**Basic Algebraic Geometry**
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**Module 1**
**Lecture 1**
**Morphism-Local Rings-Function Fields-Nonsingularity**

Okay, so welcome to this first course on algebraic geometry. So let me begin by trying to tell you what algebraic geometry is all about in its greatest generality it is trying to study the geometry of the set of common 0's over bunch of polynomials, okay.

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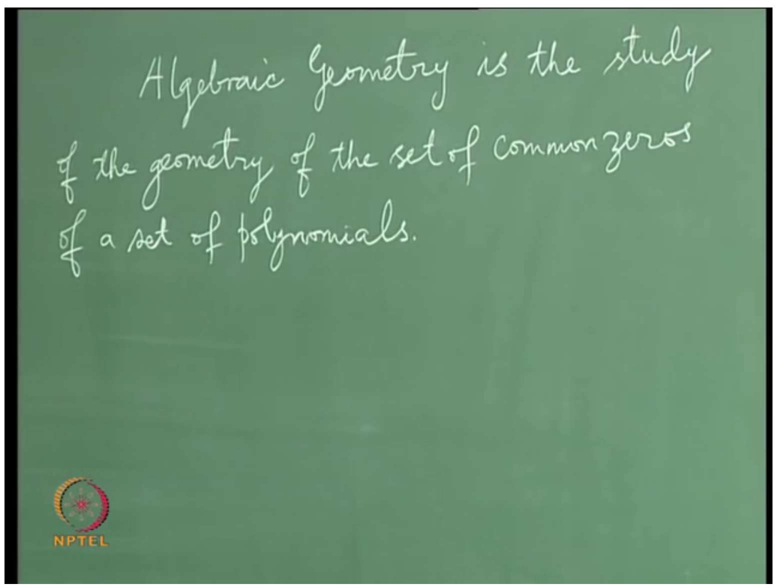


So you know let me write that down, so that is in one line what algebraic geometry is, okay. So beginning with this let us try to take the discussion further, so when I say a set of polynomials of course I should tell you where this polynomials are from, okay and then I shall also tell you by that I mean I should tell you these are from polynomials in how many variables, okay.

So you must have fixed number of variables and then I should also tell you about the coefficients of these polynomials because of course for example we are used to writing polynomials over real numbers which means polynomials of the real coefficients and if for example also with complex coefficients, okay or sometimes we also look at polynomials with just the integer coefficients, okay. So the coefficient usually come from a ring, okay.

So what is happening is that you see in this way the ring of polynomials in several variables over a given ring comes into the picture, okay.

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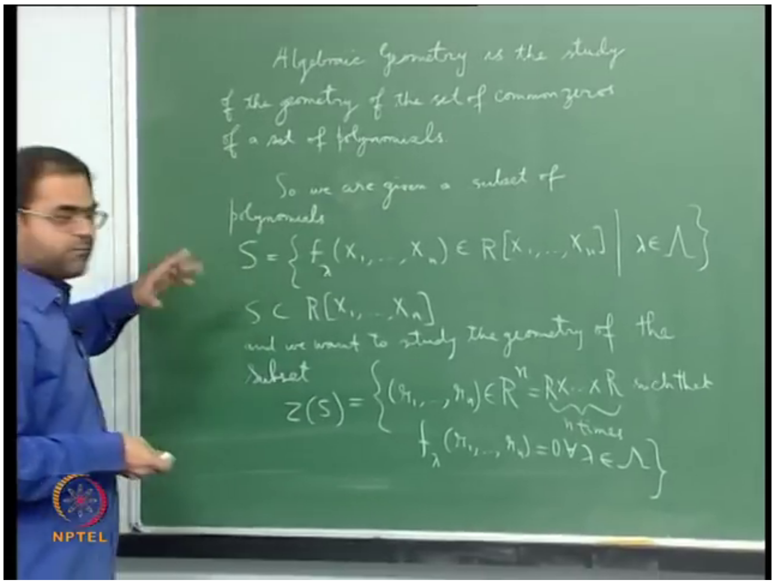


So you see so what you do is so we pick or rather we are given we should not say we pick so we are given a collection of polynomials let me say a subsets of polynomials so let me write them as f sub alpha of X1 etcetera Xn S is the set of all polynomials f sub alpha X1 etcetera Xn and this is each one of these f alpha is a polynomial in a polynomial ring, okay. So this ring is the ring the commutative ring that consists of polynomials in the variables X1 through Xn and with coefficients in the commutative ring R, okay.

