Basic Algebraic Geometry
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Lecture – 12

Capturing an Affine Variety Topologically From the Maximal Spectrum of its Ring of Functions

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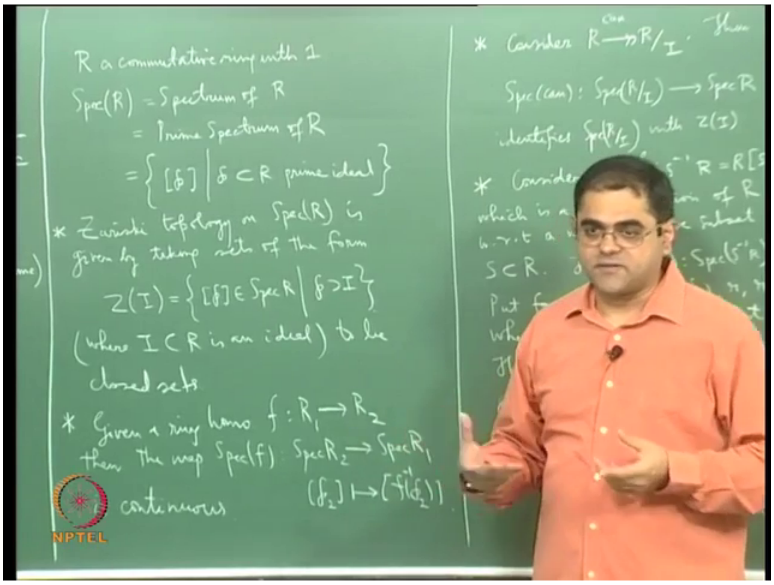

Okay so let me continue with what we were doing in the previous lecture, so let me quickly recall, we start with the commutative ring R with unit element 1 and we define this prime spectrum which corresponds to the prime ideals of R being thought of as points of the prime spectrum spec R and then we define the Zariski Topology on the prime spectrum which comes out of ideals of R given an ideal of R we define a close subset define by that ideal.

Namely it is all those, it consists of all those points in the prime spectrum which correspond to prime ideals which contain the given ideal okay and sets like this form closed sets for a topology called the Zariski Topology, these satisfies the conditions (())(02:21) for closed sets okay and this is called Zariski Topology on the prime spectrum and the formation of the spectrum is actually a functor it is a what is called a contravariant functor okay.

So it is a contravarient functor from the category of rings commutative rings with 1, the objects or commutative rings with 1 and the morphisms are ring homomorphisms which take 1 to 1 and it is a contravarient functor from this category to the category of topological spaces namely it associates to every R spec R which is a topological space and to every arrow, every ring homomorphisms, it associates an arrow in the continuous map in the reverse direction in the corresponding topological space is given by this prime spectrum okay.

We, this reversal of the arrow when you apply the spec is what is meant by its contravarients okay and it is a functor right, of course if you do not know the definition of functor you can always look it up, the idea of functor is something like a function which is generalization of a function, the idea is it has to go from what is called one category to another category and the category is supposed to consists of objects and morphisms, maps between objects.

And of course these maps have to preserve the objects have some structure the maps have to preserve the structure. So for example we are all familiar with so many categories for example category of sets is a category for which the objects is sets and the maps are just maps of sets, the morphisms of the category are just maps of sets.

We are also familiar with the category of groups for which the objects are groups and the maps are the group homomorphisms, similarly you have a category of rings where the objects are rings and the morphisms are the category are the arrows in the categories are ring homomorphisms okay and for example you can also talk about the category of topological spaces where the objects are topological spaces and the morphisms are continuous maps between topological spaces.

Similarly you also can talk about you know category of modules of a ring and so on and so forth okay, so the fact is that a functor from one category to another is that something that does the following thing, given an object in the source category it gives you an object in the target category given a morphism in the source category between two objects it gives you a morphism between the corresponding objects in the target category.

And the only thing is that the arrow of the target category may be the reverse direction if it is in the reverse direction it is called a contravariant functor if it is in the same direction it is called a covariant functor, of course another example of you know a category is the category of vector spaces over a field the objects are all vector spaces over a field and the maps are (()) (05:33) maps that is also a category.

So spec is contravariant functor from the category of rings, commutative rings with 1, the objects are commutative rings with 1 and the morphisms are ring homomorphisms with carry 1 to 1 and from this category spec is a contravariant functor into the category of topological spaces, namely the category for which the objects are topological spaces and the morphisms or the maps are the continuous maps.

