Advanced Complex Analysis - Part 2: Compactness of Meromorphic Functions in the Spherical Metric, Spherical Derivative, Normality, Theorems of Marty -Zalcman-Montel-Picard-Royden-Schottky

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Lecture No 7

When is a Function Analytic at Infinity?

Okay so what we are looking at is you know we are trying to understand the behaviour at infinity okay and somehow the idea is that if you want to study f of Z at infinity then it is the same as studying f of 1 by Z at 0, so see the question was, why is this why is this justified and well I was trying to explain that in the towards the end of the last lecture, so I will take off from there.

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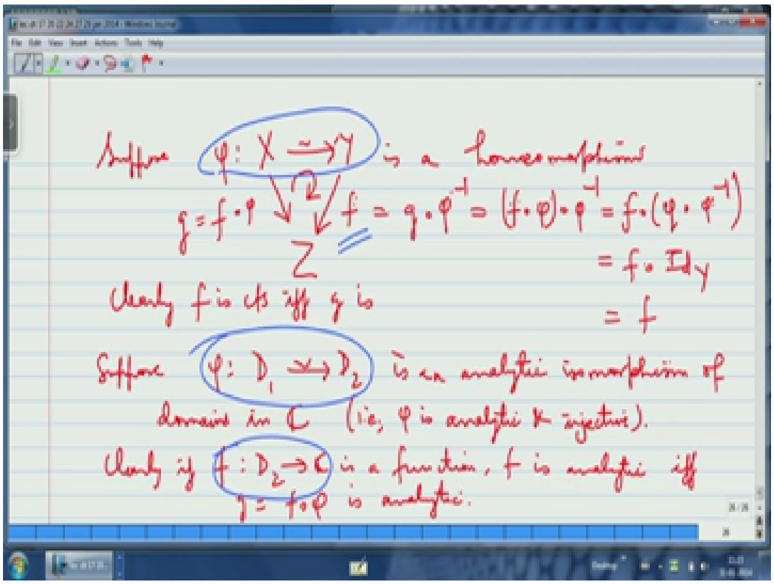

So let me let me go back to what I was saying, so you see so the idea is the idea is the following, the idea is the following. The idea is that you know whenever 2 objects are isomorphic okay then they are property should correspond alright and is not only there for example if 2 topological spaces are isomorphic okay then you expect both of them to have the same to have the same topological properties, so if one has a certain topological properties, you should expect the same thing to happen of the other okay, so for example if 2 topological spaces are isomorphic one is connected then the other should also be connected, so a disconnected topological space cannot be isomorphic to a connected topological space because continuous image of connected set is connected okay and so on and so forth.

So this this is a very general philosophy in mathematics for example if 2 vector space are isomorphic you then they will have the same dimensions okay so the properties like dimensional, properties like in topology, properties like connectedness, compactness these are all intrinsic properties and they are not supposed to change under isomorphism. Now therefore so that is one level of thinking, the other level of thinking is that especially when you are looking at an isomorphism of spaces okay is not only that the properties of the spaces, geometrics properties of the spaces should correspond namely if one of the spaces as a geometric property in the other should also have an by geometry here of course I mean you know geometry something that includes both I mean it includes topology it includes algebra, it includes analysis, it includes an interplay of all these properties.

So you expect if 2 objects are isomorphic you expect them to behave in the same way whether no matter if you look at them topologically or algebraically or analytically okay and of course you should also assume that the isomorphism is also going to be compatible with whatever structure you are looking at if you are if you are looking at topological properties, the topological structure then you expect the isomorphism to be are topological isomorphism which is a homeomorphism, if you are looking at algebraic properties you should expect the usual thing of an isomorphism which also you know behaves well with the algebraic structure and if you are looking at an analytic properties for example if you are looking at manifold theoretic properties and you should expect the isomorphism to be an isomorphism that respects their structures.

So now you see the big deal is that whenever 2 spaces are isomorphic then the functions on those spaces are also isomorphic okay, so this is a very important idea, so and function are isomorphic the sense that you know properties of functions carry over so that is what was trying to explain last time, so look at this thing so x phi from x to y is a homeomorphism of topological spaces and an your given this f which is a function on the topological space y with values in another topological space Z okay and I am just saying is a function, I am not saying that I do begin with I do not know whether it is continues or not okay but the point is that the moment you give it is very natural (())(4:53) theory that whenever you give me a function on the target set you can composite with the given map from the source set to the target set to get function on the source set.

So that is called the pullback of the function, so you know f is a function on y you can composite it with phi to get this functionality g which is now a function on the source which is x okay and then the question is that what is the connection between f and g in fact f and g should correspond to one another in this isomorphism of space which induces an isomorphism of functions okay. So in fact you know what is happening is that there is a map from functions from y of Z to function from x to Z that is a pullback map and what is happening is f is going to g okay and this is an isomorphism because phi has an inverse okay this an isomorphism and the fact is that under this isomorphism properties functions of properties, particular properties coincides, so they correspond, so for example if f is continuous then g is continuous and conversely if g is continuous f is continuous because you can get f from g and g from f because phi is invertible.

