affine space is N, okay.

So this is n dimensional, this is also n dimensional so dimensions match okay and I told you it is a matter of exercise, it is good exercise for you to check that this map is actually a homeomorphism, in fact as I was trying to point out in the last lecture let me say the following thing An plus, so An k since inside An plus 1 k as it sits inside as a subset which is given by the zero set of y, so the zero set of y is, it is a hyper surface defined by y which means you are looking at all the points where the Y co-ordinate vanishes and all the points where y co-ordinate vanishes will give you the copy of, it will give you this An okay.

And the fact is that, so this is the identification okay so An is identified with Z of y, zero set of y in An plus 1 okay and mind you this means that you are thinking of An as a, An is a hyper surface in An plus 1 and in this case you call it a hyper plane if you want okay. It is a hyper surface in An plus 1 because it is a zero set of single polynomial okay where the set of points where the y co-ordinate vanishes is precisely a copy of An right?

And, well what is happening is that you also have a projection, this is you have a projection map from An plus 1 into An and the projection map is the map that takes the N plus 1 co-ordinates and forgets the last co-ordinate okay and the statement is that you take this projection map and restrict it to this closed subset then that gives an isomorphism with this open subset okay.

So projection restricted to Z of gy minus 1 from Z of gy minus 1 to D g is a homeomorphism and you know we will see this later and in fact isomorphism of varieties this is something that we will see later because I have to define what isomorphism varieties is but then if you believe this then it is and also believe the fact that you know an isomorphism varieties has to correspond to an isomorphism of their co-ordinate rings, namely rings of functions okay then it will tell you that the rings of functions on this and the rings of functions of this have to be the same.

So that will tell you the ring of functions on this has to be isomorphic to this but that is also isomorphic to this because of commutative algebra so it will tell you that the ring of function on D g is this, it is connected to define ring of functions on D g to be this. So a nice, to see that in a very simple case what you can do is that you can simply take, you can just look at the plain A2 and then you can look at the rectangular hyperbola which is given by the zero set of XY minus 1.

This is x axis, this is the y axis okay, this x axis actually corresponds to A1 which is sitting inside A2 given by the equation y equal to 0 okay and if you take the projection onto the first co-ordinate that is you forget the last co-ordinate okay then what you will get is you will get under the projection the image of this rectangular hyperbola will be the compliment of the origin in A1 and that is precisely the affine open set D x the compliment of the point where x equals to zero.

So projection restricted to Z of XY minus 1 from Z of XY minus 1 to the X which is A1 minus the origin is an isomorphism, so this is a very simple diagram that you can always remember that tells you what is happening more generally so I have taken n equal to 1 and there is A1 sitting as a x axis inside A2 okay and A1 is sitting as y equal to 0 just like this An is sitting as y equal to 0 in An plus 1.

And then you have this projection from An plus 1 to An which is in this case projection from A2 to A1 and this projection is simply given by forgetting the last co-ordinate which is y that means projection on to the x axis okay and under this projection the zero set of gy minus 1 goes to D g for me now G is X so zero set of XY minus 1 which is the rectangular hyperbola, it projects onto the compliment of the origin because that is the only point that will be left out okay.

And the compliment of the origin is of course D x, it is set of all points where x does not vanish and that is the compliment of the origin and you get a isomorphism like this okay and the fact this is an isomorphism of varieties is a geometric fact and what does the corresponding, what is the translation of this fact into commutative algebra it is just the statement, it is just the statement that if you take the affine co-ordinate ring of Z of xy minus 1 which is k xy y xy minus 1 that is isomorphism to k x localized at x okay which is defined to be the affine co-ordinate ring of the ring of functions on D x okay.

So that tells you, I mean that gives you picture in the simplest possible case okay as to what is happening and what is happening here is same thing is happening here right, fine now so I have given you, you know I have given you two lines of justification or two lines to convince you that this definition is correct okay.

So let me recall them, one is that the functions on the open set, basic open set D g have to (()) (15:26) in evaluation of negative powers of g okay which is sensible enough because g does not vanish and when a function does not vanish it is a reciprocal is also valid as a function should be valid as a function okay so natural that you should be able to invert g and if you invert g or actually localizing at g and this is the ring of functions localize at g so that is one justification.

