Advanced Complex Analysis - Part 2: Compactness of Meromorphic Functions in the Spherical Metric, Spherical Derivative, Normality, Theorems of Marty -Zalcman-Montel-Picard-Royden-Schottky

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Lecture No 9

The Generalized Liouville Theorem_ Little Brother of Little Picard and Analogue of Casorati-Weierstrass; Failure of Cauchy's Theorem at Infinity

Okay so let us continue with whatever we were doing say we were trying to understand what removable singularity at infinity means okay and what we saw was that a function if it is entire that is it is analytic on the whole complex plane and it is also analytic at infinity which by definition means it has removable singularity at the point at infinity and of course that is equivalent to it being bounded at infinity or it is also equivalent to it having a limited infinity okay in such a function must reduce to a constant, so an entire function which is analytic at infinity has to be constant.

In other words a non-constant entire function at infinity cannot have a removable singularity has to be either a pole or an essential singularity so it has to be a real singularity it cannot be a non-singularity which is what removable singularity is, a removable singularity is actually a non-singularity in disguise, so it is a fake singularity a removable singularity is a fake singularity, it is not a real singularity okay. So an entire function which is non-constant cannot have a fake singularity at infinity, it has to have a either a pole or an essential singularity. Now I told you that this is essentially the same as Liouville's theorem I mean this is just another avatar of Liouville's theorem and so you know the point is that somehow there is a there is a generalisation of Liouville's theorem okay which is exactly you know which is exactly like the Casorati Weierstrass theorem okay see with is something that I want to say in this connection.

See let me recall, what is the big Picard theorem? The big Picard theorem is given function analytic function which has an isolated essential singularity at the point then in every neighbourhood of that isolated essential singularity function takes all complex values except with the exception of one value at most one value okay which means it might fail to take one value at the most or it might take all values okay and this it will do in every neighbourhood of an essential singularity and every value will be taken every value that it takes will be taken infinitely many times, this is the big Picard theorem or the great Picard theorem and what is the 1^{st} approximation to the big Picard theorem.

So the big Picard theorem (())(4:01) what it says is that if you take the image of or deleted neighbourhood of an essential singularity you get either the whole complex plane or you get the complex plane minus a point, so it is either the whole plane or a punctured plane, the punctured being corresponding to removing a value which it will not take okay. Now what is the Casorati Weierstrass theorem? The Casorati Weierstrass theorem is a weaker version okay which we actually prove using Riemann's removable singularity theorem okay and what is the weaker version, the weaker version is that the images dense, you take any neighbourhood, deleted neighbourhood of an isolated essential singularity and take its image onto the analytic function, the image is a set which is dense in the plane which means it is the closure of the image is the whole plane, the other way of saying that is that every complex value the function value is come close to every complex value.

In other words you give me any complex value I can find a sequence of points such that the function values tent to that complex value okay, so that is the same as saying at the images dense that the image the closure of the image contains all the points in the complex plane which is the same as saying that every complex value can be approached by values of the function as close as you want okay. So this is the Casorati Weierstrass theorem which is relatively easy to prove because you can reduce it from the Riemann's removable singularity theorem that we did okay. Now in the same way there is a there is a similar version of this for entire functions okay, so you know so again let us go back to the Liouville's theorem, what is Liouville's theorem? Liouville's theorem says that a bounded entire function is a constant okay and so if you say it in a different way, what it says is that an entire function which is not constant is unbounded okay.

What it means is at it will take values with bigger and bigger modulus because if all the values that it takes are bounded by a certain modulus that means it is a bounded function and if it is entire end bounded Liouville will say it is a constant, so if you take non-constant entire function, what it will tell you is that the values of the function can become arbitrarily large in modulus okay. Now…but then you can ask the question what is an image of an entire function and the answer to that is little Picard theorem the little Picard theorem says that the image actually, the image of an entire function is either the whole complex plane okay or it is a punctured plane namely it will omit one point at the most and that is that is the case.

For example if you take E power Z it will omit the value 0, so and I told you that this is called the little Picard theorem or the small Picard theorem and this is supposed to be we would like to derive it as the corollary of the big Picard theorem okay but then the big Picard theorem has a weaker version which is the Casorati Weierstrass theorem and similarly the little Picard theorem also has a weaker version and that weaker version is the generalise Liouville theorem and what is a generalise Liouville theorem? Generalise Liouville theorem says that you take the image of an entire function then the image is dense okay that is the that is the weak version of little Picard theorem is called the generalise Liouville theorem okay.