So what it does is that to every ring R commutative ring R with 1 it associates spec R which is a topological space given the Zariski Topology and to every arrow in the category of rings namely to every ring homomorphism which takes 1 to 1, it associate continuous map in the reverse direction of the corresponding topological spaces so it is a contravariant functor okay.

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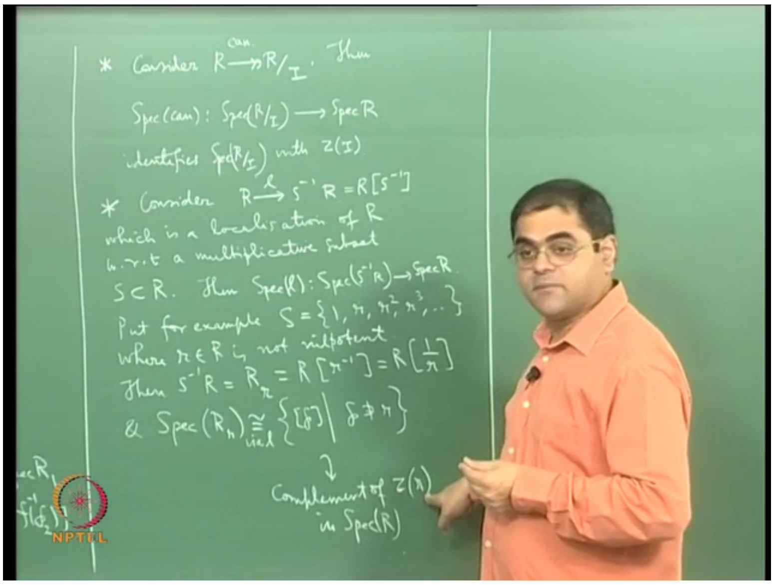

And the point that I want to recall is that if you take the closed subset Z of I that itself is a spectrum and it is actually identified with the spectrum of the quotient of R by I okay and if you look the open subset that corresponds to the compliment of the ideal generated by a single element that also is a spectrum that can be identified with the spectrum of the ring localized at that element okay.

So roughly the point is that the closed subsets corresponds to quotients okay and open subsets given by compliment of a single function or a single element the zero locus of a single element they correspond to localizations by that element okay so the fact is that just like you know in the usual algebraic geometry when you first started out on this side we put you know we put affine space and then we put closed subsets of affine space and then we had arrow reversing equivalence that went from closed subset here to radical ideals on this side.

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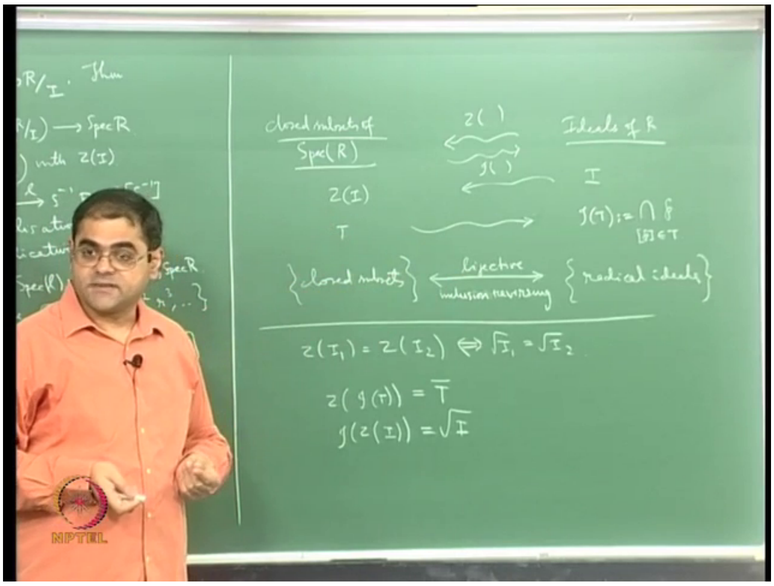

On this side we put radical ideals inside the polynomial ring in as many variables as we dimension of the affine space we start within there is a similar thing that is going on here so what you can do is actually put this side you can just put closed subsets of spec R on this side, on that side you can put ideals.