So of course I will not repeat this often we are always going to be worried only about commutative rings and the commutative rings are assumed to be with a unit element that is with 1 and we had always assumed that homomorphism is a commutative rings carry one to one, okay. So R is a commutative ring with 1, okay and X1 through Xn are variables n variables and this is the polynomial ring over R in n variables and each f sub alpha is a polynomial it is ring and you take some of these a subset of these polynomials so this is alpha is some indexing set let me put it is as lambda if you want, okay in which case I add another change it to small lambda for more coherence.

So this is S is a subset of the ring of polynomials in n variables over R, okay and we want to study the geometry of the subset is Z of S the set of common 0's of S and that is defined to be the set of all r1 etcetera rn n tuples of elements of r, okay so this just r cross r n times this is just the Cartesian product of r taken with itself n times. So these are n tuples of elements of r each one each ri is an element of small ri is an element of capital R such that the if you plug in Xi equal to Ri, okay that is if you substitute for Xi the corresponding Ri in this tuple, okay or in other words if you substitute for the tuple X1 etcetera Xn the values r1 etcetera rn in that order then you will get and when you evaluate this polynomial, okay then you will get an element of R and that has to be 0 and that should happen for every lambda, okay then and only then is a point of rn in this set, okay.

So such that f lambda of r1 etcetera rn is 0 for every lambda belonging to capital R, okay. So maybe this I will just write it as such that, okay.

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So you see so basically what is happening is that you already have two objects here on the one hand you have rn which is R cross R cross R n times, okay and on the other hand you also have the polynomial ring over R in n variables, okay. And what is happening is that I am given a subset S of the polynomial ring and I am associating to that subset the set of common 0's of that subset which is a subset of rn, okay.

So and the purpose of algebraic geometry is to study the geometry of this common set of 0's, okay. So this is the general picture, so you can see that already there are two sides to the picture there is one side which is the geometric side, okay where you have this space where you the space were you are looking at the 0's, okay. And you have here the algebraic side which consist of essentially your looking at the polynomial ring, okay. So you can already that there is a commutative algebraic side and there is an algebraic geometry side.

So algebraic geometry is all about going from this to that and that to this based on the properties on both sides, for example you could have a properties of this set which is geometric properties, okay and they would translate into some properties into some properties connected on this side and the properties that are connected on this side a ring theoretic properties or which maybe ring theoretic properties or they therefore they could be ideal theoretic properties or more generally they can also be module theoretic properties, okay because modern commutative algebra is not just the study of the rings and ideas but it is also the study of modules because the notion of a module generalizes the notion of an ideal and also that a vector space at the same time and is more versatile, okay.

So the properties on this side are geometric properties the properties on that side are commutative algebra properties and it this dictionary is setting up this dictionary which is the subject of algebraic geometry, okay. And but of course there are several things that I will have to explain to you, first of all I have not so let me write that here so this is the geometric side and this is the commutative algebraic side, okay. And in some sense therefore you can say that algebraic geometry and commutative algebra are kind of married to each other in that sense, okay there are two sides of the same coin, okay.

So but then I will have to explain to you what you mean by geometry by of subset, okay because that you know several things but for example it is at the base level it will involve some topology and then on top of that it will involve further properties topological properties and on top of that it will involve some properties connected with manifold theoretic properties, okay and so and so forth which we will try to explain but let me at this point go to something else, okay.

So first of all you know if you give me an equation like this forget even a set of equations suppose I give an single equation over a ring, okay it might turn over that this set maybe empty, okay that is the problem this set if it is going to be empty it does not there is nothing interesting to study because there is nothing to study first of all. So that can happen very easily, for example you know if you take the ring R to be real numbers and you know if I take something like just one variable, okay and if I take the and if I call that variable so I just take real numbers with one variable X1 and if I take the equation X1 square plus 1 then you know that it has no 0's in real numbers, okay because obviously you know because the 0's are non-real they are complex, okay. So in so it is very possible that if you work over a general ring, okay you are this is particular set the 0 set it turn out to be empty and then you are there is nothing to study, okay.

So at this point there is a I should say a dichotomy the subject actually breaks into or rather can be divided into two parts, so there is one question that tells you that if you are going to work over rings such that this is never going to be empty, okay then of course you have these are good rings over which you can do this kind of study, okay. The other thing is the question that what you do if this ring does not have those properties.

So the answer to that is there is an answer to both, so if you so the first question the first part is you restrict to rings which are algebraically closed fields, okay if you restrict to rings which are algebraic closed fields then this set is never empty for this n collection of polynomials, okay I will explain what this means later and you can do geometry, okay. So that is called variety theory and that is what is usually done in a first course in algebraic geometry. Then the other thing is what you do with the general ring, okay for example I would like to have I would love to work over the integers, for example you know questions like Fermat's last theorem it also involves a an equation in three variables it is an equation with coefficients in integers then you are trying to ask whether there are non-trivial integers solutions apart from the easy solutions, okay.