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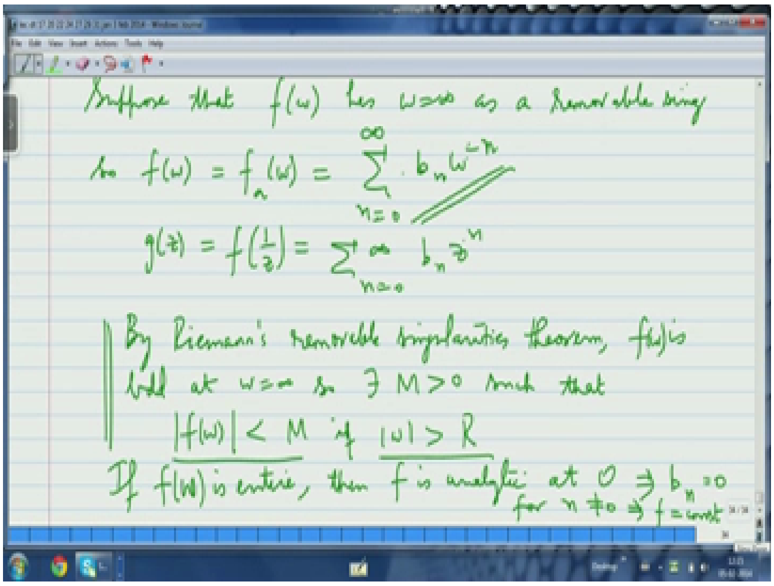

And so the reason why I am trying to bring it at this point in the in our discussion is because just to tell you that this is a circle of ideas with the with the generalise Liouville theorem is to the little Picard theorem what we Casorati Weierstrass theorem is to the big Picard theorem okay is a similar thing. Both the Picard theorem tell you what exactly the image will be okay that the image will be the whole plane or the plane minus 1 point and the both the Casorati Weierstrass theorem and the generalise Liouville theorem will tell you that the image is that the image is dense that is the point okay and how do you prove the how do you prove the generalise Liouville theorem is also it is actually the same proof as the same tactics that we used to prove the Casorati Weierstrass theorem except that you use Liouville theorem okay so let me write that down.

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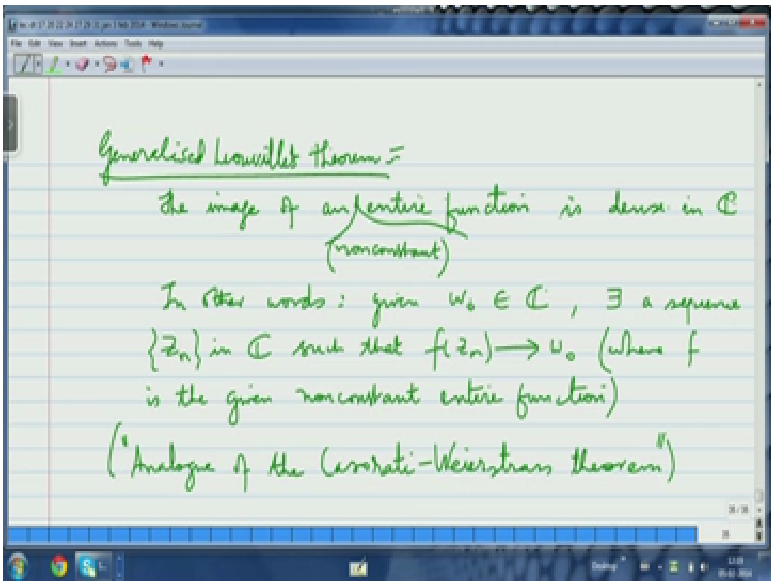

So here is the generalised Liouville's theorem the image of an entire function is dense in the complex plane and of course it is very important that all these theorems are valid only for non-constant functions okay, so whenever you should not miss saying there is a non-constant type of function because a constant function always the image is a single point and it is entire okay, so you should this non-constant should always be carried through if you want your statements to be accurate okay you might you might tend to miss it but if you miss it, it is a big miss okay, so let me write that the image of well a non-constant, so this is very important the image of a non-constant type of function is dense is dense in the complex plane, what does it mean?

