Basic Algebraic Geometry

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Lecture-13

Analyzing Open and Basic Open Sets for the Zariski Topology

Alright so what we are going to do in this lecture and the next is you know try to look at the Zariski Topology but not in terms of closed sets but in terms of open sets, see if you recall that any topology on a topological space is specified by either giving a collection of subsets which is called close sets which satisfy the axioms for closed sets for a topology or equivalently it is given by a collection of subsets called open sets which satisfies the axioms for open sets required of a topology.

And you know the axioms for closed sets and the axioms of open, for open sets are just got their equivalent to one another by using the properties of compliment, taking the compliment of a set in a larger set so De Morgan's Laws for example, so usually when we do classical analysis or topology for example if you study Euclidean Space, n-dimensional real space, then the topology is given only by using open sets.

And the open sets are given by, thought to be given by unions of you know open discs or open balls okay which turn out to be open intervals if you are in one dimensional okay, of course if it is two dimension then these are open discs and so on, but so the approach is by specifying open sets but in the zariski topology our approach has been by specifying close sets okay call the algebraic sets.

And these closed sets were given by the sets of common zeros of a bunch of polynomials in the right number of variables okay, the number of variables should be equal to the number of copies of the field that you are taking okay, now what I want to do is to now shift the focus and get into study of open sets okay in the zariski topology.

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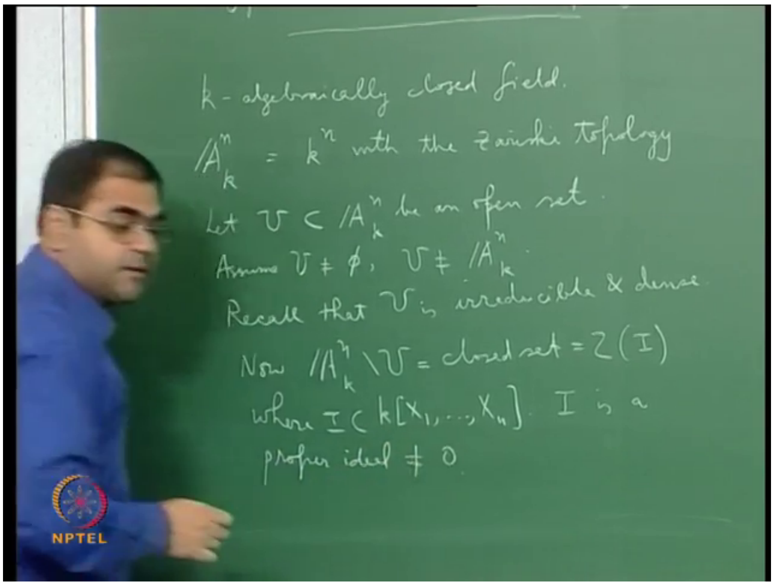

So what we have is, so open sets in the zariski topology so this is what we want to get an idea about okay. So you know k of course is an algebraically closed field and if you want as you shall you can think of k to be the set of complex numbers, field of complex numbers which is algebraically closed and then you are looking at An of k this is just kn with the zariski topology.

And of course the zariski topology comes by looking at the polynomial ring in n variables and looking at common zero loci of a branch of polynomials and calling that common zero locus that particular subset of An as a closed subset and then taking all possible subsets like that we close sets and that is how you get the Zariski Topology okay, now of course we have defined what a variety is we have closed sub variety of affine space okay is supposed to be by definition of closed subset is reducible right we have done that.

So now let, I start with an open set here and see how it looks like so let U inside An be an open set okay so by, of course assume that U is non-empty okay and also that U is not the full space which are both open of course okay because the null set is closed the full space which is its compliment is open and since the full space is closed, the null set which is its compliment is open so these two are both open sets, they are the trivial open sets okay.

But you looking at non-trivial open set, proper non-empty proper open subset okay and of course what you must understand is you must remember that any such open set U is very special in the sense that topologically it is dense okay and it is also irreducible as a topological space, so that is because I have told you that the irreducibility of a topological space force is that every non-empty open subset of that topological space continuous to be irreducible and also is dense okay.