Now this is at the this is at the topological level okay and then I am saying at letters look at it at a at the level of complex analysis for example, so suppose D 1 and D 2 are domains in the complex plane and phi from D 1 to D 2 is an analytic isomorphism namely it is holomorphic map, it is an analytic map which is injected okay and you know an injective holomorphic map is an isomorphism and then our philosophy should tell us that the functions on D 2 correspond they are in one-to-one correspondents with functions of D 1 okay. In particular if you are looking at complex value functions on D 2 they should be in Bijective correspondence with complex valued function on D 1 okay and that is again by the pullback, so if you give me a function f on D 2 with complex values then you know I get the pullback function g okay and f is analytic if and only if g is analytic and the reason for that is again the fact that phi is invertible, phi is analytic and you use the fact that the composition of analytic function is analytic okay, so now what I want to do is that you can go one step further and you can discuss singularity.

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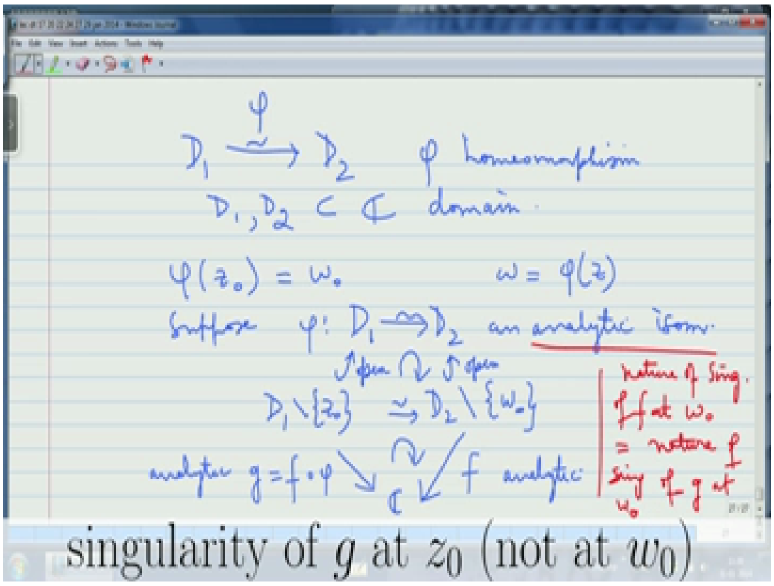

So you know so let me write this down, so let us assume that you again have this you have a map from D 1 to D 2, phi is a map from D 1 to D 2, phi is a homeomorphism okay and I am assuming that D 1 and D 2 are they are domains in the complex plane okay and of course mind you I am being a little careful, I am saying that phi is a homeomorphism am not saying that it is an isomorphism, it is holomorphic isomorphism, analytic isomorphism but I say the following thing suppose phi takes a point Z naught in D 1 to point W naught in D 2 okay, so I am writing the function phi as W is equal to phi of Z okay. Z is the is the independent variable is supposed to vary in D 1 okay and W is phi of Z it is a dependent variable it is supposed to take values in D 2 okay and I am assuming that phi takes Z naught to W naught okay and suppose that you know phi from well suppose phi from D 1 to D 2 is actually an analytic isomorphism.

So now I am you know I am assuming something more am assuming that phi is not just a homeomorphism I am assuming it is an analytic isomorphism which means in particular it is a homeomorphism okay and mind you an analytic map is continuous okay and therefore analytic isomorphism is stronger than being a homeomorphism okay. Well now you see now you can do the following thing, see suppose you already know that if you have a function on D 2 I can put it back to give you a function on D 1 but suppose I had a function on D 2 with a single point, isolated singular point a singularity at W naught okay. Then by composing with phi I will get a function on D 1 which will have an isolated singular point at Z naught okay.

So you see suppose so the situation is like this you have D 2 minus the point W naught and that is being carried over by phi to D 1 minus Z naught because you know phi is after all

Bijective it will take the complement of Z naught to the complement of W naught and Z naught will go to W naught, so basically what it is doing is that it is taking the punctured domain, punctured at Z naught and mapping it at holomorphically, isomorphically, analytically isomorphically onto the punctured domain at D 2 punctured at W naught okay and suppose you have a function, suppose you have a function f here, complex valued function okay which is analytic okay then you know if I compose this composite with phi I will get this function g which is first apply phi and then apply f okay and of course g will also be analytic okay and mind you this inside this is for an open subset.