Then you also need to solve questions like that and to solve questions like that of course there are equations over integers which to have solutions, for example x square plus 1 equal 0 is variable and equation with integer questions and it has no solutions in integer it does not have any solutions even in real numbers so how can it have solutions even in integers. So the question is how do you deal with such things so there is a part of algebraic geometry slightly more sophisticated area of algebraic geometry which deals with such things and that is called scheme theory and this scheme theory is the it is the modern language of algebraic geometry and that is usually what is covered in a second course in algebraic geometry because it involves far more machinery, okay but what we will be doing in this course is that we will be safely restricting ourselves to the cases when this set is non-empty, okay.

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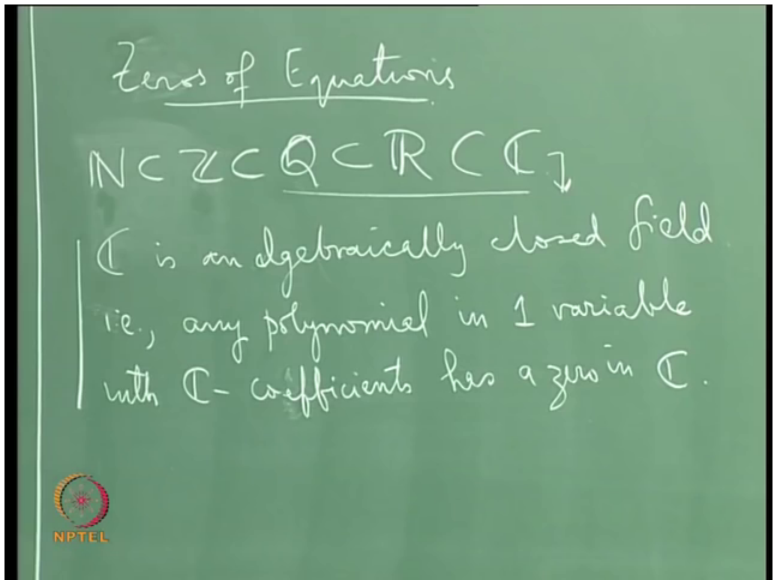


So what I am going to do is that I am going to tell you something about I am going to tell you about solving equations or rather, okay 0's of equation. So well so the first thing is you know I see what I have come across in any first course in algebra normally the usually we start with integers and then you know you extend them to rational numbers where of course you could also take natural numbers before this, okay you can well maybe I can even do that it is not a big deal I can take natural numbers then I can extend them to integers then there are there is field of rational numbers, then there is a field of real numbers, there is field of complex numbers this is how it goes and every time you have a bigger number system and that is because essentially you want to solve equations.

So number system become bigger and bigger because you have some equations for which you do not have solutions so you have to make your set of numbers bigger, okay. So for example natural numbers the counting numbers 1, 2, 3, 4 does not have 0, so an equation like x plus 1 equal to 1 which has a solution x equal to 0 will not have a solution here and an equation like x plus 1 equal to 2 I mean x plus 1 equal to or rather x plus 2 equal to 1 which is solution x equal to minus 1 also does not have a solution here so you are forced to go to integers and then you have a equation of integers which does not have a solution of integers which have solution only in rational numbers for example things like 2 x equal to 3 its variable and equation over integers, okay but the solution is 2 x equal to 3 the solution is x equal to 3 by 2 which is a rational number so you have to extend.

So finally what happens is that you see that you come to fields and you come to field extensions, okay and the point is that every time you ask the question when you get a bigger number system you ask the question well if I write a polynomial in that with that questions will the 0's the polynomial always lay there and if the answer is yes at some point then that is called an algebraically closed field, okay. So the fact is C is an algebraically closed fields, okay C is an algebraically closed field.

So and what does this mean that is so that is any polynomial in one variable with C coefficients has a 0 in C, okay. So and in fact you know if a polynomial has a 0 if you call the polynomial as fx f of x of course it is one variable so I should just call it as f of x where x is a variable and if it has a 0 which means it has a value lambda such that f of lambda equal to 0 then you know x minus lambda is a factor of the polynomial and by the divisional algorithm you know that the polynomial becomes x minus lambda times another polynomial of lower degree of one degree less, okay and in this way you can continue and well (())(20:12) polynomial of lower degree that you get that also will have a 0 in the complex numbers and you can factor that near factor out and if you do like this what it will tell you is that finally any polynomial can be completely split into linear factors, okay that is what it says.