And since An itself is irreducible okay because it corresponds to the zero ideal okay which is prime and the zero ideal in the polynomial ring is prime because the polynomial ring is in integral domain okay, so An corresponds to, the whole affine space corresponds to the prime ideal maybe the zero ideal therefore it is irreducible we proved that a closed subset in An is irreducible if and only if the ideal that it corresponds to okay namely the ideal of functions that vanish on that closed subset is actually a prime ideal okay.

That is the translation of irreducibility which is a geometric property into the ring theoretic property of primness in the polynomial ring okay. So An is irreducible and therefore any non-empty open set is both irreducible and dense okay, so recall that U is irreducible and dense, so this is something very special okay does not happen for example in the usual topology okay.

For example if you take the topology of the real line given by open sets given by open intervals and or if you take the topology of the plane or two copies of R, the real plane and give the topology to be given by open sets which are unions of open discs then you can see that a non-empty open set need not be irreducible, it not be dense okay but this is very special for the Zariski Topology.

Now the compliment of U which is An minus U is a closed set that is by definition and what is the closed set for the Zariski topology it is of the form Z of I where I is an ideal in the polynomial ring and mind you this ideal cannot be the unit ideal okay because if this is unit ideal then the zero set of that will be the null set and then the compliment of U will be the null set and that will mean that use the full space which is not the case.

And mind you this ideal cannot be the zero ideal so because if it is a zero ideal then the zero set of the zero ideal is the full space and that means that the compliment of U in the full space is the full space and that will force the set U, open set U to be the null set. So our assumptions tell you that I is a proper ideal which is not zero okay, I should say I is a proper ideal not equal to C okay.

Now you see, now let us go back and look at it carefully I mean let us go back and recall the fact that this ideal has to be finitely generated you see we have Hilbert’s Basis Theorem it says that if R is a commutative ring with 1 which is noetherian then any polynomial ring in finitely many variables over R is also noetherian okay, which means that and the noetherian property, one of the definitions of the noetherian property is that every ideal is finitely generated.

Therefore field k is always noetherian because it contains only two ideals namely the full field as unit ideal and the zero ideal therefore a field is always noetherian and therefore a polynomial ring in finitely many variables over a field is also a noetherian ring okay this is because of Hilbert's Basis Theorem or Emmy Noether's theorem and therefore this ideal I is finitely generated okay.

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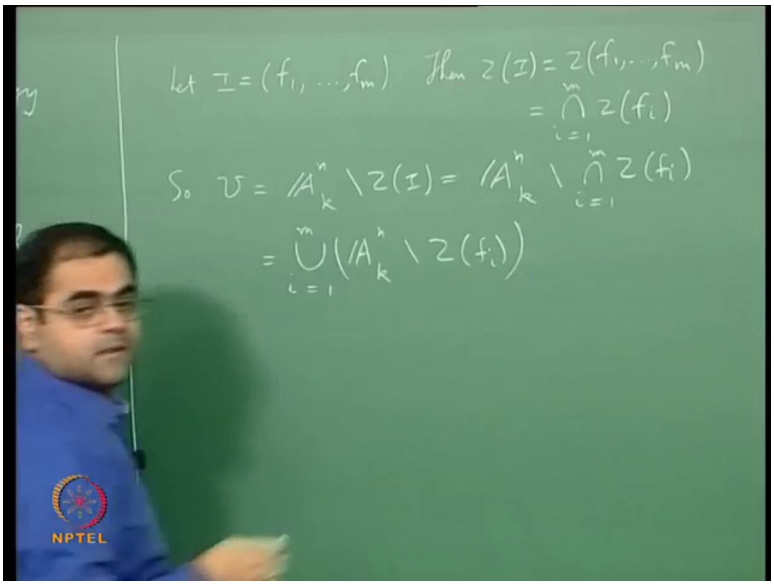

