Advanced Complex Analysis - Part 2: Compactness of Meromorphic Functions in the Spherical Metric, Spherical Derivative, Normality, Theorems of Marty -Zalcman-Montel-Picard-Royden-Schottky

Dr. Thiruvalloor Eesamaipaadi Venkata Balaji
Department of Mathematics
Indian Institute of Technology Madras

Lecture No 8

Laurent Expansion at Infinity and Riemann's Removable Singularities Theorem for the Point at Infinity

So you see what we did last time was to try to define when an analytic function is having infinity as a point of analyticity okay and so as I told you if you recall one of the standard ways of defining analyticity at the point is to say that the function is differentiable at that point and also in a neighbourhood at every point in a neighbourhood of the given point okay. So this is how you define analyticity at the point in the complex plane okay but since we are worried about analyticity at infinity okay what we do is that we tend to look at the point at infinity as singularity a singular point of an analytic function which is defined in a deleted neighbourhood of infinity means that the function is analytic or mod Z greater than R or R sufficiently large okay.

Outside a circle of sufficiently large radius the function you are given a function which is analytic and then infinity the point at infinity becomes an isolated singular point and then you want to say that function is analytic at infinity, so it will not help to say that the function is differentiable at infinity because if you write you know a differential limit it does not make sense at infinity, so what is the way out? The way out is to actually get the draw inspiration from Riemann's removable singularity theorem. Riemann' removable singularity theorem tells you that if you are looking at an isolated singularity of an analytic function at a point in the complex plane then saying that the function is analytic at that point namely that the which is essentially saying that the function can be extended to an analytic function at that point including that point okay.

This is one of the definitions of what the removable singularities okay is equivalent to requiring that the function has a limit at that point, it is which is equivalent to continuity of the function at that point, in fact it is also equivalent to the function being bounded in a neighbourhood of that point bounded in modulus of course, so this is so the moral of the story is that you can use these conditions to define the function to be analytic at infinity okay. What you can say is that go it will not it does not make sense to will not work to say that the function is differentiable at infinity, you can always say that the function has a removable singularity at infinity in the sense that the function either has a limit at infinity.

So the limit as Z tends to infinity f Z exist okay that is one condition, the other condition is the function is bounded in bounded at infinity that means there is a deleted neighbourhood of infinity where the function, the modulus of the function can be made less than positive constant okay and in these conditions are one and the same okay and why these conditions are one and the same is because of this other important philosophy that studying the function at infinity, studying f of W at infinity is the same as studying f of 1 by W at 0 okay, so and I told you the (())(4:15) or the justification for that is that Z going to W equal to 1 by Z is actually a homeomorphism of the Riemann extended complex plane onto the extended complex plane which interchange is 0 at infinity okay and further if you throughout the point 0 at infinity then you get the punctured complex plane, the complex plane punctured at the origin.

If this map Z going to 1 over Z which is equal to W is going to be an analytic holomorphic isomorphism of the punctured complex plane onto itself okay and under such an analytic isomorphism, the nature of the singularity at 0 and the nature of the singularity at infinity which is an immediate 0 they should correspond, this is a philosophy that we use. Now what I want to say is that I want to go ahead with this and so now that we have defined function being analytic at infinity okay and the weakest definition or a function being analytic at infinity that it is bounded at infinity namely there is a point I mean there is a small neighbourhood of infinity okay which should be thought of as all Z, so I said mod Z greater than R for R sufficiently large then mod f Z should be made you should be able to make mod f Z is less than M okay for some positive constant M okay so that is exactly what you want to do, so what I want to do next is I want to worry about how I want to worry about what it means to say that a function has a removable singularity at infinity okay, so we want to analyse this, right.

So let us take the let us take the…so we will look at the case of an entire function alright but even before that I wanted to say that you know this there is one more aspect that we need to be need to actually look into okay and this aspect is about this aspect is about the Laurent series okay, so you see you know in some sense of the study the isolated singularity of a function at a point in the complex plane at the point Z naught the complex plane.