So if you want I can put ideals of R so again you have, there is Z map like this which takes any ideal to Z of I which is a closed subset in spec R and then there is a map like this which is script I which takes to any subset T, it takes it to I of T okay and what is I of T? I of T is, so T is a subset of spec R so it is a certain collection of points which corresponds to certain prime ideals and I of T will simply be the intersection of all those prime ideals.

So I of T will be defined to be equal to the intersection of all those P such that the point corresponding to P is in T okay and again you will see that here closed subsets if you take the set of closed subsets that corresponds to, there is a bijective correspondence with radical ideals on this side that is ideals which are equal to their radical and this will also be arrow reversing, sorry this will be inclusion reversing.

Namely larger the ideal, smaller the closed subset, larger the closed subset, smaller the ideal, in particular the ideal that corresponds to the unit ideal that will give rise to the null set the ideal which corresponds to 0 will give you the whole space okay and you have statements that are very the statements here which corresponds to the kind of statement that we got in the case of affine space okay.

So here we got it between closed subset of An and ideals of the polynomial ring in n variables over K we have similar statements here, so here also you have you know if you have statements like Z of I1 is equal to Z of I2 if and only if well rad I1 is equal to rad I2 okay you have statement like this then you have also the statement that Z of I of T is just T bar the closure of the set T and you also have the nullstellensatz in very ring theoretic form which is I of Z of I, script I of Z of capital I is rad I, this is the, mind you this statement analogous in the case for the affine varieties was a very deep statement it was a nullstellensatz but it is true in this sense for any commutative ring okay it is already there, it is God given.

So you have this very beautiful thing happening you started the commutative ring R with 1 then there is automatically this beautiful picture which on one side translates from ideals to closed subsets okay and in fact as I told you it is more this is at the level of R but if you want to go outside of R and you want to consider other rings gotten from R like for example if you want to consider quotient rings of R and you want to consider localizations of R then you will end up with going to the closed, the quotients of R will correspond to the again closed sets okay.

And the localizations of R by single elements will correspond to the closed sub, the compliments, the open subsets given by compliments of those closed subsets corresponding to that single element okay the ideal generated by that single element so you get very beautiful you know this is, you my call this as a geometric side, you can call this is as a commutative algebraic side but the beautiful thing here is the geometric side has been cooked from the algebraic side that space spec R itself has been cooked up from R, I mean there is no in analogy here when you compare it with here.

We started with polynomial ring in n variables and that was being thought of as acting on as functions on the affine n space okay we already have a space here but so you had a space namely Km and you had polynomials on that space and using these polynomials you define the Zariski Topology here and you got all these nice correspondences and equivalences but in this case what you are doing is you are starting with the ring okay you are cooking up a space using that (())(13:41) spectrum and then you still get this beautiful correspondence all these properties which are just very very analogous to what you have got in this case okay.

So this is at a first point you should therefore feel that you know you should be able to catch the space from the ring of functions okay so I will tell you what the arrow, what is this max spec okay it is very simple so what you do is that you give me anything on this side namely you give me a finitely generated K-algebra which is an integral okay.

For example say this something like this then of course this is also commutative ring with 1 alright take its spectrum okay take spec of this the spec of this will contain all prime ideals okay but what you do is you take a smaller subset namely you take the max spectrum and what is max spec, it is not prime ideals but it is maximum ideals so you know every maximal ideal is prime ideal and converse is not true so you restrict to only the subset of maximal ideals okay.

That will be a subset of the prime spectrum the maximal spectrum is denoted max spec it is a subset of the prime spectrum is given by spec and since a subset of a topological space automatically gets a induce topology what will happen is that max spec will also get a topology it will get a Zariski Topology induced from the topology on spec okay.

Beautiful thing is if you take max spec of this along with that topology it will be exactly homeomorphic to Z of P okay so in particular what I am saying is if you take max spec of this you will simply get back affine space upto homeomorphism, see we have already seen this in the nullstellensatz I have already told you that max spec of this is actually Kn because every point in Kn corresponds to a maximal ideal in the polynomial ring that is exactly what the nullstellensatz says.

So what the nullstellensatz says is that max spec of this as a set is exactly affine space the point with co-ordinates lambda I corresponds to the maximal ideal given by XI minus lambda I with generators XI minus lambda I right that is what the nullstellensatz says, but that is just identification of max spec of the polynomial ring in n variables with An with Kn in fact it is just set theoretic identification.