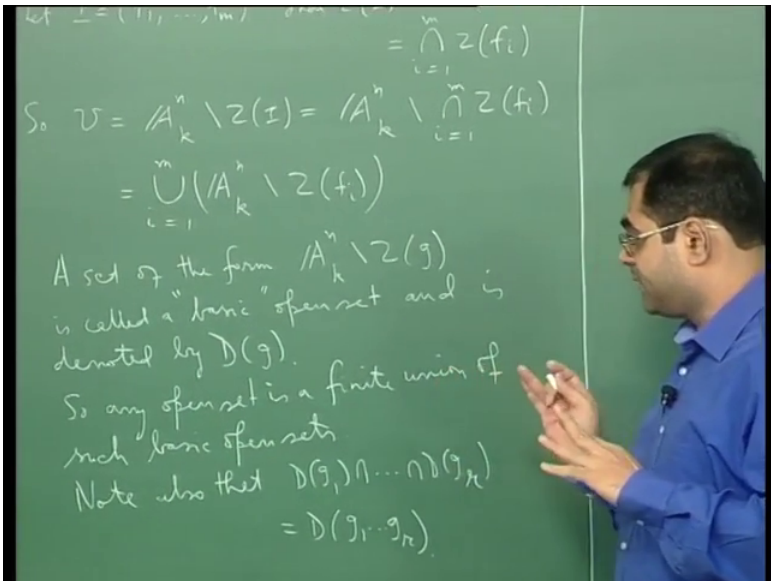

So let I be generated by f1 etc to fm okay, so let I have finitely many generators f1 through fm okay then what is Z of I? Then Z of I is going to be just Z of f1 to fm and that is which is actually equal to you know intersection I equal to 1 to m, Z of f i okay, so the Z of I is set of points in the affine space which are common zeroes for each of the polynomials f i okay, i running through 1 to m and the common zerio locus is just gotten by taking zero locus of each of these sets in intersecting okay.

Now what does this tell you about U? So what will U be? U will be the compliment of Z of I so it will be compliment of this and by De Morgan's laws this is the union I equal to 1 to m of An k minus Z of f okay, so you get this expression which expresses any non-empty, non-trivial open set as a union of open sets of these types but the open sets of these types are special in fact they are the building blocks for the Zariski Topology for the open sets.

Mind you Z of f i is a hyper surface okay it is essentially a hyper surface because it is defined by a single equation okay it is defined by a single equation of course for example we assume that say f i is actually irreducible okay then Z of f i is a hyper surface okay it is defined by a single equation and what this is? This is a compliment of hyper surface okay.

What is this locus? This is the locus where the particular function f i does not vanish, it is the compliment of the locus where f i vanishes. Z of f i is a locus of points where f i vanishes and this is a compliment of that locus okay and so it is a compliment if for example the f i's are irreducible then this is actually a compliment of a hyper surface ,the hyper surface defined by f i, and these sets are very special they are called, they will turn out to be the basic open sets okay.

So a set of the form An minus Z of g is called a basic open set and is denoted by D of g so this is the notation D of g. D of g is a locus where g does not vanish, it is a compliment of Z of g which is a locus where g vanishes okay and this is such sets D g are called basic open sets. Now what we have just seen above is tells you that any open set, any non-trivial open set is a finite union of basic open sets okay.

So any open set is a finite union of such basic open sets okay, now you see there is a, if you have gone through a first course in topology there is a statement, I mean there is a notion called what is meant by a collection of basic open space, a collection of subsets of a topological space is called a collection of basic open sets if there is a collection of open sets such that any other open set can be written as union of open sets in this collection.

So the condition for collection of subsets to be basic open sets is that they should be a collection of open sets and any open sets should be written as union of such basic open sets and you can see that in that sense also any open set is writable as a union in fact it is a finite union we get more, any open set is in fact not just a union of basic open sets but it is a finite union of basic open sets okay.

And the other beautiful thing is if you take the you know, if you take the intersection of basic open sets okay that will also continue to be if you take finitely many basic open sets and take their intersection that will continue to be a basic open set okay, so note also that D of g1 intersection D of gm, r if you want to take r too it is the same m I can, it can be something else gr is actually D of g1 product gr okay.

This is quite clear because you see the D of, taking intersection of all the D of g i's is trying to look at those points where none of the g i's vanish and none of the g i's vanish at a point if and only if the product does not vanish at that point okay, so you see these basic open sets have the property that you take finitely many of them and intersect them the result then, the resulting subset is again a basic open subset of the same type okay they are closed under finite intersections and they are finite unions give you all possible open sets okay.

So this justifies the terminology basic open set from the topological point of view okay so what all this tells you is that the open sets for the Zariski Topology are built up by simply taking finite unions of basic open sets, basic open set is just given by the locus of a non-vanishing of a single polynomial okay fine.