One of the ways of studying this is by looking at the Laurent expansion of the function centred at Z naught that means you expand the function in positive and negative powers of Z minus Z naught that is the Laurent expansion you get the coefficients are the Laurent coefficients and the Laurent theorem says that there is such a Laurent expansion okay which is in general valid in an annulus is alright and then you know that the nature of the Laurent expansion to be more specific the nature of the principal part or the singular part of the Laurent expansion tell you what kind of singularity Z naught is, so you know that if the Laurent expansion as only negative powers I mean it has only finitely many negative powers of Z minus Z naught know it is a pole if it has no that is if the principal part has only finitely many terms then it is a pole okay.

If the principal part does not exists namely if the principal part is 0 then it is a removable singularity okay and if the principal part has infinitely many negative powers of Z minus Z naught then it is an essential singularity okay this is this are trying to classify singularity based on the Laurent expansion okay which is something that you know. Now the question is what is an analog of this when Z naught is the point at infinity, not when Z naught is the point at the complex plane but when Z naught is the point at infinity, so how do you get a handle on this? How do you get a handle on this?

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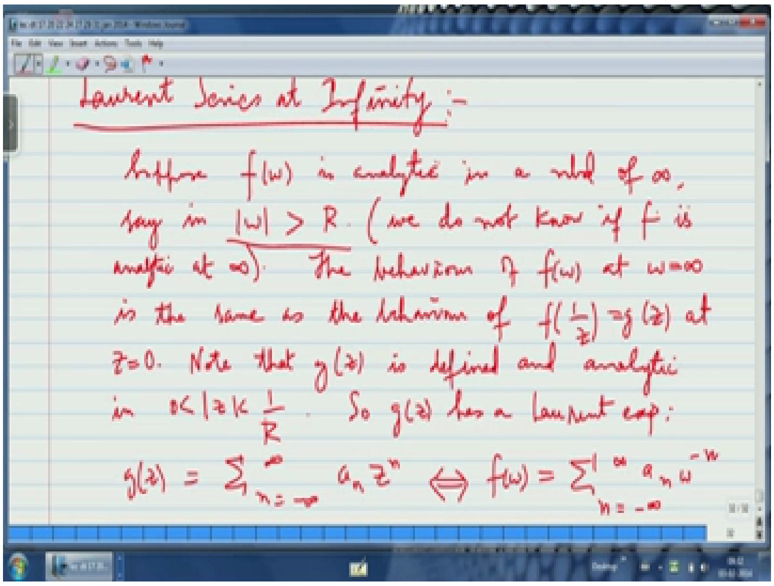

So you see so let me write that down, so my point is what is this Laurent series at infinity which is rather funny thing again you know you must remember that the way you deal with infinity is you have to be very clever, there are certain things you can do with infinity, there are certain things you cannot do with infinity so for example when you want to define a function to be analytic at infinity you do not take the road of trying to define it differentiable at infinity because derivative at infinity is not defined is not easy you cannot define it so easily, right.

Of course you can go to another level of abstraction called what is called of what is called Riemann's surface okay and you can make the you can make the complex plane along with the point at infinity namely the extended complex in into a Riemann's surface and once you have Riemann's surface and you have a holomorphic function then you can talk about derivative at any point by you seeing local coordinates okay but then we do not want to get in to that amount of generality you can look up that point of reasoning if you look at my video course on Riemann surfaces which is available on the web okay but then let us not go to that level of abstraction okay.

At this moment at this level of our exposition we do not try to define derivative at infinity but trying to say it is analytic at infinity you get away by using a drawing inspiration Riemann's removable singularity theorem okay by simply for example just requiring a function continues at infinity okay, so in the same way if you look at the Laurent series at infinity, how do you define this? So see the idea of so you have to think like this, the idea of what is a Laurent series see the whole point about Laurent series, a Laurent series is a generalisation of Taylor series and what is the Taylor series? Taylor series is trying to express an analytic function in terms of simple analytic functions.