And this is the property of the field of complex numbers which makes it algebraically closed and in general this is the definition of what and algebraically closed field is and algebraically closed field is a field such that you take any single polynomial in one variable over that field then all its 0's are in that field that it means that you can find all the 0's in the field itself you do not have to extend the field, you do not have to extend your number system to a something bigger that you will get the 0's, okay.

So basically what it tells you is that you know if your so what it tells you if you go back to our setup what it tells you is that if I take R equal to C if I take my ring to be C which is in fact a field, okay and if I take n equal to 1 then I have C here and here I will have C of X, okay polynomial ring in one variable and then it tells you that you give me if you give me a single polynomial if this set S is single polynomial then the 0 set of that is non-empty in fact it will have the number of 0's will be equal to the degree of the polynomial but of course you have to count the 0's with multiplicity okay you have to the count the 0's with multiplicity after all the polynomial maybe x minus 1 to the power of 5, okay.

And then well if you want to count the 0's with multiplicity then you can think of them as five 0's but well if you think of it as a subset here you get only point namely the point 1, okay. So what this tells you is if you are working on algebraically closed field and there is only one variable involved and you are looking at only one polynomial then you end up in a situation where this set is 1 set is non-empty, okay.

Now the question is mind you our original question is we are not looking at a single polynomial we are looking at a bunch of polynomials, okay and this (())(22:33) even be finite this collection of polynomials will not even be finite. And not only that we are not looking at a polynomial in one variable we are looking at the polynomial in several variables, okay what the algebraic closure property tell you is that if you are looking at the polynomial in one variable and if your coefficient ring is an algebraically closed field then if you look at the 0's over single polynomial then it is not going to be empty, okay but we want something very very general to happen, okay the answer to that is so you might expect that should you put something more some more conditions than just the field being algebraic closed for such a thing to happen and the answer is no, the amazing answer is yes the amazing answer is this itself will ensure that so long as this set S is good in decent way if R is an algebraically closed field then Z of S can never be empty, okay and this is a very deep fact and this is called the this is one form of the Hilbert Nullstellensatz, okay.

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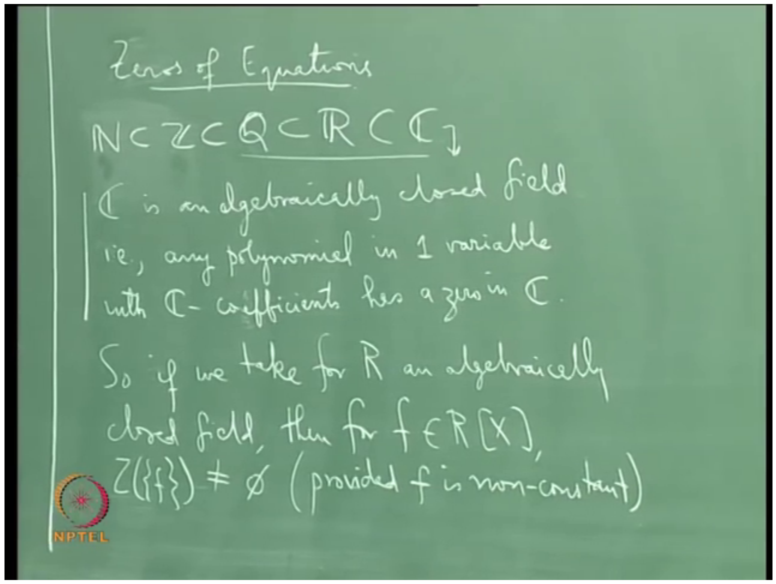


So let me write that down so fact so if we take a for R a algebraically closed field then for f in Rx Z of f is non-empty of course provided f is non-constant of course you know when you are looking at the 0's of polynomial suddenly you are not looking at a constant polynomial, okay what that means is if you are looking at a constant polynomial if that constant is non-zero then there are no zeros and if the constant is 0 then the whole space satisfies the citation, okay.

So if you take a non-zero constant and consider that as a polynomial then it has no 0's so the 0 set is empty if you take 0 as a constant polynomial then this 0 set is a whole space, okay. So what it says is that if you take for R an algebraically closed field and this set S to be a single term consisting of only one polynomial which is of course non-constant then the 0 set is non-empty. So you ensure that this is non-empty so you can do some geometry, okay but then here is the important thing so what we want is not for one variable X but we want it for several variables and we do not want it for a single polynomial in several variables but we want it want it for a whole collection of polynomials in several variables, okay.