Now that is not the, that is just the beginning of the story in fact the whole philosophy in the most sophisticated form of algebraic geometry is that not only do these basic open sets define the most sophisticated possible object called a scheme in algebraic geometry okay but the fact is even the functions are built by looking at the functions on such small pieces so you see the, by now I think you should have noticed that our focus has started shifting to looking into rings of functions okay.

See the last lectures what we did was we assigned to every affine variety, its co-ordinate ring, the affine co-ordinate ring of functions which is simply the polynomial ring which is a ring of functions on the affine space in which the close sub variety sits divided by the prime ideal given by the ideal of vanishing of that irreducible closed subset okay so this is what we called as the affine co-ordinate ring of an affine variety okay.

And I told you that some, I gave you an indication in the last couple of lectures that the association of a variety to its affine co-ordinate ring is a equivalence of categories, it can also be thought of as a bijection okay so the set of isomorphism classes of varieties is bijective to the set of isomorphism classes of affine co-ordinate rings okay and affine co-ordinate rings are of course given by absurdly they are defined as finitely generated k-algebras which are integral domains okay.

So the moral of the story is I told you that you know the whole affine variety is completely controlled by its ring of functions which is in line with the segment of Felix Klein that this geometry of this phase is controlled by the function on the space so the whole point about algebraic geometry in going from the geometric side to the commutative algebraic side is to completely is to associate to the space its being of functions okay.

So the fact is that not only does, not only do this basic open sets form the building blocks of any open sets but also that the very functions on your space they also come by gluing together or putting together okay just like you put these sets together to get an open set, the fact is to get a function on that open set, you will have to put together functions on the sets like this which you put together to get that open sets okay.

So the fact is this, these are not just basic open sets in the topological sense, they are basic open sets even in the function theoretic sense, even they functions on these will dictate the functions on the union okay so the whole, so what I am trying to tell you is that I am just trying to tell you that it is very important to look at if you want to study functions on open sets okay then you should study functions on basic open sets okay that is what I am trying to tell you.

And why do you look at functions on open sets because you will have to look at functions if you want to go to the commutative algebra side and all this time we were looking at functions on irreducible closed subsets which are sub varieties and the functions where the ring of functions, where the corresponding affine co-ordinate ring which are finitely generated k-algebras which were integral, were domains okay.

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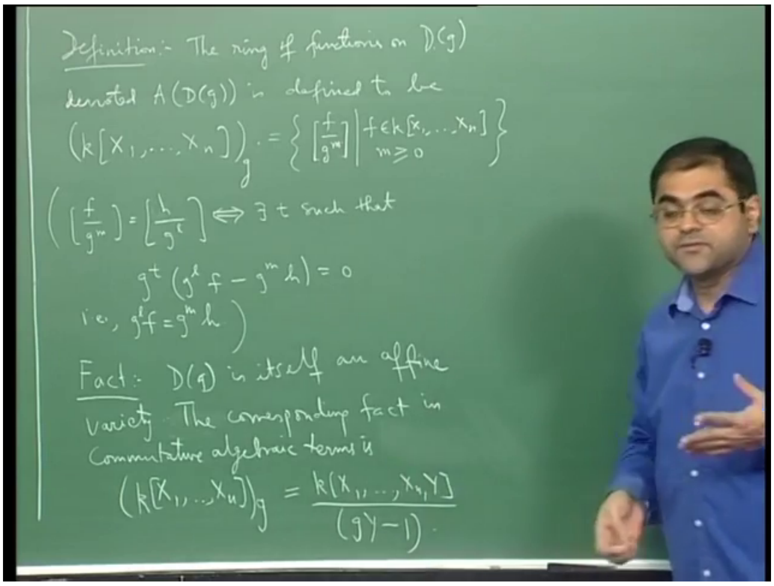

Now if you try to do that open sets they first appear to start with is to, try to understand what are the functions on set like this, a basic open set like this okay so to that, so to get into that so let me make a definition so okay, so definition, the ring of functions on D g is namely denoted A of Dg is defined to be the polynomial ring which is the ring of functions on the ambient affine space on which in which you are considering D of g, localized at g.

So look at this definition okay, the definition says that the co-ordinate ring, the affine co-ordinate ring, the co-ordinate ring of functions is this polynomial ring localized at g okay and if you remember if you go back and go back to commutative algebra what does means is this is, as a set this is, this consists of equivalence classes of the form f by g where f is a polynomial in those many variables and maybe I will have a power of g and m is greater than but equal to 0.