Functions which are simplest possible analytic functions, so you know if when I say Taylor series of an analytic function at a point Z naught in the complex plane I am simply expanding the function in powers of Z minus Z naught and I know Z… the powers of Z minus Z naught they are they are simple function they are simple polynomials and I am trying to expand I am trying to write the function a given analytic function as the limit of polynomials, after all a power series for that matter any functional series is by definition when it converges is just the limit of partial sums okay and if you take a Taylor series or a power series then the partial sums are all polynomials okay and it is the limit of these polynomials that gives you the given function okay which is the limit of the series.

So the purpose of a Taylor series is to expand a function as a series in terms of simple function is that is the idea okay and this is this is at the point where the function is analytic okay but suppose a point you are in question is not a point of analyticity suppose it is a point where the function as an isolated singularity then then what comes in is the Laurent expansion, the Laurent expansion says that well you can still get an expansion of the function in the form of the series but then now you will have to allow also negative powers okay.

Now so in general we think of philosophically we think of the Laurent series as a generalisation of the Taylor series and the guiding philosophy is that number 1 is that it allows you to expand the function in terms of simple functions that is point number 1, point number 2 is that the Laurent series and the broken up into pieces, there is one part of the Laurent series which consist of positive and 0 powers of Z minus Z naught Z not is the point where you are looking at the Centre of the series that is called the analytic part of the Laurent series okay and then there is also the part of the series that involves the negative powers of Z minus Z naught which you call as the singular part or the principal part of the Laurent series.

So you see the general idea of Laurent series is that it breaks the function into 2 functions, it breaks the function into 2 pieces it expresses the function as a sum of 2 pieces. One piece is the analytic part of the function at that point in the neighbourhood of the point, the other piece is the principal part of the function at that point which is not analytic at that point okay. Now using these 2 guiding philosophies you can also define what a Laurent series at infinity means okay and well the point is that the point is as follows, so you suppose f of Z is analytic at Z equal to infinity or let me not even start with analytic at infinity.

Let me just say let me say this analytic at neighbourhood of infinity suppose f of Z is analytic in a neighbourhood of infinity say in and you know for obvious reasons let me do the following thing let me not use Z let me use W okay say in mod W greater than R okay so I am using W as a variable because I will always you know when I want to study W at infinity I will rather study you know there is one of the tactics that we have been using is that you study W equal to 1 by Z at 0, so that is why I want to reserve W for 1 by Z okay, so suppose f is analytic at neighbourhood of infinity say in mod W greater than R, so of course by this I do not mean the function is analytic at infinity mind you okay, so you have to be a little careful when I say function is analytic in a neighbourhood of a point in the complex plane it is understood that the function is also analytic at that point okay but then when I am saying f of W is analytic in the neighbourhood of infinity I am not necessarily meaning that it is also analytic at the point at infinity. The point at infinity could is a singularity okay it is an isolated singularity I did not know whether it is analytic or not okay.

So let me let me state that we do not know we do not know if f is analytic at infinity okay. Now what you do with this? Of course you know let us go by the philosophy that to study f of

W at infinity you study f of 1 by Z at 0 okay. The behaviour the behaviour of f of Z, f of W at W equal to infinity is the same as the behaviour of f of 1 by Z which is g of Z mind you f of 1 by Z is the same as W, f of W where W is equal to 1 by Z at Z equal to 0 okay. Note that g of Z is defined in defined an analytic define an analytic or holomorphic in a deleted neighbourhood of the origin which is just given by 0 less than mod Z is less than 1 by R so which is actually writing this mod W with greater than R in terms of Z okay putting W equal to 1 by Z okay, so and 0 less than 0 strictly less than mod Z strictly less than 1 by R is a deleted neighbourhood of the origin it is a circle I mean it is an interior of a circle with the origin removed radius one by R okay.

Now but then you know now you are looking at the point 0 in the complex plane and we have Laurent's theorem since now 0 is an isolated singularity g of Z okay and the idea is that a nature of the singularity of g at 0 should be the same as the nature of singularity of f at infinity okay that is the idea okay and of course you know why that is correct because Z going to W which is Z going to 1 by Z is an isomorphism okay of deleted neighbourhood, alright. So now g has a Laurent expansion okay so g has a Laurent expansion has a Laurent expansion.