The other justification is D of g is also an affine variety it is isomorphic to an affine variety for which the ring of functions is this which is also isomorphic to this from the sense of commutative algebra, so this is another justification. Now I will give you yet another justification and this is the justification essentially that goes along the lines of the fact that you must have an isomorphism of affine varieties as equivalent to an isomorphism of their affine co-ordinates rings okay and so let me recall something from the previous lecture okay.

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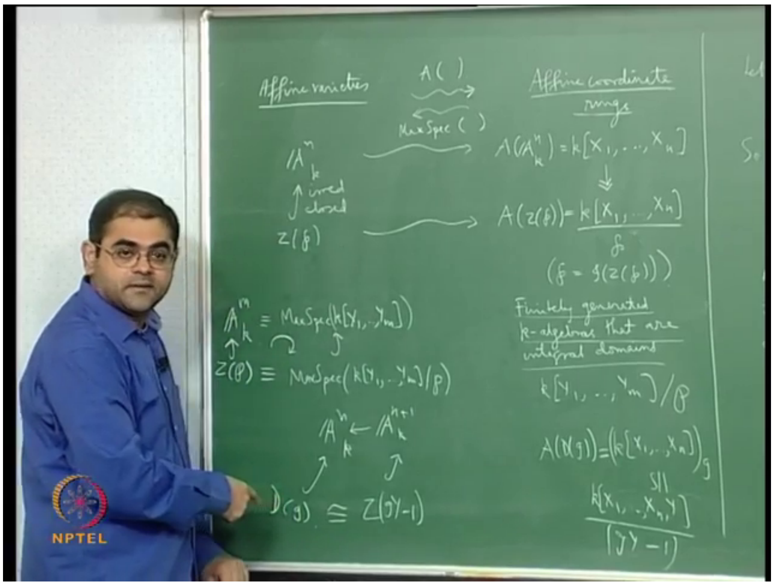

So you see we did the following thing what we did was we put on one side you know we put affine varieties okay and on the other side we put affine co-ordinate rings which are just co-ordinate rings of rings of functions of affine varieties okay I mean they are always called as co-ordinate rings because they essentially are based on the variables which are thought of as co-ordinates giving co-ordinates on the ambient affine space and the word affine is used because they are all affine varieties okay they are all considered inside affine space alright.

And so what is the definition here the definition here is my original definition on this side was an irreducible closed subset of affine space so you know if I started with affine space An k well I would end with the affine co-ordinate ring A, so you know this direction the association is given by A of, okay so A of An is just the polynomial ring in n variables okay.

And if you give me a irreducible closed subset Z of i or Z of p for p a prime ideal in the polynomial ring if you take the zero set of p okay that will give me irreducible closed subset of An okay, so this is irreducibly closed and we call such an irreducible close subset as an affine variety, irreducible close subset of some An and for it the corresponding affine co-ordinate ring or ring of functions was defined to be A of Z of p is equal to the co-ordinate ring, the ring of functions on the ambient affine space, the larger affine space divided by the ideal of that variety which is just p okay where of course p is ideal of Z of p okay this is how we define.

Now you see and in fact I have also told you that this fact that this is an irreducible closed sub variety is reflected by showing that from here to here you have a quotient because it is just quotient by the prime ideal P okay and I was trying to tell you that you know you have a more general picture that is on this side whatever you have there is a reflection of that here, this is a geometric picture, this is commutative algebraic picture and I told you that the equivalence comes because of an arrow that is going in this direction and I told you that arrow is actually max spec is given by max spec okay.

So what I told you was well you give me a general affine co-ordinate ring so what is the definition of general affine co-ordinate ring? A general affine co-ordinate ring is defined to be something like this it is a finitely generated k-algebra which is a integral domain namely it is a polynomial ring in n variables, in any number of variables not necessarily n, any number of but finite number of variables divided by a prime ideal, why do I want divided by prime ideal, because I want integral domain and why finitely many variables because I want a finitely generated k-algebra right.