And of course this equivalence class is, I put a square bracket this square bracket denotes equivalence class and you know f by g power m the equivalence class is equal to let me see h by g power l if and only if there exist t such that g power t into g power l f minus g power m h is equal to 0, that is g power l f should be equal to g power m h okay and of course, so in all these thing I must remind you that you, when you localize at a single element in commutative algebra it means that you are taking the multiplicative subset to be the set and taking the power of this element along with the element 1 okay.

So and then it is also important to make sure that the element is not nilpotent okay the multiplicative subset should not contain 0 and if the multiplicative subset is going to contain powers of g and 1 and 1 is being thought of as 0th power of g if you want okay then no power of g should vanish and the fact is no power of g will vanish because the ring here is an integral domain it has no 0 devices in particular it is reduce, it has no nilpotence.

So the condition that g is not nilpotent is not necessary it is automatic okay. So and well so leave alone this probably not so nice looking bunch of equations basically what it says is the set of functions on this locus where g does not vanishes is just given by taking the usual polynomial functions and multiplying them with powers of g inverted okay so this f by g power m can be thought of as f into 1 by g power m which is f into g power m inverted okay.

And that is a very sensible definition because you see on this locus g does not vanish okay therefore reciprocal of g makes sense so 1 by g on this locus if you evaluate 1 by g it is going to give you a non-zero scalar, so 1 by g is certainly a valid function where g does not vanish I mean this is very, this is a very standard thing that we come across always if you whenever a function is non-zero then the reciprocal of the function is also a valid function with the same property's original function.

For example if a function is continuous then if wherever it is not zero the reciprocal of that function makes sense and that is also continuous similarly if a function is holomorphic where the function is non-zero, the reciprocal of that function also turns, becomes holomorphic. If a function if differentiable and if you look at the points where it is non-zero okay, of course you always assume the points where it is non-zero is an open set which will be true because where the function, the points where the function vanishes will always be a closed set because the functions will be continuous basically.

And if you have a function with a certain property then at the locus open locus where the function does not vanish the reciprocal of the function is also have the same property, so you should expect that 1 by g should be good enough only thing is it is not a polynomial but it is a reciprocal of a polynomial and then you see therefore the functions on this this basic open set are given by actually rational functions.

Rational function is a quotient of two polynomials, the point is that the denominator polynomial is always some power of g, of course it may be a honest polynomial, m can be 0 okay or it could have powers of g in the denominator okay. Now this is a definition, now what we need to do is I need to convince you about two things, I need to convince you about that this definition was correct in by looking at it in another way.

And that involves trying to tell you that this D of g which is an open subset of am okay is actually also an affine variety, there is another avatar of this D of g which makes it an affine variety it in fact becomes a close sub variety of An plus 1 okay and this fact is you would have already encountered in commutative algebra when you look that properties of localization, so you know so the fact is D of g is itself an affine variety, this is a geometric fact okay I will explain why that is correct okay.

And this is a geometric fact the corresponding fact in commutative algebra is that, so let me write that, the corresponding fact in terms of commutative algebra is k of x1 etc xn localized at g is actually equal to or maybe so let me write it here, let me rub that out, k x1 through xn localized at g, localization is actually equal to k x1 through xn, then add one more extra variable y modulo gy minus 1 this is that.

So the fact that these basic open sets are actually affine varieties okay affine varieties means they must be close subsets of affine space, irreducibly close subsets of affine space what you must understand is in the affine space in which you have started considering them they are not close they are open mind you, D of g is being considered in Am and in An, D of g is an open set it is a non-trivial open set okay.

So it cannot be a closed set because you know it is irreducible so if it is closed then it has to be whole space okay so because it is, if it is closed it is also dense so if it is closed it has to be equal to its closure and its closure has to be the whole space so it will have to be equal to the whole space so if you considered D of g in An it is certainly pakka open set, it is not a closed set.

But the fact is that in a higher dimensional affine space mainly An plus 1 it sits as a close subset okay so and this is, this geometric fact and the corresponding commutative algebraic factor related as we will see and the fact is that in that bigger affine space this is the affine co-ordinate ring of that subset which is the, which is given by the co-ordinate ring of the full space divided by the prime ideal that corresponds to that subset, that closed subset.