But what the claim now is, is that if you take max spec of this with the Zariski topology induce from spec then this identification of max spec of k x1, etc xn with Kn will actually be an topological isomorphism homeomorphism from affine n space to max spec of k x1 etc xn so it is not, so the nullstellensatz is not just a set theoretic bijection it is a topological isomorphism so homeomorphism okay.

So in other words even if I do not have this side I can get back my affine space simply taking max spec of polynomial ring in n variables so that is a beautiful thing so this whole side I can reconstruct from here by simply applying this max spec and then it is a matter of exercise to check that this followed by this is the identity and this followed by this is identity so that these two are actually inverse associations okay.

And of course when you take that you will have to worry about isomorphisms okay but that can be done okay, so in fact to check that these two are inverses of each other in a rough way you can do that checking but then I need to tell you what are the morphisms on this so I need to go on and explain morphisms of varieties okay.

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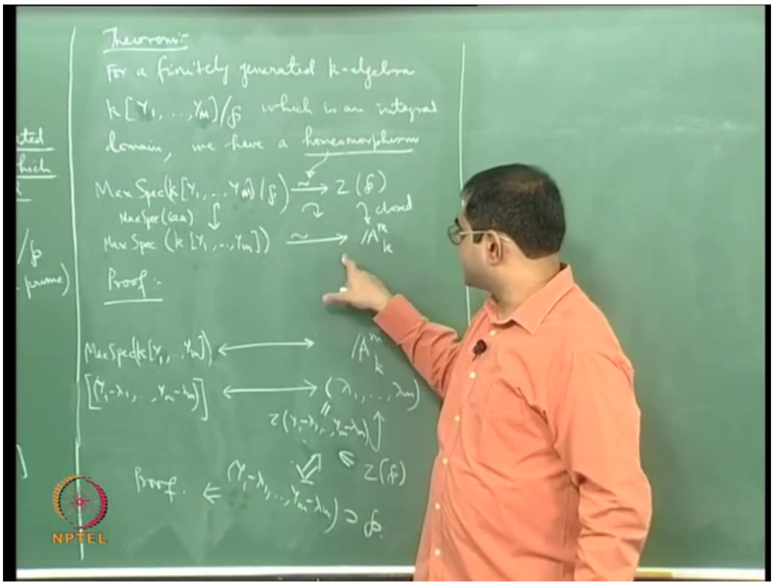

So let me write that, for a finitely generated k-algebra, k y1 by p which is an integral domain we have a homeomorphism max spec k y1, yn by p which is an integral domain, we have a homeomorphism, max spec k y1, yn, mod p to Z of p which is considered as a closed subset of Am so maybe I, let me use the m here, here also put m here Am, okay. So the point is that it is a homeomorphism okay and how does one see it, let me try to explain that, so this is exactly how you go from here to here okay.

So you know if I start with Z of P inside Am where p is the prime ideal in polynomial ring over k with m with m variable which is the affine co-ordinate ring of functions on Am larger space okay then what is the ring of functions on the Zp, it is the quotient of the ring of functions on Am modulo p mainly it is the, so that is what you get when you go from here to here and then when you come back what you will get is max spec of k of y1 etc ym by p and what that statement says is that, that is exactly Z of p.

How is that true? That is very very simple, see what happens is that so, what is the prof for this, I will just outline that so if you want theorem it is a theorem in its entirety but it is actually a corollary because all these are all more and more grandiose versions of the nullstellensatz okay I mean which you get by using after putting all these Zariski topology and so on and so forth and using f commutative algebra okay.

So these are all various grander and grander versions of nullstellensatz, so let me explain this so you know this is the nullstellensatz from Am to max spec k x1, sorry y1 etc ym that is your map like this which is given by you know give me a point lambda 1 etc lambda m that corresponds to the ideal y1 minus lambda 1 and so on ym minus lambda m this is the ideal, this is the maximal ideal in k x1, k y1 etc ym and then I take the point I put this square bracket to say that I am considering it as a point of the maximal spectrum okay.

So this is, this bijection is simply because of nullstellensatz which tells you that you know every ideal of this form is maximal that is true for any field and conversely where field is algebraic closed every maximal ideal is of this form that is the nullstellensatz okay. Now what you do is you go to the subset which corresponds Z of f okay just think what are going to be the points here which are going to lie here, mind you the point with co-ordinates lambda 1 through lambda m lies in Z of p if and only if every function in p vanishes at each of those points that is the definition.