So on this side the more general definition of affine co-ordinate ring will be finitely generated k-algebras which are integral domains okay so you know, so let me write that here if you take a finitely generated k-algebra, there is not any space here so let me write it in the next line, finitely generated k-algebra that are integral domains okay, this is what you should put on this side in general and what will that be, that will be some k of well Y1 etc Ym modulo some well some p which is a prime ideal okay and what is this come from, this comes from affine variety that something that you can very easily see.

What is an affine variety? You take max spec of k Y1 etc Ym, this is nothing but Am okay this is what we saw in the last lecture I told you that this is a, what I proved was this is a isomorphism as topological spaces okay but and I told you that that is only half the story, in fact it is an isomorphism of even varieties but there is something that I will keep telling you but that something that I will justify later because I have not defined what isomorphism is on that side okay.

But just take it for that so if you believe that then this is an isomorphism varieties okay but mind you at the satirical level this statement is just Nullstellensatz that corresponding to a point with co-ordinates lambda I, you are associating the maximal ideal given by xi minus lambda I generated by xi minus lambda i and that is the Nullstellensatz, that every maximal ideal is of this form okay.

And let me recall that max spec was suppose to be the maximal ideals in this ring and the, which is subset of the full spectrum called the prime spectrum which consists of the prime ideals and that spectrum itself had a Zariski topology and therefore the maximal spectrum which is a subspace of that, subset of that topological space got induced topology and with respect to that induced topology this identification became not just a bijective but it became actually a homeomorphism of topological spaces.

And the most strongest statement is that this in fact an isomorphism of varieties okay, now in this what you do is you look at max spec of this quotient k of Y1 Ym mod p then you know this sits inside as a closed subset here and that close subset actually corresponds to that is the identification of Z of p which is identified with this diagram commutes okay so this the very identification that associates to every point of Am, the maximal ideal in the polynomial ring corresponding maximal ideal, a unique maximal ideal in polynomial ring in m variables will associate to every maximal ideal here I mean to every point here.

A maximal ideal of the polynomial ring which contains the ideal p because the maximal ideals in the quotient are precisely the maximal ideals in the parent ring which contain the kernel okay that is a correspondence. So this is what is happening in this case okay, this is what we saw in the last lecture.

Now what I am going to do is, I am going to slightly modify see just like I am modifying the objects on this side, I am not saying that they are affine co-ordinate rings when I say they are affine co-ordinate rings it means that I am already starting with something here and taking its affine co-ordinate ring but instead of that if I want to independent define it on this side, I simply define it like this, finitely generated k-algebras that are integral domains okay.

So in particular what happens is that these guys something like this does comes from here okay but of course the way I defined it, it is a completely, the definition is completely commutative algebraic it has got no geometry in it okay, I am not making any reference to this side I am not saying that these are affine co-ordinate, they are the rings of functions of some affine varieties but they turn out to be okay.

Similarly on this side what I am going to do is I am going to call a variety, affine variety if it is isomorphic to an affine variety okay now that is again in a way like begging the question but I have not defined what a general variety is but let us assume that the moment you naively accept meant by an isomorphism of varieties and then you say that any variety which is isomorphic to an affine variety should also be called an affine variety.

Suppose you make that definition then the beautiful thing is that if you look at in An if you look at this open subset given by D g okay, the basic open subset that given by D g mind you this is not a closed sub variety of An, it is not a closed sub variety of An. Then this also turns out to be an affine variety okay and I told you roughly the story as have done that is that this is isomorphic to the set Z of gy minus 1 in a bigger affine space, space of dimension 1 more okay.

And there is a projection like this and under this projection which forgets the last co-ordinate this is identified with this okay, that is what I have explained here, that is exactly what I have explained here okay, now if you believe that then it tells you that even open subset of basic open subset should also be the thought of as affine varieties because they are isomorphic to affine varieties okay.

So if you go by this then what I should get on this side is A of D of g and A of D of g is well I have defined it as k x1 etc xn localized at g and well there is no problem with this isomorphism because this isomorphism in principal should also give an isomorphism of rings and that isomorphisms are already there from commutative algebra. You have an isomorphism of this with k of x1 etc xn y, you add extra variable y and divide by gy minus 1.