So let me explain this, so what you do is, so let us try to understand this, so here is geometric segment, here is the corresponding segment in algebraic terms and of course when I write here maybe I should say isomorphic okay, this is an isomorphism as k-algebras if you want okay so I have written equal to but actually it is strictly speaking I should say they are isomorphic (())(33:57) okay and of course that isomorphism comes because of universal properties.

The isomorphism in one direction, the isomorphism consists of two homomorphism which turn out to be inverses of each other, the homomorphism from this side comes because of the universal property of localization, the homomorphism from this side comes because of the universal property of the polynomial ring okay so that is how the isomorphism comes.

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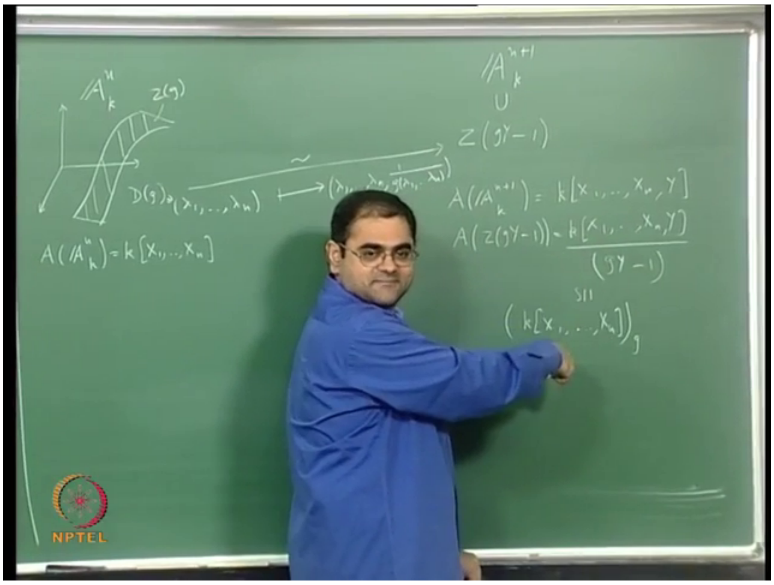

So let me do the following thing so what you do is, so here is, let me draw a diagram it is not a very nice diagram but anyway let me draw it. So here is my An and in An in the space An which will have several co-ordinates I mean several dimensions but I am drawing something like a 3-dimensional thing and then g, the locus of g will correspond to hyper surface, so this is Z of g okay so I am just drawing a diagram that will help you to think but it is not accurately correct.

And it is a compliment of Z of g which is D of g, it is everything outside the hyper surface okay. Now what you do is you take and you see here the ring of functions is A of An k which is k of x1 through xn and g is of course is a non-constant polynomial if you want you can take g to be irreducible but it does not matter, then what you do is you look at An plus 1 so here is, so An plus 1 there is one extra co-ordinate alright and well I am not going to really draw a picture but I am going to do the following thing I am going to look at the zero set of gy minus 1 inside An plus 1 okay, So what does that mean? That means that I am taking the affine co-ordinate ring of An plus 1 to be just k x1 through xn, so I am taking the same n variables plus I am going to add another variable y so this is now n plus 1 variables okay and the way I have written it since I put an extra variable this an is actually sitting as an n dimensional plane inside this n plus 1 dimensional space which is, so this An is actually sitting inside this An plus 1 okay.

And it is the locus given by y equal to 0 when I put y equal to 0 then I get the, I cut down by 1 dimension okay and the corresponding sub variety that you get here is actually this An that is what it means to take this variables to be x1 through xn okay and if you look at the polynomial gy minus 1, gy minus 1 is an irreducible polynomial okay this is something that you check, the polynomial g y minus 1 is irreducible alright.

And therefore the ideal that it generates is a prime ideal alright the ideal that it generates is a prime ideal and therefore the zero set of that ideal is an irreducible closed sub variety alright, so this certainly is an affine variety, this certainly is an affine variety in this affine space of dimension 1 more okay and what is the affine co-ordinate ring of this affine variety? What is the ring of functions on this affine variety?

It is by definition equal to the polynomial ring of the ambient variety divided by the ideal of this variety, the ideal of this close subset which is the prime ideal gy minus 1 okay, see gy minus 1 is an irreducible polynomial and an irreducible polynomial in the unique factorization domain, an irreducible element in the unique factorization domain always generates the prime ideal so this ideal is a prime ideal okay.