Let us say it includes different ways, it means that any complex value and be approached by function values that is one way and that if you want to say it more clearly you can say you can find a sequence of points such as the function values at those points approach the given value okay. So let me write that in other words, in other words given W naught in the complex plane there exist sequence is at the n in there is a sequence in the complex plane such that the f of Z n the sequence the image of the sequence under f distance to W not where f is the given non-constant entire function okay. So this is another way of saying what I said, now this is the generalise Liouville theorem and this is…

So let me write this this is the analog of the Casorati Weierstrass theorem okay this is the, so I will write it as analog of the Casorati Weierstrass theorem okay and I am still putting this analog of Casorati Weierstrass theorem in quotes in some sense but it is actually Casorati Weierstrass theorem but you know the only thing is the case where it is actually a Casorati

Weierstrass theorem for infinity as an essential singularity justifies this name actually and I will explain that to you later okay but what I want you to at this point to understand is that statement of the generalise Liouville theorem is similar to this statement of the Casorati Weierstrass theorem and these are both weaker version of the stronger theorems, the stronger version of the Casorati Weierstrass theorem is the big Picard theorem, stronger version of generalise Liouville theorem is the little Picard theorem okay that is the point.

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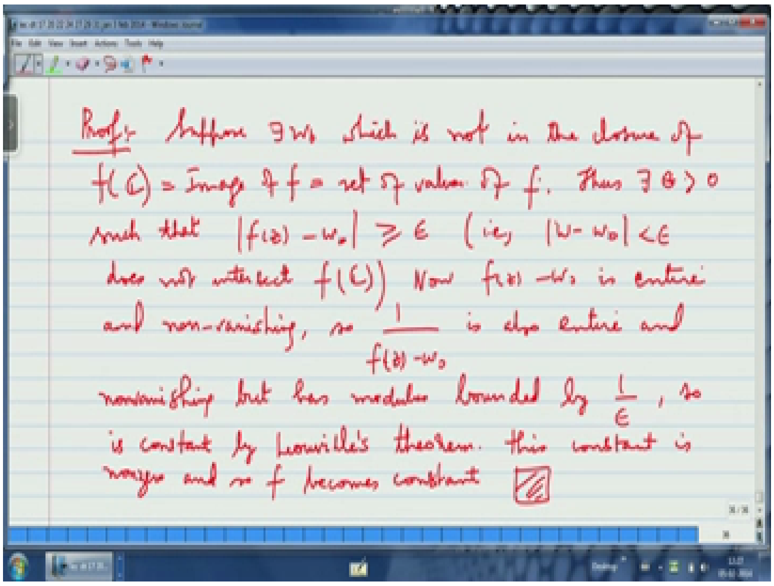

Now let us try to prove the proof method is exactly the same as we have been doing for the Casorati Weierstrass theorem. Proof, so you prove by contradiction, you assume that there is a value which the function does not come close enough to and then get to prove that the function is a constant and you are done okay, so suppose there exist W naught which is not which is not in the closure of f of C, f of C is actually the image of F. The image of f is this all this set of values of f okay, of course f is entire okay and of course let us assume that f is entire non-constant okay.

Suppose there is a point here suppose there is a value W naught which is not in the closure that means f this W naught cannot be approached by function values okay that means there is a neighbourhood surrounding W naught which is which does not intersect the image okay, so W naught is not in the closure of the of the image means W naught is outside the closure of the image okay and the closure of the image is the closed set okay and anything outside a closed set is an open set and anything which is in an open set is contained in a small disk which is also in that open set, so I can find a small disk surrounding W naught which does not intersect the image okay.

So let me write that down thus there exist an Epsilon greater than 0 such that mod of f Z minus W naught is greater than or equal to Epsilon okay I mean this is this is same as saying that the disk mod W minus W naught less than Epsilon does not intersect f of C, so this this does not intersect f of C means that it complements f of C can at most intersects its complement okay and well but now you know what to do? You see again the trick is that you know F, f of Z is analytic everywhere it is entire, so f of Z minus W naught is also analytic it is also entire okay because it is just the constant minus W naught added to an entire function and you know some of analytic function is analytic, a constant function is analytic for f minus W naught becomes a entire function but it never vanishes, it is always greater than Epsilon.

So it means that it is reciprocal which is defined because it does not vanish and you the moment analytic function does not vanish, it is reciprocal is define and that also turns out to be analytic, so one by f Z minus W naught also turns out to be analytic okay and because the denominator which is f Z minus W naught doesn't vanish and that is because it is always greater than (())(16:14) Epsilon in modulus okay and Epsilon is a positive quality right, so I get is function 1 by Z minus W naught which is on the one hand analytic and on the whole plane plus I also get that its modulus are bounded by 1 by Epsilon, so it is an entire bounded function and Liouville's theorem will tell you that it is a constant okay and so one by f Z minus W naught will become a constant that constant cannot be 0 okay because if that constant is 0 your f Z minus W naught not be a finite quantity okay.