These two are of course isomorphism okay that is something we have seen that okay and by the way I should tell you that this trick of looking at g which is an affine open as a hyper surface in a affine space of 1 dimension more is called Rabinowitsch Trick, it is a trick of inverting g okay it is just trying to say that, see it is a beautiful thing what it tells in commutative algebra is localization by a single element is a quotient.

See k x1 through xn localized at g means you are inverting a single element g, it is a localization okay whereas what you have here is the quotient of polynomial ring in one variable more by suitable ideal and these two rings are the same so it says that, it does not say every localization is a quotient what it says is localization at a single element is always a quotient okay.

And that geometric content of that is that any basic open set is actually an affine variety and that is the reason why people call sets this form as basic affine open that is the word they use, they also add the adjective affine, they say basic affine open because it is not just a basic open set it is actually an affine variety in its own right under this identification.

Now you know if you want to believe things what you should expect is if I take max spec from here to here I should get back my D g okay so if I take max spec of this localization I must get D g if everything is correct okay if everything fits into the picture properly and that is the case okay, so in fact if you apply max spec of, max spec to this you do get D g and how is that true that is just because of some basic facts from commutative algebra which you must have come across in the first course in commutative algebra namely the following.

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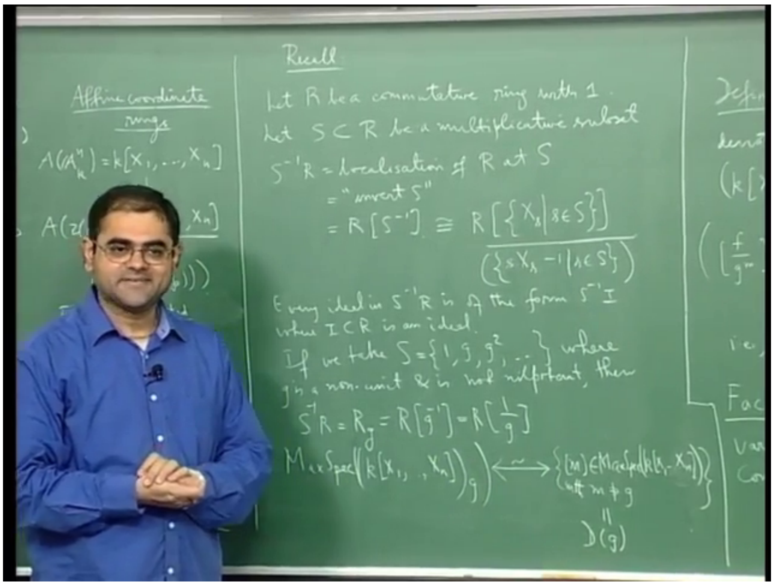

Let R be a, so I recall let R be a commutative ring 1, let S in R be a multiplicative subset, let this be a multiplicative subset, so that means that S is a subset of R which contains 1 okay and it is closer to multiplication and it does not contain any nilpotent elements okay, it does not contain 0 in particular right and then we have the localization S inverse R this is the localization of R at S what it means is just invert S okay.

And sometimes people also write it as R S inverse okay and the reason why we write it as R S inverse is actually because this is isomorphic to R of, you take the polynomial ring in R in as many variables as there are elements of S okay and then you go modulo s times Xs minus 1, the ideal generated by all these s times Xs minus 1 where s is in S, that is what it is, I mean the point is that I want to invert, how do I invert S, how do I invert an element?

What I do is, I add a variable and then I add a variable Xs, X small s corresponding to the element small s and then I kill small s times Xs minus 1 because when I kill this what I am trying to do is I am in the quotient ring s times Xs will be equal to 1 and when a product of elements is 1 it means that each of these elements is a unit, so therefore s has become a unit that means I have inverted s okay this is what is happening.