So what is the Laurent expansion it is a of Z equal to Sigma n equal to minus infinity to infinity a n Z power n, this is the Laurent expansion of g okay and mind you I am the Centre of the expansion is the origin normally if the centres is the point Z naught and you have to use powers of Z minus Z naught but here Z naught is 0 so use powers of Z and the point that is a Laurent expansion at there are negative powers of Z included as well that is why this summation is running from minus infinity to plus infinity and well so this is the g as it is. Now you know what you must understand is that you know if you look at if you look at this what does it mean for F, so what this will tell you is that see this is valid for mod Z less than 1 by R Z naught equal to 0 okay and if I plug-in see of course instead of Z I can put 1 by W okay instead of Z can put 1 by W and g of 1 by W is just f of W okay.

So this is the same as writing this is equivalent writing f of W is equal to Sigma you know n equal to minus infinity to infinity a n W power minus n okay. So I can simply replace Z by 1 by W's, so Z power and becomes W power minus n and summation will run from again minus infinity to plus infinity the only thing is that because I change my variable the powers the power of the nth power of the variable has a negative subscript okay right, so this is well this is the Laurent expansion but now so you know the point is that the Laurent expansion, so this gives you a clue as to what you should call the Laurent expansion of f at infinity you can very will call this expression that you have written for f as a Laurent expansion at infinity okay because it is in line with a philosophy that it is expressing it as a series okay in terms of simple functions, the functions are just powers of Z okay, so you can very will call this in on the right this expression of f W as Laurent expansion of f at infinity that is fair enough but then there is a little bit more to be seen.

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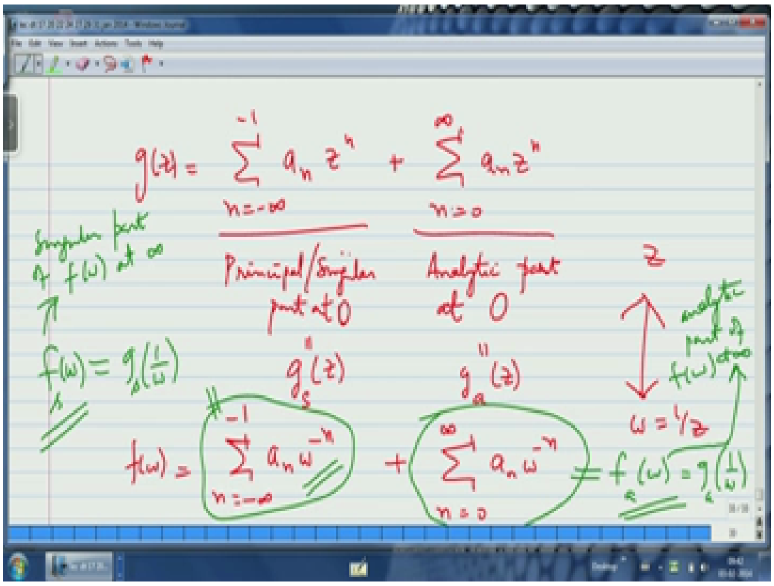

You see if you write if you look at g of Z the Laurent expansion split into 2 pieces as I told you, the Laurent expansion split into principal part is a single apart plus an analytic part okay and the principal part or the single apart consist of negative powers of set, so you know let me write this like this, so this is Sigma n equal to minus infinity to minus 1 a n Z power n plus Sigma n equal to 0 to infinity okay a n Z power n okay there are 2 pieces and this fellow here is the so what I have written here is the principal part is the principle or the singular part at the at the origin okay and this part is the analytic part at the origin okay, so you know if you want let me give symbols to these things, so this is so if you want this is g S of Z, g S of Z is the single apart okay and this is g a Z is the analytic part alright.