So the constant is nonzero and once it is nonzero the reciprocal of that constant will become f Z minus W naught, so it will tell you f Z minus W naught is a constant so f itself be a constant and that will contradict if with the assumption that f is non-constant okay and that ends the proof and if you really see this is exactly the same roof as the same technique of proof as we probably you know the Casorati Weierstrass theorem okay. So let me write this down now f Z minus W naught is entire and non-vanishing, so 1 by f Z minus W naught is also entire and non-vanishing but has modulus bounded by 1 by Epsilon, so is constant by Liouville's theorem and of course the fact that one by f minus W naught does not and this will tell you that this constant cannot be 0 okay because after all this this constant is the value is the function itself okay.

So this constant, this constant is nonzero because one by f Z minus W not is equal to that constant and 1 by f Z minus W naught and never vanishes and so f becomes constant and that is the end of the proof, so I am not the way you should read this proof is that either you assume that f is a constant and get to the end of this proof which will say that you have got a contradiction or you assume that f is a function an entire function which misses a value I mean which stays away from a value then it has to reduce to a constant that is what this proof says okay, so this is the generalise Liouville's theorem okay and I just want to say that this is an this draws an analogy with the Casorati Weierstrass theorem okay fine.

Now you see so what we have now at the moment you have at the moment the idea of a function adding a removable singularity at infinity okay and then of course you can ask all you can ask regular questions that you would ask usual questions that you can ask for a function which is analytic at a point okay, so for example so to begin with suppose you have a function which is analytic at a point in the complex plane in the usual complex plane then what you know about? What are all the things that…what are all the basic things that you know about such a function?

So the 1st thing is that you know that it is infinitely differentiable okay in fact all its derivatives exist and they are also analytic at that point okay this is one thing. In the 2nd thing is you have Clausius theorem that you take a neighbourhood sufficiently small neighbourhood to the point stop in fact you take you take any disk surrounding that point or even a domain surrounding that point containing that point where the function is analytic and the integral over of the function over a simple closed curve such that the function is analytic on the curve and in the interior of the curve also will always be 0 that is Clausius theorem essentially and then of course you have that the function can be expanded as a Taylor series, this is Taylor's theorem okay and you have a converse criterion or a necessity which is given by Morera's theorem which says that if you have a continuous function and if the integral over a very simple closed curve is 0 then the function is analytic okay.

So these are all the things that you know about analytic function. This is all know for a function is analytic in a domain in the complex plane but now we are interested in an analyticity at infinity okay, so you can ask all these questions of function which is analytic at infinity okay you should see what happens, so the 1st thing is there is one point we are to be very careful about. See you cannot define the derivatives of the function at infinity, it is troublesome because you know I cannot really write limit Z tends to infinity f of Z minus f of infinity by Z minus infinity means it really does not make any sense okay but there is a tricky way of doing it.

See the tricky way is well you how we define a function to be analytic at infinity, we cleverly said that it has to have a removable singularity at infinity and for having it, making it have a removable singularity at infinity, we found that either you say it is bounded at infinity or it has limited infinity or it is continuous at infinity, you just put one of these weak conditions okay and that all these 3 are good enough and they are powerful enough is because the inspiration comes from the Riemann's removable singularity theorem, so you can ask this question suppose a function f is analytic at infinity then are all its derivatives also analytic at infinity okay, so and the answer is yes, the answer is yes so it is funny you are saying that the function is analytic at infinity, all the derivatives are analytic at infinity what you really do not go and define the derivative at infinity, in fact there through this the derivative at infinity will always be 0 okay except at the worst it can be your constant okay I mean at the best I mean it can be a constant all the time most of the time is 0 okay, so you know so let me 1^{st} answer that question.