And this is again something that you would have learnt in our course in commutative algebra it is very easy to verify okay, again the map from here to here comes because of the universal property of localization, the map from here to here will come because of universal property of the polynomial ring okay, now okay now you see the point is what are the ideals in spec inverse R, I mean what are the elements of spec of S inverse R what are the ideals in S inverse R so the fact is that the every ideal in S inverse R is of the form S inverse of I where I in R is an ideal okay.

So every ideal in the localization is given by localization of an ideal in the original ring okay now what is localization of an ideal? It is just you take the ideal and invert elements that you have to invert okay, so localization of an ideal is particular case of even more general thing namely its localization of a module so in fact if m is in R module then you can also make sense of S inverse m, S inverse m is simply the localized module.

S inverse m will become a module over S inverse R and S inverse m will just be a module such that you are allowed to multiply not only by ring element you are allowed to also divide by elements of the multiplicative set which is equivalent to multiplying by their reciprocals which exists because they are units in the localized field okay.

So this is a fact from commutative algebra that every ideal is of this form and in fact if you want a non-trivial ideal then this I, if you recall this ideal I should not meet S okay this ideal I should not meet S and further every prime ideal in S inverse R also will be localization of a prime ideal in R that does not meet S. So this characterization of ideals in the localization coming as localization of ideals in the original ring, it will not only hold for ideals, it will hold for prime ideals, it will also hold for radical ideals okay this is a fact but what we are interested is in the fact that it holds prime ideals okay.

So the moral of the story is that if we take S to be the subset 1, g, g squared where g is a non-unit and is not nilpotent then we usually denote S inverse R as Rg this is a notation sometimes you also write R of 1 by g, Rg inverse or R of 1 by g in keeping with this notation R of S inverse okay and what are the ideals in Rg they will be the ideals in R localized at g okay and in particular the ideals should not contain g.

So the prime ideals in Rg will be precisely the prime ideals in R which do not contain g okay and now if you apply this to, if you apply it to this localization what will it tell you, it will tell you that the max spec of this ring the localization of the polynomial ring in n variables at a single polynomial g, the maximal spectrum of that mainly the prime, I mean the set of maximal ideals there.

The set of maximal ideals would be precisely those maximal ideals in the original ring it will be in correspondence with the maximal ideals in the original ring which do not contain g okay so what all this will tell you is that max spec of k x1 through xn localized at g will is can be identified in a bijective correspondence with the set of all m in max spec of the original ring of the polynomial ring such that m does not contain g, this is what you will get so let me erase a little bit of this so that I can get some more space right, so I will have x1 through xn, max spec of this with m, g not in m okay so let me draw line right, this is what it is.

So the maximal spectrum of this localization will actually be all those points which corresponds to maximal ideals in the original ring of which this is the localization such that m should not contain g okay, now you see what you must understand is that, let me see whether I am saying that correctly, so the ideals in the localization will correspond to the localization of ideals in the original ring that do not meet the multiplicative subset, so this is correct. So this m will meet this S if and only if some power of g is in m and that is the same as saying g is itself in m because m is a maximal ideal so it is prime.

So this statement is right okay but then what you must understand is what is this set, see if m is not, if g is not in m it means that the point m is in D g so this is actually D g what is D g? D g corresponds to points in the affine space where g does not vanish okay but what does a point in affine space corresponds to? It correspond to maximal ideal okay and the fact that g does not vanish at that point means that g should not belong to that maximal ideal.

So this set is a same as D g okay this set is exactly the same as D g so the moral of the story is that if you apply max spec to this you end up getting D g which also tells you therefore that this is the correct co-ordinate ring if you take A of D of g and then apply max spec you get back D g that is the third justification for the definition of A of D of g to be the localization at g okay.

So these are three justifications as to why this definition is correct okay but then all this is only to get you a feel of how things are going it helps you to understand that you can make sense of the ring of functions on an open set. See because my aim is to go from the geometric side to the algebraic side my aim is always to associate rings of functions and from the commutative algebra side if you want to the geometric side I will always look at the maximal spectrum that is how we are doing it.

So on the geometric side, the open sets are also important so if you give me an open set I need to know what are the ring of functions, what is the ring of functions on that open set okay. Now that is a question that to answer that question first of all you break the question down in two pieces, first you realize that in any open set is a finite union of basic open sets and then the idea is that if you knew what are the functions on the basic open sets then using this you can get an idea of what will be the functions of open sets okay.