So that is our deep requirement and that is what (())(26:09) and that is what the Nullstellensatz as it is called Hilbert's Nullstellensatz that is what it promises.

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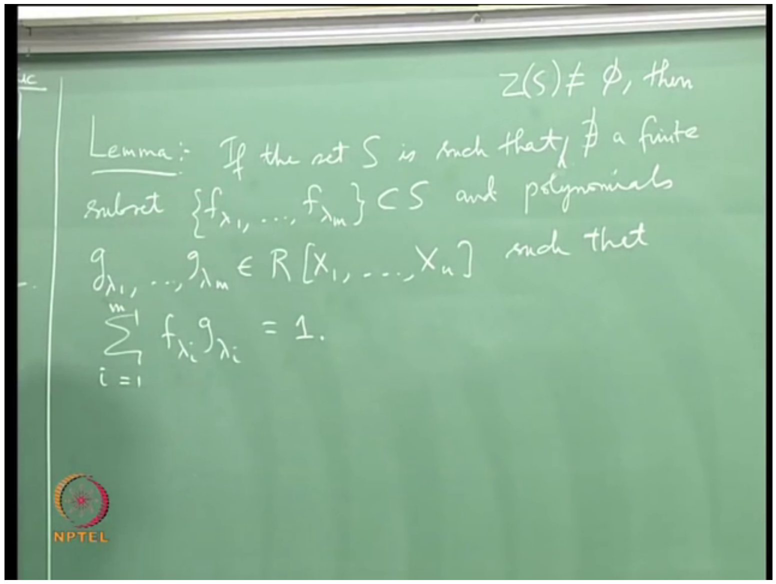


So let me write the following down, so first let me say the following team suppose so here is a very simple let me call it a lemma I will write a very simple lemma if the set S is such that does not exist a finite subset let me write f lambda 1 except lambda m and polynomials g lambda 1, g lambda m in the polynomial ring such that sigma f lambda i g lambda is equal to 1 i equal to 1 to m then so I should say and only then, sorry I should say the following this so there is one part of the lemma which is very trivial there is the other part of the lemma which is actually the Nullstellensatz when R is an algebraically closed field, so let me say that correctly I have to modify this statement a little bit, yeah so let me make a small modification if the set S is such that is that of S is non-empty then that cannot exist a finite subset such that this is equal to 1 so this is a statement, okay.

So I am saying that suppose so you see the whole point is a following the whole point is we are trying to look we are trying to study this set of common 0's, okay we do not want that to be empty we do not want that to be empty and the statement I am making is if it is non-empty then no finitely no finite subset of S can generate 1, okay no finite subset of S can generate 1 that means you cannot get finitely many elements of S and finitely many polynomials from the ring of which this capital S is a subset of such that you take multiply them and then add them you get one that cannot happen that is obvious because you see if the 0 set is non-empty that means there is a value small r1 etcetera small rn in the 0 side and this value small r1 etcetera small rn when you substitute it in each of this f lambdas it is going to vanish, okay.

So in particular if I substitute it in this relation on the left side, okay the left side is going to vanish and I will get 0 equal to 1 and in a ring if 0 equal to 1 the ring is a 0 ring, okay it has only one element which is 0 ring and suddenly we are not interested in working with 0 ring, okay. So if the ring is not the 0 ring then what this tells you is that if the 0 set is non-empty this can never happen, okay it is obvious it is a very very simply thing, okay but the converse to this that the converse to this holds when the ring is algebraically closed field is the is one form of the Hilbert's Nullstellensatz.

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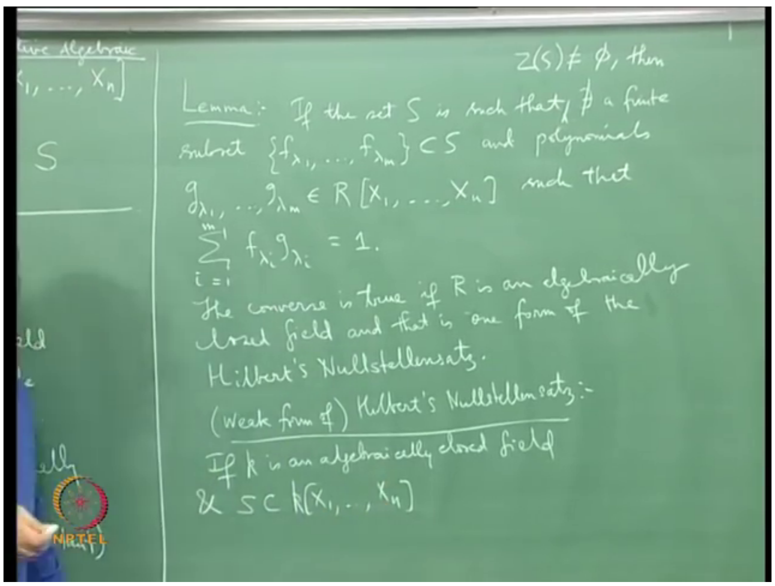