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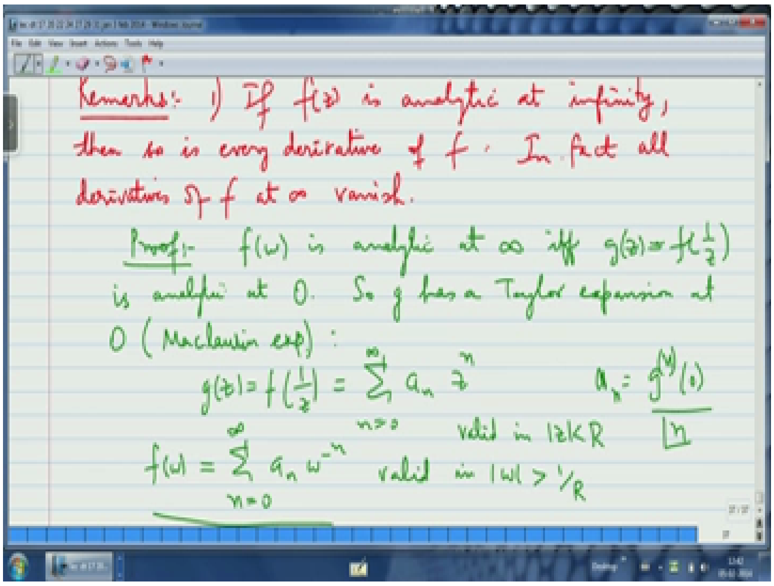

So remarks number 1 if f of Z is analytic at infinity then so is every derivative of f okay, so if f is analytic at infinity then all its derivatives are also analytic at infinity that is the that is the 1^{st} statement okay and the truth is that the moment you say something is analytic at infinity, its value at infinity as to be defined and the fact is that all the derivatives vanish okay. In fact in fact in fact all derivatives of f at infinity vanish okay except possibly f itself not vanishing at infinity it may be a constant and f is (())(24:33) if you want to think of f as 0 derivative of itself okay then the 0 derivative can (())(24:39) can be a constant which is nonzero but all other derivatives have to be 0 okay so why is this true? So this is true because of basically there are many ways to see it, the easiest way to see it is by looking at the by looking at this philosophy that if a function is I mean the behaviour of function at infinity f of W at infinity is the same as the behaviour of g of Z which is f of 1 by Z at 0 okay.

So if you use that you will see that the then you use the theorem that for a point in the complex plane if function is analytic at a point in the complex plane then it is infinitely differentiable there. All the derivatives are also analytic at that point okay and that gives you a result okay, so let me explain that so proof is f W so I will change this Z to W, f W is analytic at infinity when only if g of Z is equal to f of 1 by W1 by Z is analytic at 0 and 0 okay and the point is that you know so g of Z is analytic at 0, so g is given by a Taylor expansion okay, so g has a Taylor expansion at the origin, so this is the otherwise called the MacLaurin expansion, the MacLaurin expansion is actually the Taylor expansion at the origin at 0 okay and what is the Taylor expansion, it is just g of Z is equal to this is f of 1 by Z and that is Sigma and equal to 0 to infinity a n Z power n.

This is what the Taylor expansion is and you know that the a n are the derivatives, the a n are the derivatives of well you know a n is just the nth derivative of g divided by factorial m, nth derivatives of g at 0 divided by factorial n this is what the a n are, so you know that okay and this g upper n let me put the bracket, the bracket is supposed to be derivative, see round bracket upper around bracket n (())(27:31) okay. Now you see now see the point is that this is where is this valid? This is valid in valid in mod Z is less than R okay where R is radius of convergence.

R is the radius of convergence of this power series in Z okay you know whenever power series converges, it converges in a disk with Centre the center of the power series and the radius is call the radius of convergence okay and this radius of convergence is actually the distance from the center of the of the series to the nearest singularity of the function that is what the radius of convergence is For example of the function has no singularity then the radius of convergence is infinite that is what happens when you write out Taylor series for an entire function okay the fact that you write out Taylor series and you get an in finite radius of convergence is a proof is equivalent to saying at you have the function that you are really dealing with is actually an entire function okay.

So you have this, now let us put W equal to 1 by Z and you will get a similar expansion for f of W in a neighbourhood of infinity, so what you will get is you will get f of W equal to Sigma well n equal to 0 to infinity a n W to minus n valid in mod Z mod W greater than 1 by

R okay this is what you will get because I just put Z equal to 1 by W that is the relationship between Z and W okay and if I put that I will get mod W greater than 1 by R and you easily recognize that mod W greater than 1 by R is a neighbourhood of infinity okay in the extended complex plane alright and there you have this function and this is kind of this is very nice for f because you see when you write I told you that this is how you should read this as an expansion function at infinity okay which is good.