So that is why all this is important right, so I hope that convinces you that this definition is valid and that these basic open sets are actually basic affine open sets in fact they are affine varieties they can be identified as affine varieties in an affine space they are of 1 dimension high okay, fine so there is one more thing that I need to tell you and this has got to do with a version of compactness, the usual compactness which comes for free in the case of zariski topology.

And that is the reason why it is called, it is not called compactness in zariski topology it is called quasi-compactness that just comes for free and then you can then you can expect that as it is the case that the corresponding notion for compactness in algebraic geometry is very different, it is called completeness or properness okay so.

So let me explain this, see the usual topology what is this, what is a definition of compactness of a subset, you have topological space, you have subsets, when do you say the subset is compact? So the definition goes the most general definition goes by using open covers, the definition is that if you have an open cover of that subset, then out of that open cover only a finite sub cover will suffice that is if you are given a collection of open subsets whose union contains this given subset which is supposed to be compact then from that collection of open subsets you can just take only finitely many whose union will also contain that subset which is supposed to have a property of compactness.

So the condition for compactness in topology is every open cover has a finite sub cover, admits a finite sub cover okay now the beautiful thing in algebraic geometry is that this is there for free okay, so what happens in algebraic geometry is that you give me a collection of, you give me any open cover okay of a subset then a finite sub cover will always be enough okay and the reason is as follows.

You give me an open cover, the union of all the elements in the open cover will be an open set okay, the union of all the element in that open cover will be an open set but you, we have already seen that any such open set is a finite union of basic affine opens okay and therefore what will happen is that this open set is a, it is expressible also as a finite union of basic affine opens. Now you intersect each of these finitely many affine opens by that open cover okay and use the fact that an open subset of a basic open set is again a basic open set okay and therefore what will happen is that.

See essentially the fact that any open set can be covered by finitely many basic open set will tell you that any open cover admits a finite sub cover okay and therefore the moral of the story is that you get any open cover admitting a finite sub cover very trivially in zariski topology okay and for that reason this property is not called compactness okay but it is called quasi-compactness so let me write that.

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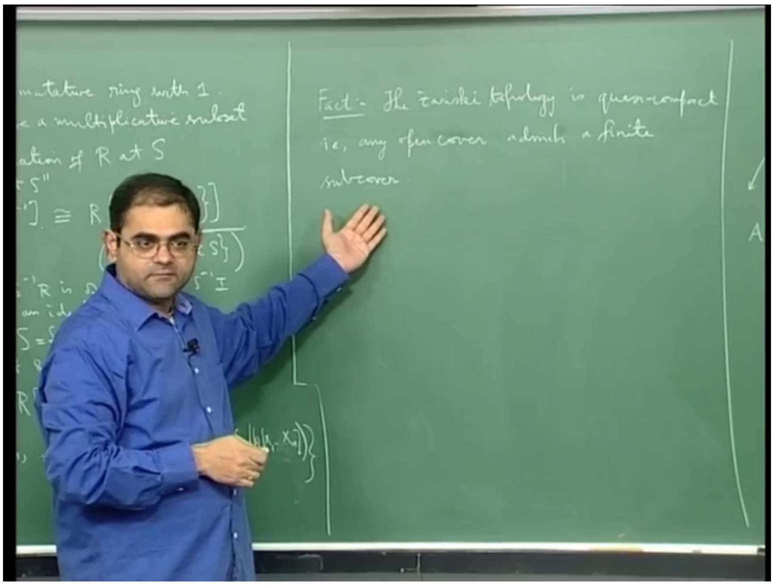

So fact, the zariski topology is quasi-compact okay that is, any open cover admits a finite sub cover whose union is the same as the union of the original open cover okay and this is, this fact just follows from the fact that any open set is finite union of basic affine open sets okay, so I will, there is also way of looking at this from the commutative algebra point of view which I think you would have come across in an earlier course in commutative algebra but nevertheless I will try to recollect that okay I will do that in the next lecture. So let us stop here.