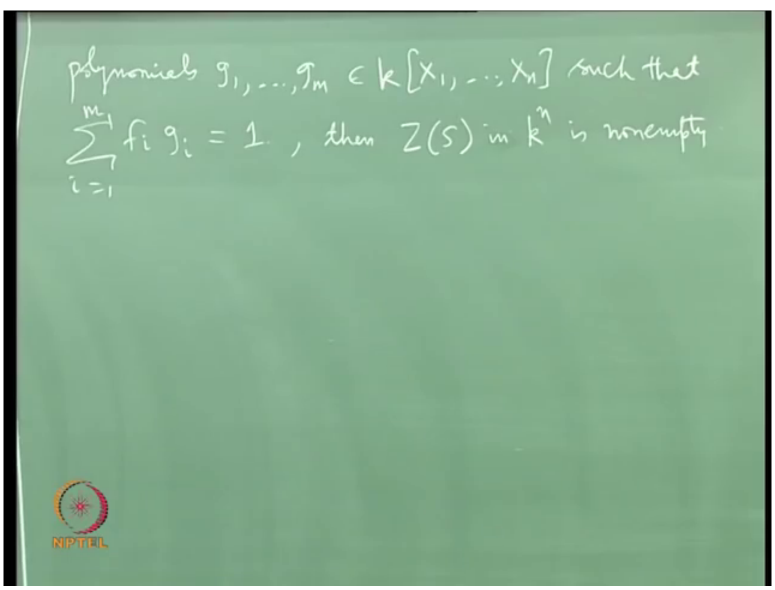

So let me write that the converse is true if R is an algebraically closed field and that is the that is one form of the Hilbert's Nullstellensatz. So maybe so let me keep it like this let me not state it the converse will be that if the subset S if R is an algebraically closed field and if the subset S of polynomials in the polynomial ring over R in n variables is such that no finite subset of that can generate 1 then this 0 set defined by that subset is non-empty, okay that is one form of the Nullstellensatz which is usually called the weak form of the Nullstellensatz, okay.

So maybe I will write it down, so what is that? So if k is an algebraically closed field and S is a subset of polynomials over k in n variables of course small n is greater than or equal to 1, okay then such that that does not exist a finite subset let me write f1 etcetera fm in S and polynomials so let me continue here g1 etcetera gm so this is m again in the same ring such that sigma fi gi is equal to 1 i could gone to m, then the 0 set defined by S in kn is non-empty you can find solutions. See what you must understand is that the Hilbert Nullstellensatz is a grand generalization of the property of being algebraically closed you know if you put in the Hilbert's Nullstellensatz if I put small n equal to 1 that I am looking at polynomials only in one variable and if I take the subset S capital S to be a singleton set, okay then this statement is obviously true if for an algebraically closed field.

See what you must understand is Hilbert Nullstellensatz when you put n equal to 1 is true just by the definition of algebraically closed field because what happens when you put n equal to 1 and when you take the set S to be a singleton consisting of only one polynomial and then if you have this condition that then this condition will become that polynomial multiplied by no other polynomial gives you 1 which is the same as saying that the polynomial is not constant, okay then you are saying that this 0 so that polynomial in k1 which is just k exist.

So you are just saying that every polynomial has a 0 which is the definition of what an algebraically closed field is. So what you must understand is that Hilbert Nullstellensatz is a grand generalization of the definition of algebraically closed and it is a very very important theorem and this is the theorem that guarantees that if you are working over algebraically closed fields then you can daily do geometry, okay.

So and this is the called the weak form we will come to the stronger form later on, okay and I will see later on if I can hint at a proof of the Hilbert's Nullstellensatz usually a proof of the

Nullstellensatz is given in course in commutative algebra but there is also a way of looking at that proof completely in algebraic geometric terms. So this is something that keeps happening that you must always keep at the back of your mind there are many things that can be said in this on the language on this side which is the algebraic geometry language and this there are same things can be said with the language on this side which is the language of competitive rings and modules and ideals and things like that, okay.

So proof here will involve a you know ring theoretic arguments ideals and homomorphisms and modules and things like that, whereas the same proof when you translate it here it will have geometric mean and its this it is trying to go from one to the other that really enriches both the sides, okay. So okay so incidentally I should tell you what this Nullstellensatz is. So this is German you know Hilbert's Nullstellensatz was a German mathematician and of course one of the greatest of all time and Null stands for 0 and stellen stands for position or point and satz means theorem or statement.