Now let us make this change of variable which is Z going to W which is equal to 1 by Z okay and you know now watch carefully what is our philosophy? Our philosophy is that if you change available from Z to 1 over Z okay then the behaviour at 0 at Z equal to 0 should correspond to the behaviour of W equal to 1 by Z at infinity therefore if you go by this if I transform g S of Z to W which is 1 by Z, what I should get should be the singular part at infinity okay because g S of Z is a similar part of singular apart at Z equal to 0 if I transform g S of Z I putting Z equal to 1 by W what I should get is a singular apart at infinity okay and similarly if I transform g a of Z by putting Z equal to 1 by W I should get the analytic part at infinity.

So what do I get…see basically what I get is I get f of W you see is so I will get Sigma n equal to minus infinity to minus 1 a n so I will get a W to the minus n plus Sigma here I will get n equal to 0 to infinity a n W to the minus n and if you watch carefully so let me use a different color at this point, if you watch carefully now you see this guy here this corresponds to…what is this? This is just g S of 1 by this is g S of 1 by W okay because I have put Z equal to 1 by W and but g s was a singular part and therefore g s of 1 by W should also be the singular part so this should be in principle this must be equal to the single apart of f at f of W okay and watch carefully this singular part of f at W has what powers of W? It has positive powers of W okay it has positive powers of W because n is negative.

So W to minus n as positive therefore the moral of the story is that if you write if you write the Taylor series if you write a positive power series in a variable at infinity corresponds to a singular part okay and that is very believable because as you go to infinity okay the partial sums which a polynomial are going to go to infinity, so infinity is a pole actually therefore at least for the partial (())(24:26) okay it is not bounded at infinity, so it is correct okay so the whole point is that when you look at the…so the Taylor series at infinity should be thought of I mean when you look at the Laurent series at infinity the principal part should will look like a Taylor series at the origin it is because it is exactly that by the transformation Z going to 1 over Z okay.

So this is the this is the similar part of f and then and then this guy here this is this is f a or W which is g a of 1 by W and this is the analytic part at infinity okay so you see f of W is also split into a singular part f S W plus f a of W which is an analytic part at infinity okay and mind you the analytic part at infinity contains all the negative powers including the constant because I have put n equal to 0 is also here, so a not is here in the analytic part of infinity and so you see when you look at the variable at 0 okay then the analytic part consist of the non-negative terms and the single apart consist of the negative terms but when you look at the variable at infinity the analytic part consist of the negative terms and including the constant and the principal part consist of the positive terms this is the this is exactly what happens and it is correct okay.

Now why do you thing that this analytic why do you think that the negative powers are the portion of the expansion which involves the negative power is analytic at infinity? That is correct because you see as variable approaches infinity the negative power approach 0, so you see that it is bounded essentially okay so it is analytic and because our definition of analytic at infinity is either that it should be bounded or it should tend to a limit okay and no positive power of a variable will ever tend to will ever be bounded or will ever tend to 0 if you let the variable go to infinity okay.

So everything is fine so you know so here is the so here is our definition our definition is you take a function is an analytic in a deleted neighbourhood of infinity okay right out its Laurent expansion okay basically the Laurent expansion is the Laurent expansion of gotten by changing the variable to its reciprocal okay and then what you do is you take the positive part of the Laurent expansion okay in the original variable and call that as a singular part okay and the negative part including the constant term is what is called the analytic okay so now we have this clear definition of what a Laurent series at infinity should mean okay, fine.

Now once you have made this definition of Laurent series at infinity what you need to know is that whether this fits in well with the theory that you do on the finite complex plane, so for example you can ask this question suppose you know for a point Z naught in the finite complex plane a function is analytic at that point if you assume to begin with that that point is an isolated singularity the function is analytic at that point if and only if you write the Laurent expansion about that point it has no singular part its principal part is 0 okay so if you go by that philosophy for the function f which is defined in a neighbourhood of infinity I mean for which infinity is an isolated singular point the function will be analytic at infinity if the Laurent expansion at infinity as no singular part that is what should happen.