You see what is this, this is z of y1 minus lambda 1 dot dot dot ym minus lambda m this is what it is this is just zero set of this is just the 0 set of this maximal ideal okay so and you know z of this is contained in this belongs to Z of p I mean Z of, this is a subset of Z of p if and only if p is inside this maximal ideal okay.

In other words, yeah if and only if this maximal ideal contains that is because you can apply the script I if you apply script I okay to z of y1 minus lambda 1 etc ym minus lambda n contained in Z of p, if you apply script I both sides, you will get I of Z of that which is just radical of that which is same as that maximal ideal contains I of Z of p which will be radical of p which is p because radical of p is p itself because p is prime, this is just I am using the nullstellensatz.

So what this will tell you that is that this point belongs to this if and only if this is if and only if by the nullstellensatz the maximal ideal y1 minus lambda 1 etc ym minus lambda m contains p okay and what does this mean, this means that this is in Z of p as far as zariski topology on the spectrum is concerned so it is an element of max spec k of y1 etc ym mod p.

See what is max, a maximal ideal which contains p the set of maximal ideals which contain p or precisely the set of maximal ideals in the quotient by p the set of maximal ideals of a quotient of a ring by an ideal is precisely the set of ideals of the ring which contains that ideal okay.

There is a correspondence between a ring, the ideals in the quotient of a ring by an ideal and the original ring and what is the correspondence, an ideal in the quotient corresponds to an ideal which contains the (())(25:10) and under this correspondence prime ideals go to prime ideals, maximal ideals goes to maximal ideals it is an inclusion preserving correspondence, so the maximal ideals in k y1 etc ym mod p are precisely the maximal ideals in k y1 etc ym which contain p okay.

So that will tell you that so this will tell you this will imply the proof okay, so you know if you want to complete this diagram what is happening is that here I have max spec of k y1 etc ym which is identified with Am by the nullstellensatz and you know from this to this there is a quotient map and there map in this direction is the max spec of the canonical quotient map. See there is a canonical quotient map from k y1 etc ym to its quotient by p.

If you apply spec to that you will get, the arrows will get reverse you will get a map from max spec of the quotient to the max spec of the parent of the quotient and it is this map which is at closely immersed namely it is an identification of this with a close subset of this what is that closed subset with which this is being identified? That closed subset is this Z of p, that is what it means this diagram commutes.

This is the nullstellensatz and this is also derived from the nullstellensatz so it is actually the nullstellensatz for affine space you restrict it to a closed subset that is exactly the statement of this theorem, it is simply nullstellensatz you know restricted to closed subset nothing more than that okay so that tells you how you get this bijection okay now to make, to show that it is a homeomorphism okay you will have to show that this is a, it is a continuous map in both directions okay.

Now that is something that you can very easily check okay so for example let us show that this map the identification due to the nullstellensatz okay, so the homeomorphism I am talking about is this one, let us first show that this bijective map which is the nullstellensatz is actually a homeomorphism let us show that that is also pretty easy to see okay so let me do that first.

So it is very clear that I have this commutative diagram, I have this commutative diagram alright because of what I explained so you have this bijection, you have this bijection this bijection is actually coming from here to here and this bijection is the nullstellensatz I just show that this is homeomorphism okay, once I show this is a homeomorphism, this is a closed subset, this is a closed subset okay and it will follow immediately that this is also a homeomorphism okay.

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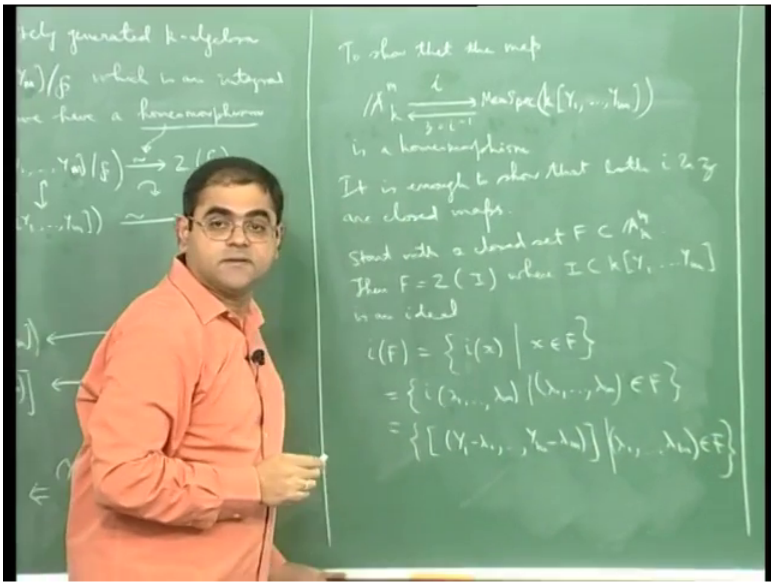