And now the more beautiful thing is that there is in fact you can define a map from here to here which is a bijective map okay, the map is given as follows you give me set of points lambda 1 through lambda n, you give me set of points here in this open set, so what it means is that a set of points in An is in this open set if and only if g does not vanish at this point that means if you plug in lambda 1 through lambda n for x1 through xn in g you should get a non-zero scalar okay.

And you know what you are going to send it to its very simple, you are simply going to send it to lambda 1 through lambda n and the last co-ordinate will be 1 by g of lambda 1 through lambda n send it to this point, now this point is a point with n plus 1 co-ordinates, the first n co-ordinates are just as these n co-ordinates okay and the last co-ordinate is such that this point satisfies the equation gy minus 1 equal to zero because the last co-ordinate y is 1 by g.

I have had g is applied to the first n co-ordinates okay so you can check that this is a bijective map okay already you know if you take two (())(40:03) like this and they go to the same one there okay, then they have to be the same here and every point there corresponds to a point here you can get the point here by simply forgetting the last co-ordinate, you restrict to the first N co-ordinates so what I am saying is that the map from An plus 1 to An is a projection, projection onto the first n co-ordinates okay.

And if you project this you will get D g okay you will get D g that is what is happening. Now you can do little bit of calculation and show that this map is actually homeomorphism okay you can try that out, you can show that this map is a homeomorphism the fact is that this map is a not just a homeomorphism, it is actually an isomorphism of varieties okay but that is a fact that we will have to check later on because I have still not defined for you what is meant by a morphism of varieties okay.

Because an isomorphism of varieties is an invertible morphism of varieties, it is a morphism of varieties which has an inverse which is also morphism of varieties okay but just grant that for the moment the fact is that this is an isomorphism of varieties and the philosophy is once you have an isomorphism of varieties that should give rise to an isomorphism of the corresponding affine co-ordinate rings okay.

So if you believe that it will tell you that the affine co-ordinate ring of this must be isomorphic to the affine co-ordinate ring of this but what is affine co-ordinate ring of this? The affine co-ordinate ring of this is this okay but what is this isomorphic too? This is isomorphic to the localization of the polynomial ring in n variables at g, this isomorphism comes from commutative algebra okay so if you believe that it is fair to take the ring of functions on this to be this which is what our definition was okay.

So what you must understand is that if you believe that these two are isomorphic as varieties okay evidence for which is that you can check as an exercise that these bijective map is actually an isomorphism in the topological sense namely a homeomorphism mind you this D g is a subset of An, An has a zariski topology, this is an open set so it has a induced topology okay and Z of gy minus 1 is a close subset here that also has an induced topology, zariski topology and I am saying this bijective map is not just a bijective map it is a homeomorphism, it is continuous in both directions okay.

And that tells you that these two spaces are homeomorphic but it does not stop there they are actually isomorphic as varieties and if they are isomorphic as varieties they are, the philosophy is that they are co-ordinate rings, the rings of functions have to be isomorphic therefore you can the ring of functions on this has to be isomorphic to the ring of functions on this which by definition is this, by earlier definition but that is isomorphic to this because of commutative algebra and therefore it is correct in that sense to take the ring of functions on D g to be this okay.

So this is a kind of (())(43:31) argument which involves certain things that need, that will be checked later okay so this is some justification as to why you should define it like this, of course the other justification is where g does not vanish 1 by g and powers of 1 by g will also make sense as functions and therefore most general functions you can write should be of this form okay.

And I should also tell you that this geometric picture is actually a translation of this commutative algebraic isomorphism okay, this is the co-ordinate ring of this, this is the co-ordinate ring of that and the fact that these two rings are isomorphic as k-algebras is actually reflection of the fact that these two fellows are isomorphic as varieties okay so it is commutative algebra this is the commutative algebra statement these two being isomorphic as varieties is the algebraic geometric statement and they are just you know each is equivalent to the other provided you make the right definitions alright.

So that is to convince you why this definition is correct okay. Okay so I will continue trying to tell you about the points that lie here okay with respect to the Nullstellensatz okay so that will also give you an idea that this is, that will also give you another justification as to why this should be defined as a ring of functions on D g okay so I will do that in the next lecture.