So if actually so if you translate it properly it means Hilbert's theorem on zeros on zeros of polynomials of a bunch of polynomials, okay fine.

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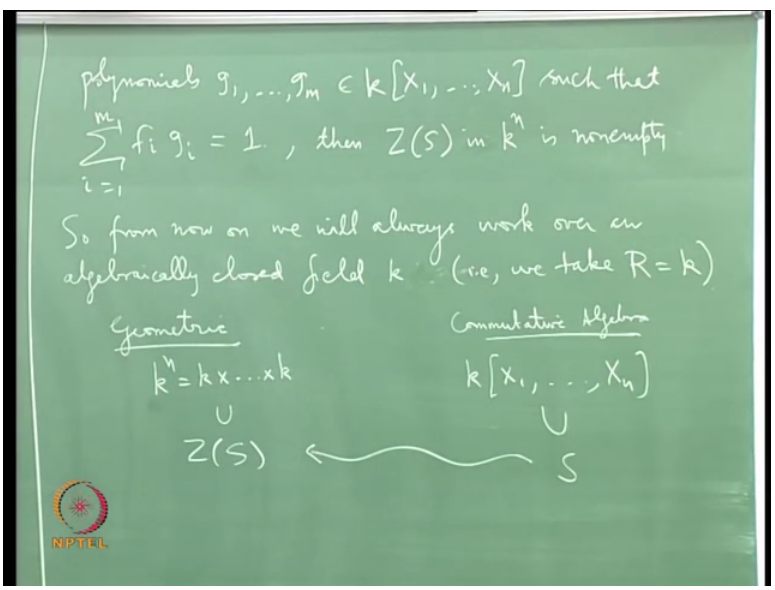


So okay so what we are going to do is so from now on we will always work over an algebraically closed field k that is we take R equal to k. So if you want for convenience you can even think of the algebraically closed field as complex numbers so that you know if it makes easier for you to think about things and visualize things, okay.

So what we are going to do so our picture becomes so our picture goes from that generality into something more concrete so we have the following picture on the commutative algebra side we have k of x1 etcetera xn this is the polynomial ring in n variables and on the geometric side we have kn which is k cross k cross k n times, okay. Now well how do you think of kn you are used to kn from linear algebra I mean they always one would the simplest way one would look at it as the vector space of dimension, okay for people who have done module theory Rn can be thought of as the end dimensional free module over R, okay. So in fact we do not use the word dimension from modules so I should amend my statement to Rn is the a free module of rank and over R, okay so dimension is usually reserved for fields so kn is an n dimensional vector space over k a module over a field is a vector space, okay.

So but it is not the vector space properties we are interested in, so you see the vector space the properties of vector space are here this 0 vector and then you know have additional vectors and so on and so forth. So in that sense you know the vector space studying the vector space properties is literally studying linear algebra but that is not what we are interested, what we are interested in is actually trying to study the points of the space, okay without any regard to the vector properties, okay so you so it means that all points of the space are alike I mean if you take the plain and through away the vector space structure that means you take the plain and how do you get the vector space structure you have to first have an origin because for a vector you need initial point and then you need a terminal point.

So normally what we do is we have an origin and every other point to every other point we associate the vector which starts at the origin and goes to that point we call it the position vector of that point and then we study all these question vectors and add them as usual with parallelogram law and then so on and so forth. But what we are going to do now is not call any not worry about doing all this we gonna think all points as equal as one and the same. So think of kn but do not think of the axis, do not think of the origin as origin in.

So you just think of it, for example if n is equal to 2 think of the plain but without the two axis there is point called the origin but that point is not you do not try to for every other point you do not try to draw position vector and think of it as a vector. So you know this process of trying to think of kn as a space not of points but not as space of vectors is what makes it into what is called an affine space, okay.

So the affine space kn is different from the vector space kn in the affine space we are only worried about the point not about the vectors, okay. So you will see often in algebraic geometry people use a word affine and that is the whole point, okay. So the first important thing is do not ever think of this as vectors, okay though it is a vector space of dimension n but that is not what we want, right? So well so this is the affine space literally and given a subset S to get the 0 set of, oops yeah that is right yeah, so here is our this is the picture we are going to look at and we are going to see what is there on both sides, okay.

So the first thing that I am going to do is so that brings us to the following question what are we going to do with this set we have thrown away the vector space structure is just a set and what I am going to do with it. So the first step is to give it a topology, okay and that topology is called as Zariski topology, okay so I will explain what that topology is.