Now does that happen? It does you see here is my function f W defined in the neighbourhood of infinity its singular part is this is this part which consist of positive powers of W okay, if that singular part is not there okay that means that I can express f only in terms of negative powers of W then that is of course analytic at infinity cause all these go to 0 as W goes to infinity okay. So the moral of the story is that our definition is correct, so you know the point is that sometimes you may have to make definitions based on certain philosophy and then you have to check whether it matches with what happens, what you expect to happen as morally correct okay, so from this it is very clear that a function is analytic at infinity if and only if its singular part at infinity vanishes okay.

If you take the Laurent expansion at infinity okay then its singular part vanishes so let me write few things so this is the this is called the singular part of f W at infinity and this guy here is called the analytic part of f W at infinity and now you know if you if you look at it in a very simple way you know when you are looking at infinity? The good functions are negative powers of W because they go to 0 and the bad functions of positive power is W because we go to infinity so if you expect a function to be good at infinity it should be expressible only in terms of negative powers of W and that is why we negative powers of W along with constant that part is analytic part at infinity okay.

So the definition is very clear and with this definition you see that Riemann's removable singularity theorem is also valid in its various forms at the point at infinity a function namely a function which has infinity as an isolated singularity has that singularity as a removable singularity if and only if it is bounded in a neighbourhood of infinity, if and only if it tends to a limit at infinity and that is also equivalent to saying that the Laurent series at infinity as no single part they are all equivalent okay, so you get the same version of the theorem as you would get in the case of a finite point, a point in the usual complex plane okay, so everything fits well. The only thing that does not work is trying to define a derivative at infinity that does not work okay, fine. So now what I am going to do is I am going to ask this question as to what it means to having the removable singularity at infinity for example for a good function For example a function like an entire function okay so what does it mean and you will see that travel throughout connections with Liouville's theorem and so on so see so let us analyse this.

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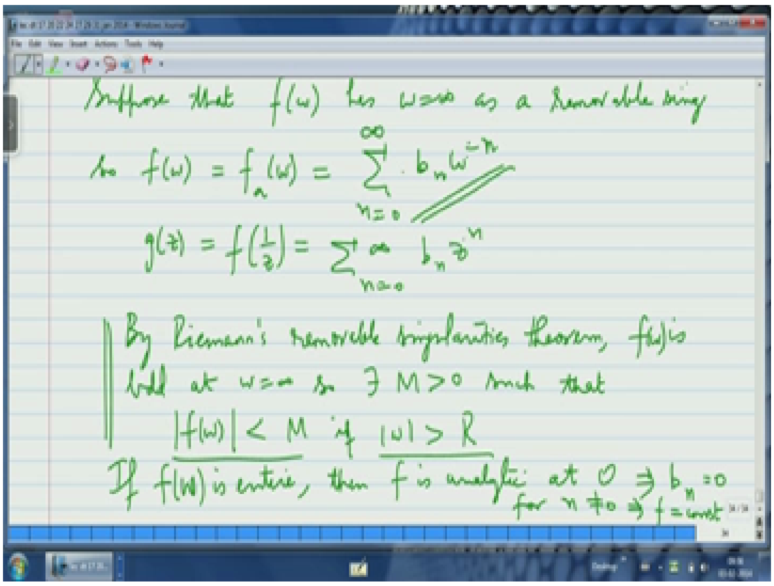

Suppose that f of W as W equal to infinity as removable singularity okay so f W is f analytic so it has no singular part it has only the analytic part of it expansion and that is what this is writable as the you know the analytic part of the expansion at infinity will involve a negative powers of the variable and also the constant term so it will be n equal to 0 to infinity if you want I can call it as b n W power okay so this is the analytic part at infinity, now let us analyse what it means say that the function is For example you know entire, suppose…so I am looking the following case suppose I have an entire function and suppose it is analytic at infinity, what happens?