See if you want to expand a function at infinity you use the powers of the variable just as you will when you want to expand it at 0, so the only thing is that at infinity it is the negative power that behaves well okay at 0 it is the positive power that behave well K so you see that this expression has only negative powers of W, it does not have any positive power of W and that is the proof that it behaves well at infinity because you know if I let if I let W tends to infinity then this expression every term this expression is going to go to 0 and this it is going to go to 0 uniformly because you know whenever these things converge whenever see this converge they all was converge whenever power series converges or Laurent series converge they always converge normally, they converges uniformly on compact sets okay.

Therefore see the fact is that this is very well-behaved okay the moment you see only negative powers of the variable in series you must really understand that you are looking at a function which is actually analytic at infinity okay, so For example the simplest case is if you if you are looking at 1 by Z okay which is the same as 1 by W if you want if you think of W as a variable, 1 by W is good at infinity it is bad at 0 okay it has a pole at 0 whereas (()) (31:21) is very good. Similarly if you take one by W Square that is a bad at 0 but it is good at infinity all the negative powers are good at infinity okay.

So the fact that your f has an expansion a series of negative powers of W tells you that it is good at infinity okay and the fact I want to say is that if you well now if you if you take this expansion for f okay and you can differentiate this expression for f the series for f of W term by term that is because of normal convergence okay and if you do that what you will get is again expressions of the same type okay because when you differentiate of course you know when I put n equal to 0 I will get a naught and what is a naught? a naught is the value at the center see if you write a series, power series centred at Z naught okay and you plug in Z equal to Z naught what will you get, you are supposed to get the constant okay if you if you write out a power series in Z minus Z naught it will be of the form a naught plus a 1 times Z minus Z naught plus a 2 times Z minus Z naught square and so on.

When I plug in Z equal to Z naught which is the center of the series all the terms except the 1st vanish and I get the constant term, so the constant term is the function value at Z naught at the Centre okay and in the same way if you look at this expression or f of W at infinity in the neighbourhood of infinity you see a naught is what you get when you again W equal to infinity okay you plug-in W equal to infinity means you take the limit as W tends to infinity, what happens is only a naught survives and this is in this is in perfect analogy that when you as you go to the center of the expansion what you get is the constant term okay and so in some sense what I want you to understand is that this this expansion a negative powers of Z is like the good expansion that is the reason why it is called analytic part at infinity okay.

All the negative powers along with the constant they form the analytic part at infinity and the positive power of the variable they form the singular part at infinity okay this is exactly upside down or what happens at the finite complex number okay, so well now what I want to tell you is that this f can be differentiated, you can differentiate f and if you look at all those derivatives, the derivatives the moment the moment you differentiate it even (())(34:10) the constant will go away okay and then you further differentiate it the constant is not going to remain even the 1st derivative of constant will go away and mind you when you and you can do the differentiation term by term okay if you do it term by term you are only differentiating negative powers of W and if you differentiate negative powers of W you will get further negative powers of W you are not going to end up with a positive power.

So the moral of the story is that you can keep on sharing this as many times as you want and you are going to get functions which are analytic at infinity because you are going to get analytic in our neighbourhood of infinity and they will all go to 0 at infinity because in all only negative powers therefore you see that all the derivatives exist and they are all 0 and that is the that is the remark okay, so I am just using the fact, I am just using the simple fact that you know whenever a functional series converges normally that is it converges uniformly on compact subsets of a domain and if you know that every in the series is analytic then the series converges to an analytic action and the derivatives and be computed by doing term wise differentiation okay so that is all I am using okay.

So that is the 1st remark, so you know it is very uneasy you must understand that we do not define what derivative at infinity is okay but we indirectly define function being analytic at infinity as being continuous at infinity are being bounded at infinity are having limit at infinity and then we get that all the derivatives also are analytic at infinity and we get in fact also that all the derivatives is 0 okay so you see it is very funny you are not able to define derivatives at infinity okay but other derivatives at infinity you are getting an expression they are all 0 okay.

So well then let us go to the 2nd remark okay so in this regard let me actually tell you that in some sense Clausius theorem fails okay Clausius theorem will fail for function analytic at infinity okay and the idea is very simple you see in the function f of W equal to 1 by W okay 1 by W is one of…is the best I mean all the negative powers of W are the best functions at infinity okay. Now if I integrate 1 by W over a curve contains 0 then I am certainly not going to get 0 because 0 is a pole but 1 by W is analytic at infinity okay, so Clausius theorem will fail so the moral of the story is at you have to be careful when you try to apply integration theorems in you want to work with the point at infinity you have fully little careful okay.