This inside this is an open subset, the complement of a point is always open and this diagram also commutes okay and well the fact is that what this tells you is that f has an isolated the point W naught I do not know whether the point W naught f is analytic or not okay but in the deleted neighbourhood of W naught for example the domain D 2 minus W naught is deleted neighbourhood of W naught and there I know f is analytic, so W naught is an isolated singularity of f and what this diagram tells you is that a function g that you got by pulling back f via phi is also having an isolated singularity at Z naught okay and now you can you can believe that if you believe in the philosophy that under a pullback functions by an isomorphism okay properties of function should coincide, you can believe that nature of the singularity of f at W naught should correspond should be exactly the same as the nature of singularity of g at Z naught okay.

So it is natural to expect that if that f as say a removable singularity at W naught then g should have a removable singularity at Z naught if f has a pole at W naught of a certain order then g will also have a pole at Z naught of that order and if f has an essential singularity at W naught then g will have an essential singularity at Z naught and the converse will also (()) (12:33) okay so properties of f this the nature of singularity of f at W naught should correspond to the nature should be exactly the same as the nature of singularity of g at Z naught okay, so this is something that that is very easy to understand and why is this why is that why is that true? That is just true because D 1 to D 2 is an analytic isomorphism okay it is because it is an analytic isomorphism that this is happening.

So let me write this in a let me use a different color okay, so well so you see well let me write this here nature of the singularity I am abbreviating it as sing of f at W naught is equal to nature of singularity of g at W naught okay and see this is this happens basically because you can you can see this in a moment you see, so why is this true? So you know let us look at 3 cases, suppose f is suppose f has a removable singularity at W naught okay then what it means is that you know by Riemann's removable singularity theorem you know that the limit of f as W tends to W naught exists this is also the same as saying that the f extends to an analytic function at W naught okay and is equivalent to saying that f is continuous at W naught okay and if f is continuous at W naught by composition with phi it is clear that g is also continues as Z naught okay.

So it is very clear that if f has a removable singularity at W naught then g has a removable singularity at W naught and the converse is also true, if g has a removable singularity at Z naught then f will have a removable singularity at W naught because you can if you compose g with phi inverse you get f okay so this tells you that f as a removable singularity at W naught if and only if g as removable singularity at Z naught okay. Now what about the case of W naught being of pole if f has a pole at W naught then a limit as W tends to W naught of f is infinity okay and you know you see by continuity it should happen that g will also have a pole at Z not.

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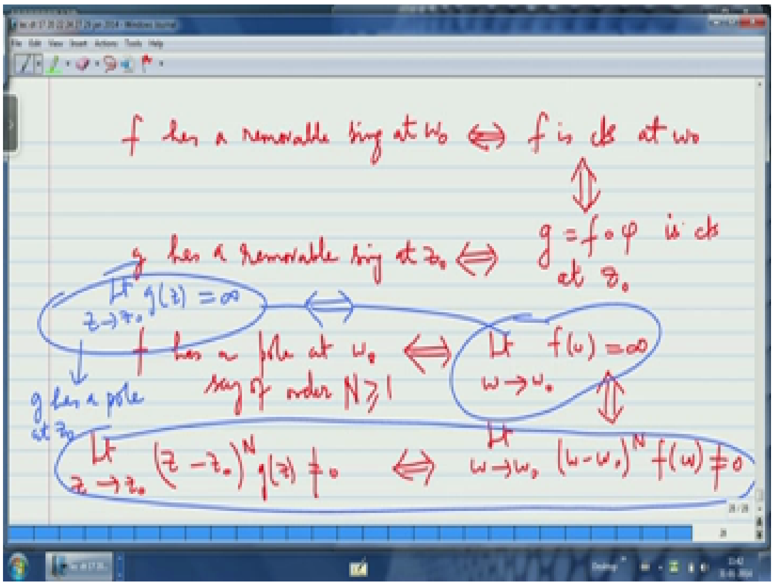

It is continuous at Z naught okay and g being continuous at Z naught is the same as saying that g has a removable singularity at Z naught okay you have this. Now what is the situation when f has a pole at W naught f has a pole at W naught if and only if you know say of order N greater than equal to 1, what is the condition for a pole? It is well limit one condition is of course limit Z tends to I mean limits W tends to W naught of f of W should be infinity this is this is one of the conditions and that is also the same as saying that limit W tends to W naught of W minus W naught with the power of N times f of W is non-zero okay this is exactly the condition that f has a pole of (())(16:36) okay and you know if you well if you translate if you translate this limit when realizing that as W tends to W naught if and only if Z tends to Z naught because phi is a homeomorphism okay and you know under a continuous map the image of convergence sequence is again a convergence sequence okay the image of a limit is again a limit alright.