We will see that it will reduce to a constant okay and that is just another avatar of the Liouville's theorem okay so how do you see that see suppose f has a removable singularity at W equal to infinity then f of W have this expansion which is analytic at infinity then you see g of Z which is f of 1 by Z where I put W equal to 1 by Z what I will get is I will get Sigma n equal to 0 we infinity I will get B n Z power n which is you can see that that is clearly Taylor series at the origin it is a power series at the origin okay centred at the origin, so it has to represent an analytic function and that is correct because f is analytic f W is analytic at W equal to infinity if and only if g Z which is f of 1 by Z that is analytic at Z equals to 0. Let us go back to Riemann's removable singularity theorem okay saying that f is saying that f is analytic at infinity is the same as saying that f is bounded at infinity okay so it means f is bounded in a deleted neighbourhood of infinity alright.

So by Riemann's removable singularity theorem f is bounded so I am using bdd as an abbreviation for bounded at W equal to infinity, so f of W if you want so that exist an positive constant M greater than 0 such that you know mod f of W is less than M for if well if mod W if mod W is greater than R okay so this is this is what bounded at infinity means. In a neighbourhood of infinity the function in modulus can be bounded by positive constant okay and this is this is equivalent to infinity being good point namely it is equivalent to infinity being a removable singularity okay. Now watch, see for mod W less than or equal to R, so mod W greater than R the modulus of the function is bounded by M okay and look at mod W less than or equal to R. I am using the assumption that so I am putting this extra condition that f is entire and mind you I am saying f is entire as a function of W okay.

So I am trying to look at an entire function which is having a removable singularity at infinity, so f of W itself is an entire function even for W finite okay, so you see if f of W is entire okay then you know f is analytic at 0, so is analytic at 0 okay f is analytic at 0 because the entire function supposed to be analytic at every point, at every finite point, so if when I say f of W is entire as a function of W it should be analytic at for all values of W in the complex plane in particular it should be analytic at 0 and if it is an analytic at 0 and you know if you write out it should tend to limit as W tends to 0 okay but then look at this expression, look at this expression these expression mind you normally this expression will be will be valid it is supposed to be a Laurent expansion at infinity.

So it should be valid only in a neighbourhood of infinity but since the function is entire it is valid everywhere, it is valid on the whole on the whole complex plane, so it is valid at 0 also okay and if it is valid at 0 you can see all the b n for n nonzero they all should be 0 okay because the moment I get negative power of W at W equal to 0 it is not going to give you a finite limit is going to go to infinity because is going to become like a pole okay so the moral of the story is that if you assume f is entire then f is analytic at 0 and this will imply that all the b n is 0 for n not equals 0 and this this implies that f is a constant okay, so that is very obvious so all am trying to say is that if you have an entire function which has a removable singularity at infinity then it is a constant.

What is the Contra positive of that? The Contra positive of that is supposed you have a non-constant entire function then infinity is certainly not a removable singularity, for a non-constant entire function infinity cannot be a removable singularity because the only entire function entire functions which are analytic at infinity are constants okay and you can the reason why I got into this this stuff about modulus is because I want to say that this is actually another avatar of Liouville's theorem see because you see, look at this look at this stuff that I have written in between see f is analytic at infinity so outside a circle of sufficiently large radius mod W greater than R, mod f W is bounded okay but if you look at the if you look at the interior of that circle and the boundary of that circle I will get mod W less than or equal to R and you see mod W less than or equal to R is a compact set it is both closed end bounded and I have assumed f is entire so it is continuous, you know in a continuous function on a compact set is bounded because the image of a compact set under continuous map is again compact and compact is will imply bounded.

So what this will tell you is that, there is a bound for f even in mod W less than or equal to R that combined with the bound for mod W greater than R will tell you that f is an entire function which is bounded on the whole plane and then Liouville's theorem will tell you that f is a constant, so that is the point and I want to tell you, I want to tell you that see this argument that an entire function which is analytic at infinity is constant is actually another avatar of Liouville's theorem okay it is actually another avatar of Liouville's theorem that is what you have to understand, so the moral of the story is that whenever you are looking at a non-constant entire function infinity is certainly a singularity it is a (())(40:13) it is not a removable singularity, so it can either be a pole or it can be an essential singularity it cannot be removable singularity okay and the only exemptions are constants which are very uninteresting okay, so I will stop here.