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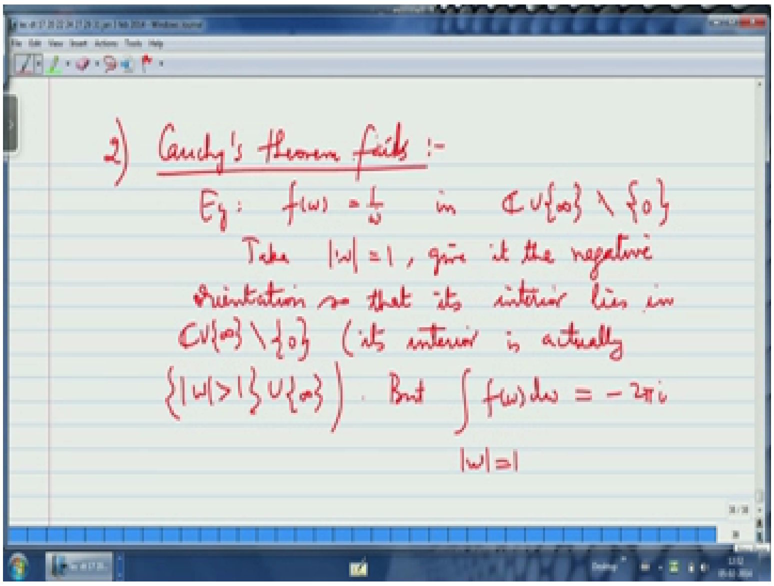

So let me write this here Clausius theorem fails which is in a way sad because but then you cannot you should see this as inevitable because you cannot define the derivative at infinity okay (())(37:32) right so what is the example, the example is you take f of W equal to 1 by W this function in you take you take the extended complex plane C union infinity okay and mind you it is also analytic at infinity its value at infinity is 0 that is you make it continuous at infinity by putting f of infinity equal to 0 okay and it is defined in the punctured extended plane okay which is the exterior you throw out 0 (())(38:05) infinity okay this is a mind you this is an open set in the extended complex plane given the you know 1 point compactification topology as we have seen earlier.

Now you take this function now it is not true that (())(38:21) every closed curve is 0 okay, so well take mod W equal to 1 okay you take mod W equal to 1 for that matter this is unit circle okay and what is Clausius theorem? The usual Clausius theorem is take a function which is analytic on and with then a simple closed contour okay and you integrate over the contour you should get 0 okay. Now the point is that if you take mod W equal to 1 if you give me the positive orientation if you give me the positive orientation then the interior of the unit circle will be the interior of the curve as usual and the exterior of the unit circle which is the neighbourhood of infinity will be the exterior of the curve okay, so and if you give it be positive orientation okay then is the function analytic in the interior?

No cost if you give mod W equal to 1 the positive orientation then the interior is the interior of the unit circle with a mod W less than 1 it contains W equal to 0 where it is not analytic, so you should not give it the positive orientation if you want it to be analytic, so what you do is you take mod W equal to 1 but put the negative orientation okay that is a curve for which the interior of the curve which will be the exterior of the unit disk will be a domain where the function is analytic but still if you calculate the integral over that I am going to get minus 2 pi I and that is not 0 so Clausius theorem fails okay.

So let me write this down take mod W equal to 1 gave it is negative orientation so that its interior lies in C union infinity minus 0 its interior is actually is given by mod W greater than 1 this set along with the point at infinity and this is the interior okay that is actually the exterior because if you change the orientation you know the interior and exterior will get interchange and again now this point let me recall 1st course in comics analysis, what is an interior and exterior? So the ruler is the following you say the interior of the region is actually the region that lies to your left as you walk along the curve okay.

So that is the role so you know if I take the unit circle and if I walk along and give it the positive orientation and if I walk along it which means I am going to walk in the anti-clock wise sense then what lies to the left is the inside of the unit circle okay which is the interior okay and if I give it the clockwise orientation means I am going to walk clockwise around it and then the interior what lies to my left will be there exterior of the unit circle and that is the reason what lies towards the left will be the interior and that will be the exterior of the unit circle okay. It will be so let me again say this just to relieve you of some confusion.

The interior of mod W equal to 1 if the clockwise orientation is the exterior of the unit circle okay because it lies to the left okay, so fine so now you know if you take but integral over mod W equal to 1 fw dw is going to be minus 2 pi okay and mind you I am getting this minus because this mod W equal to 1 is given the clockwise orientation you know you have done this computation with anticlockwise orientation in a 1st course complex analysis and you always get 2 pi i but since I have changed the orientation to clockwise I will get a minus because you know after all if you change, if you change the orientation of the part of the integration then the integral will change by a minus sign you know that okay so the point is that the Clausius theorem fails okay.