So to show that this is a homeomorphism that map, let me write this map Am to max spec k of y1 etc ym is a homeomorphism, let me first show, so I will first show that this arrow is a homeomorphism okay then I will show this arrow is a homeomorphism okay I will deduce that this arrow is a homeomorphism and I am done okay.

So how do I show something is of homeomorphism? I will just have to show that you know it is already a bijective map one way of showing it is a homeomorphism is to show that both the map and its inverse are closed okay, so you know the condition for a map to be continuous is that the inverse image of every open set is open and that is also equally, that is also equivalent to requiring that the inverse image of every close set is closed okay.

And therefore it will check that there is a homeomorphisms is enough to show that this map, this arrow is closed and it is also which means that maps a closed set to a closed set and to also show that the reverse arrow is also closed because it is already a bijection okay so let me give this some name let me use i and let me call that inverse map in this direction as small z which is i inverse okay.

And the reason is why I want to use i is because if you give me a point lambda 1 through lambda m I am what I get on this side is actually the ideal of that point which is y1 minus lambda 1 through y1 minus lambda m considered as a point of the maximal spectrum which is a subset of a prime spectrum okay and the map from this direction is, give me a maximal ideal I am just looking at the zero set of that maximal ideal which is a single point.

The nullstellensatz actually says that the zero sets of the maximal ideals are exactly the single point source okay. So that is the reason for this notation i and z, so it is enough to show that both i and z are closed maps it is enough to show that they are closed maps so in other words what does it mean it is enough to show that the map is said to be closed every image of a closed set is closed okay.

And it is equivalent even showing that they are open maps but we always work only with closed sets because that is how the zariski topology is defined, it specify that the zariski topology, the topology is specified by only defining closed set okay so we always try to do all the checking with respect to closed set, so you see so let us start with the closed set here and let us see what its image that is.

So start with a closed set F inside Am then you see what is F, F is z of some ideal I where you know I inside k y1 etc ym is an ideal this is the definition of closed set in the zariski topology now what is i of F, i of F is look at this it is a set of all maximal ideals, so let me write i of F is just i of x where x belongs to F by definition is the image of the set under a map and so this will be equal to i of, how does a point, so you know this x in F will have co-ordinates.

So you know it will be i of lambda 1 etc lambda m where lambda 1 through lambda m is a point of F so this will be, but i of this is supposed to be the maximal ideal corresponding to this point considered as a point of the maximal spectrum is a subset of a prime spectrum so what I will get is I will get the maximal ideal y1 minus lambda 1 and so on ym minus lambda m such that lambda 1 etc lambda m belong to F this is what I will get okay but what is F?

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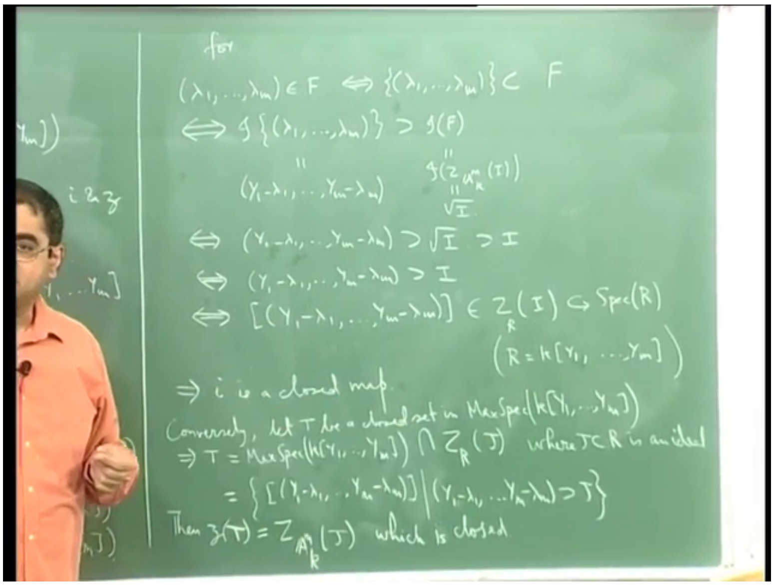

F is by definition, F is Z of i we will have to use that and we like probably apply the nullstellensatz so you know lambda 1 through lambda m belongs to F if and only if the single term point lambda 1 through lambda m is a subset of set F that is if and only if you know I can take ideal of the subset so I of lambda 1 lambda m contains I of F okay because you know when you apply script I the inclusion is reversed.