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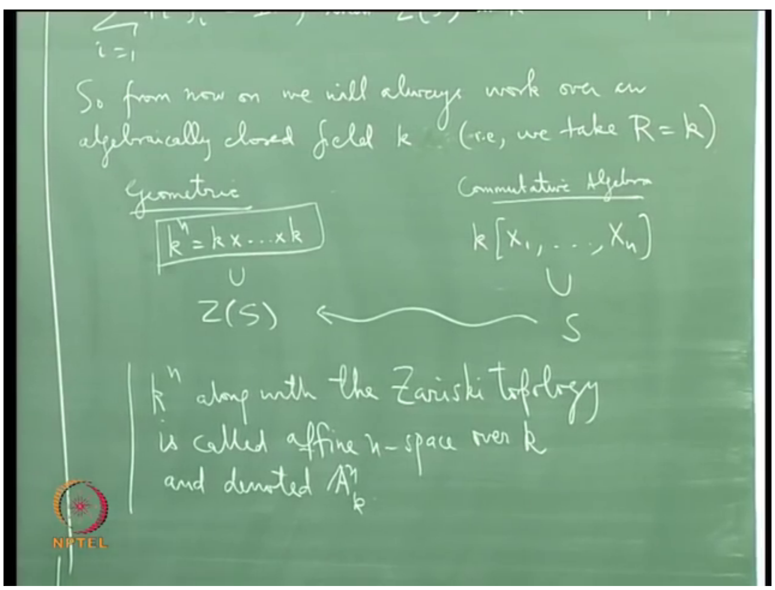


So well so let me write that so let me write the following thing kn along with say Zariski topology is called affine n space over k and denote it kn, okay. So the first thing that we do to get some geometry on the left side is first of all have at least the topological space, okay. So we make kn into a topological space, what does that mean? It means that you have to specify a topology on it and specifying a topology on it means what you try to give a collection of subsets which you make all as open subsets and this collection has to satisfy the (())(44:17) for a topology and what are the (())(44:18) for a topology the whole space should be an open set the null set which is the empty set should be an open set an arbitrary union of open sets should be open a finite intersection of open sets must be open.

So these are the four conditions for a collection of subsets of a space or of a set to make it into a topological space, okay. So I have to explain what the open sets are, okay but then you know when you go to topology there are two ways to approaching a topology, one way is by open sides the other ways is by using close sets, okay because close sets are just commonly means of open sets, so I can also give a topology on a set by giving a collection of subsets called closed sets but then the conditions will be complimentary, okay because of De Morgan's laws so the conditions will be that the whole space is a closed set then the empty set that is the null set is a close set any if you take finite collection of closed sets their union is again a closed set, okay.

And if you take an arbitrary collection of closed sets there intersection is again a closed set these are just translations using De Morgan's laws for the corresponding schemes for open sets. So basically what I will have to do is that I have to tell you either I should either give you a collection of open sets or I should give you a collection of closed sets which satisfy the corresponding schemes. And what happens in geometry is that is easier to begin with by looking at collections of closed sets and guess what, what are going to be the closed sets the closed sets are going to be just common zeros of functions of this type.

So you see the so the idea is a following you look at this space d look at kn think of this as functions on kn, of course you take any polynomial here and you take a point here if you evaluate the polynomial at that point you get a value. So certainly a polynomial the elements here which are polynomial that are certainly functions on this space and what you do is you call a subset here to be closed if it is the if the 0 locus common 0 locus of a bunch of polynomial functions which is exactly what this is, okay.

So the moral story is that if you declare subsets of kn of this form 0 loci of a bunch of polynomials common 0 loci of a bunch of polynomials call them as closed sets then you get the Zariski topology that makes this kn into Zariski topology. So what that will tell you is that what are the open sets, open sets will be loci where certain where functions do not vanish the loci where the functions vanish for example the loci where a single function vanishes is a closed set and the compliment of that locus where this (())(47:37) does not vanish is an open set, okay and this is in tune with our common sense, okay because if you take a real valued function on some subset of Rn or you take a complex valued function with on a subset of n dimensional complex space Cn.

Then the set points where the function takes the value 0 is going to be a closed set provided the function is a continuous function because it is a point is always a closed subset and if a function is continuous the inverse image of a closed subset is closed therefore the set of points where the function is 0 is exactly the inverse image of the point which is 0 and that has to be closed, okay. So it agrees with our usual intuition so the idea is to give the Zariski topology in that way, okay. So we will look at that in more detail in the next lecture, okay.