So you should not expect anything out of Clausius theorem here, so for a function which is analytic at infinity Clausius theorem ills then the next thing I want to talk about is about Morera's theorem okay, so thankfully Morera's theorem works and Morera's theorem works just for the case just because of the fact that Riemann's removable singularity theorem works and basically Morera's theorem works because we cheated by saying that analytic at infinity is same as continuity at infinity okay.

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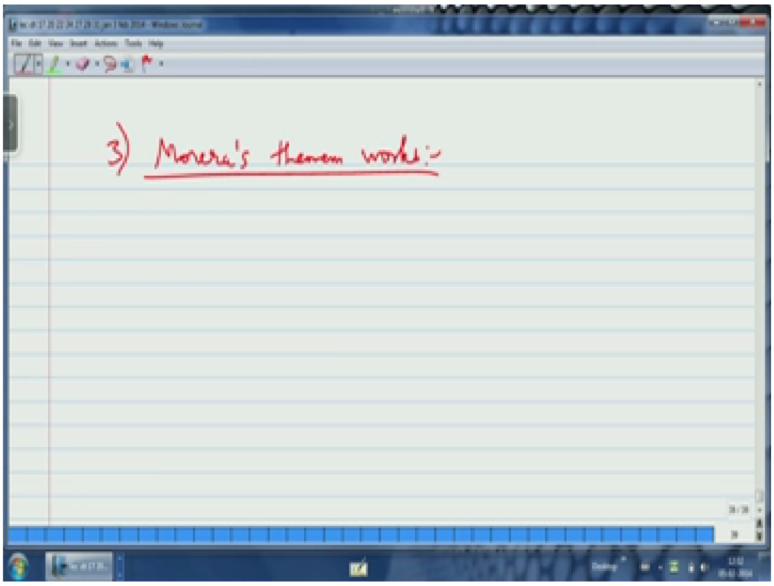

So let me say that Morera's theorem Morera's theorem works so what is this so you know so let me recall what is the usual Morera's theorem or a domain in the complex plane mind you Morera's theorem is supposed to be it is supposed to be a converse Clausius theorem okay now the only beauty about Morera's theorem is that whereas in Clausius theorem you always try to apply it to simply connected domain okay you do not allow any holes and that is because cautious theorem says integral over a curve is 0 provided the function is also analytic inside the curve okay there should be no point inside the curve where the function has a singularity or is not analytic.

So there should be no holes in the domain of analyticity with inside that curve okay whereas Morera's theorem is valid even for non-simply connected domain is this is the beauty of Morera's theorem is slightly stronger in that sense but it is converse to Clausius theorem, so what is Morera's theorem for a domain in the complex plane it is just that suppose I know I have I have a continuous function on a domain in the complex plane and suppose I know the integral over every simple closed curve is 0 then the function is analytic okay and the proof actually is very easy, what it does the proof is actually that because the integral 0 okay you can fix the point and then you can define an anti-derivative.

The anti-derivative will be dependent of the path it can be defined as an integral, the integral is independent of the path because the hypothesis of Morera's theorem is that the integral over any closed curve is 0, the integral over any closed curve is 0 mind you it is equivalent to integral being independent of the path okay and therefore you can define an anti-derivative and is anti-derivative will be a function whose a derivative is the given function okay but the moment, the fact that anti-derivative as the derivative as the given function 0 that the anti-derivative is analytic and you then use the theorem that the derivative of an analytic function is analytic therefore the original function which you are assumed to be continuous is also analytic, so this is how Morera's theorem works okay.

Now you will have to modify this this more or less works also for the for a domain in the extended complex plane okay, so of course you are you have a domain the extended complex plane and suppose integral over any closed curve is 0 and suppose you have a function which is continuous okay then you forget the point at infinity for whatever is left Morera's theorem still works and tells you the function is analytic, so infinity is becomes a singular point but then it is continues at infinity and we have cheated by saying that continuity at infinity is as good as analyticity at infinity so it becomes continues everywhere I mean it becomes analytic everywhere and you are done okay.

So Morera's theorem works or a domain in the extended complex plane also, so it works at infinity Morera's theorem works at infinity okay. So only thing you should be careful is about is that you should not try to integrate over a curve which passes through infinity which does not make sense, okay. By a curve we always mean a curve in the finite complex plane okay not involving the point at infinity, okay. So I will write this down in more detail in the next talk.