But this is, but what is I F, a point, it is just y1 minus lambda 1 etc ym minus lambda m and what is I of F it is just I of Z of capital I which is rad I okay it is just rad I so what you are going to get is, so that so the condition becomes if and only if the y1, the ideal y1 minus lambda 1 etc ym minus lambda m contains rad I okay but you see this is equivalent to saying that this contains I okay.

Because you know because if an ideal contains I, if one ideal contains the other then the radical of that ideal will contain the radical of the other okay so this is equivalent to saying that y1 minus lambda 1 etc ym minus lambda m contains I itself okay, of course if y1 minus, if this ideal contains rad I it also contains I because rad I contains I, conversely if this contains I then it will also contain rad I because you know what is rad I it is all those elements some power of which is in I.

So if some power of an element is an I then that power of an element is in this but this is a maximal ideal so it is prime so that element has to be in this so these two are equivalent because the thing on the left side is a maximal ideal which is actually prime it is a primness okay, so but what does this tell you but this is if and only if the point corresponding to this y1 minus lambda 1 etc ym minus lambda m this the point corresponding to this spectrum is in Z of I because this is how the Z of I is defined in the spectrum of the ring okay.

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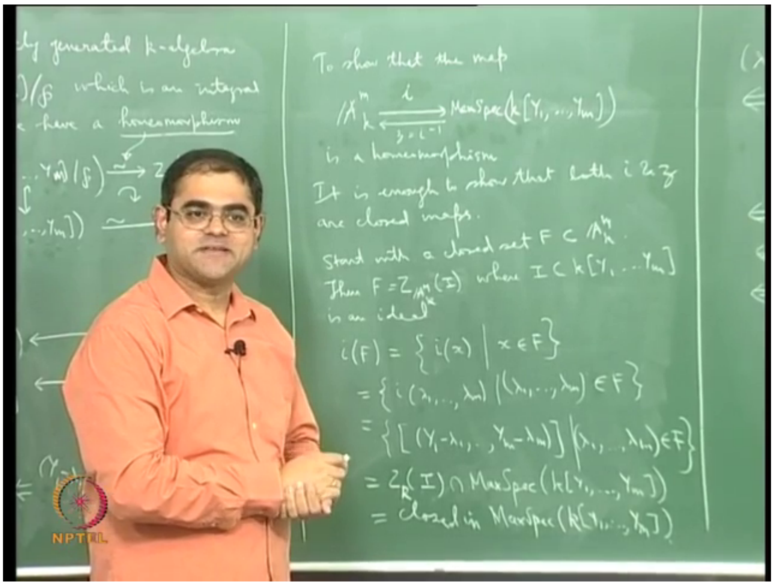

So in other words what is this? This is just Z of I and so I of F is Z of I which is a close subset so you know I should say if Z of I intersection of the maximal ideals okay which is closed in max spec okay, mind you if I take Z of I in the spectrum it will consist of all those prime ideals which contain I, but I want I am looking at only maximal ideals which contain I so I will get Z of I intersection the max spec.

And what I wanted to understand is there is a small element of confusion this Z of I is the zero set of I in if you want I will put a subscript here I will put Am okay this is zero set the ideal I in Am right whereas this 0 set of in spec R so there is a difference between this Z I and that Z I okay so let me do that so this is, so I need to write that let me put S p, so maybe I will write it little larger so that you know you see the difference so here is, this is Z in Am k of I okay.

And this is Z, I will put Z sub of R of I okay where Z sub R of I this is being considered in spec R where R is of course k y1 etc ym okay so and this Z is Z sub, this is Z sub Am, so let me correct that also okay so the zero sets are being considered in different zariski topologies, one is the zariski topology in the affine space, the other is the zariski topology in the prime spectrum but the beautiful thing is it is the same ideal.

You start with the closed set here that is given by a ideal, it is the same ideal that comes from that side (())(40:07) that makes the image of that a close set it is the same ideal okay, so that tells you that I is a closed map okay, so this is the reason for that last equality so this implies that I is a closed map and in principal one should be able to reverse this whole argument and show that I inverse which is Z is also a closed map.

So how will you show that that is also pretty easy, so what you do is conversely let T be a closed set in this, in max spec k y1 to ym okay so what does this implies T is actually max spec k y1 through ym intersected with a closed subset of the bigger space which is a spectrum of k y1 to ym okay and closed subset in that is the zero set of an ideal so it will be Z sub R of some J where J inside R is an ideal okay.

And what is this, so this so what is Z sub R of J it is all those prime ideals in spec R which contains J if I intersect with a maximal spectrum I will get maximal ideals in spec R which contains J so this will be the set of all maximal ideals in this which by the nullstellensatz is the form, are of the form of x y1 minus lambda 1 ym minus lambda m these are all those maximum ideals such that the corresponding maximal ideal y1 minus lambda 1 dot dot dot ym minus lambda m contains j okay.

And this is what T is you can see very clearly that Z of T is just small z of t is Z sub Am of J so it is closed so let me write that, have a little space so let me write it anyway then small z of T is just Z in Am of, Am sub k of J and which is closed okay. So what is z of T it is going to be you know all those points of Am which corresponds to maximal ideals which contain J, that means it is going to be all the points of z of J in affine space okay.

So z of T will just be Z of Am k of the ideal J, so what this tells you is that the inverse map of i name small i which is small z that is also closed therefore this is a homeomorphism okay so the moral of the story is therefore that this lower arrow which is the nullstellensatz which says that every maximal ideal corresponds to a point okay that identification is actually homeomorphism of the (())(44:37) spaces.

And now it is a matter of routine checking which I want you to do to show that you know under this homeomorphism okay if you, see you already know that this diagram commutes okay you know that this is a subset here this is a subset there and you know that this bijection restricts to this bijection, these are closed set, these are closed set okay, a continuous map restricted to a subset will be continuous for that subset with the induce topology.

So what it will tell you is that this the arrow in this direction will be continuous okay and because this is a close subset here, that is a closed subset there and similarly the reverse arrow will also be continuous so that will also tell you that this is also a homeomorphism so it is not just a bijection, so that will prove the theorem that you have a homeomorphism between Z of p and max spec of this and of course this Z of p is Z in Am okay.

So moral of the story is that if I start with Z of p here in Am I take its affine co-ordinate ring, I get this quotient ring, if I again take its max spec what I get is something that is homeomorphic to the Z of p okay, I get back Z of p, so it tells you that if I go like this and come back I get something that is not exactly the same as this but something that is isomorphic to this, isomorphic in the sense on this side at least you have been able to prove that it is a topological isomorphism.

But in fact we are going to prove, we are going to define what is meant by a morphism of varieties and we will show that this map and for that matter this map you can also think of it as a in the most general sense you can even think it as a morphism of affine varieties, isomorphism of affine varieties okay so the moral of the story is if I start with Z of p then I go the affine co-ordinate ring then if I come by applying this A and then if I come back by applying max spec what I get is not just something that is homeomorphic to Z of p.

In fact we will get something that is isomorphic to Z of p, isomorphic as affine varieties and what is isomorphism means it should be an invertible morphism so I should define what is meant a morphism of affine varieties and that is something that I will do, so I am saying that still the picture is not over, it is on this side what we have got in this theorem is just a homeomorphism but it is not just a homeomorphism, it is even an isomorphism of varieties okay.

And that is the story if you go from here to there and come back okay and if you go this way okay, you will also have to show that I mean the same kind of argument should convince you that if I start with this okay and if I take max spec of this okay then max spec of this can be identified with this because of the literally the same argument and then if you take the affine co-ordinate ring of that I will get something that is isomorphic to this okay.

Only thing is that the Y I is instead of calling the variables as y1 through ym I might call them as if you want you know say t1 to tm or some other names I could give but in any case I will get exactly a polynomial ring in m variables divided by a prime ideal in that polynomial ring which will go to this prime ideal under an isomorphism between this polynomial ring in the y's and the other polynomial ring that I started that I am thinking of.

So the point is if you go start from here go there and comeback what you will get is an object upto isomorphism varieties and conversely we will start with an object here, go there and comeback what you will get here is an isomorphism of rings okay, so that is what happens okay, so I will stop here and probably in the next lecture we will try to understand what happens for an open set and try to understand how to define morphisms.