So you know so if you translate that you did you will actually if you if you change the variable it will tell you that Z minus Z naught to the power of N g of Z is nonzero if you lead Z tends to Z not, okay. So the easiest thing to do is the following thing, what you do is you take this this is the easiest probably, is the easiest thing to do what you do is you put W is equal to phi Z okay if you put W is equal to phi Z you will get limit phi Z tends to phi Z not, f of phi of Z but f of phi of Z is g Z okay and the limit phi of Z tends to phi of Z naught is the same as limit Z tends to Z not, so you get limit, so this is the same as saying that limit Z tends to Z not, g of Z is infinity okay. See this is this is plainly equivalent okay so this is what I want, so this is what we want, so this is this is plainly equivalent to limit Z tends to Z naught g of Z is equal to infinity.

It is just you just make a change of variable from W to Z so you will have to so you put W is equal to phi Z okay then you will get f of phi Z, f of phi Z is just g Z by definition okay and W tends to W naught will read phi Z tends to phi Z naught but phi Z tends to phi Z naught is equivalent to Z (())(18:34) to Z naught because phi is a homeomorphism so this is the same as this and add this condition limit Z tends to Z naught g Z is infinity is will tell you that it is a pole Z naught is a pole of g okay and you will have to do a little bit more work to for example compare Laurent series say that the poll is exactly at the same order on both sides okay, so you see therefore essentially I am just using the fact that phi is a homeomorphism I am not using anything more than that alright, so well so let me write this here g has a pole at Z naught and a little bit more work will tell you that the pole will also have order m okay fine.

So you know you can if you try to if you try to prove this thing below okay that may not be so easy to do at the face of it okay, fine. So anyway what this tells you is that if f has a pole at W naught then g has a pole at Z naught and conversely okay and of course you can get the converse because instead of phi you can use phi inverse which is also a homeomorphism okay and well and then the last case is the left out case if you will have an essential singularity at W naught if and only if g has an essential singularity at Z naught this is just by tautology it is just biologic logic because essential singularities are singularities which are neither removable nor poles okay and you already shown that removable singularity is correspond you showing that polls correspond therefore essential singularities which is complement of these 2 should also correspond okay.

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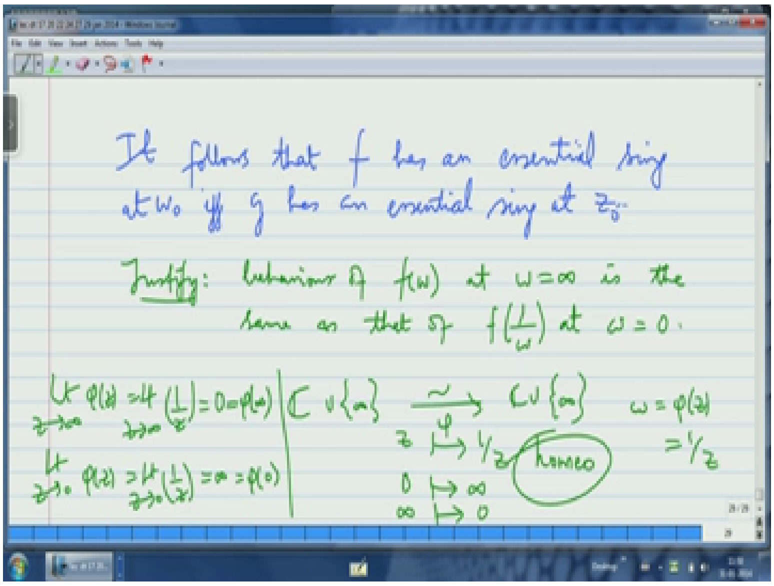

So let me write that down it follows that f has an essential singularity at Z naught at W naught if and only if g has an essential singularity at Z naught okay this is this is very clear and in fact if you want we can also say it in another way, what is the condition that what is the condition that a function has an essential singularity that point one condition for example is at the limit as pro set point does not exist. If limit Z if limit W tends to W naught f W does not exist then f must have an essential singularity at W naught and again you know if you if you use a substitution W equal to phi Z and remember that phi is a homeomorphism it is very clear that the limit as W tends to W naught f W will not exist if and only if limit as Z tends to Z naught g of Z does not exist and this will tell you that W naught being an essential singularity or f is the same as Z naught being an essential singularity for g okay.

So essential singularity corresponds but of course here I am using the fact that the limit does not exist and where did that come from that basically came from application of actually if you go back it is an application of Riemann's removable singularity theorem which says that the limit exists then it is removable okay and if the limit exists and is infinite then it is of pole okay and if the limit does not exists then it is an essential singularity and all these ifs are actually if and only ifs okay fine. Now having said all of this, now how do we deal…I want to get back to try to you know tell you the story that saying that f of Z studying f of Z at infinity at Z equal to infinity is the same as studying f of 1 by Z at 0 okay and what is the, why is that justified in the light of this argument, so it is justified in the following way, so let me write that down you know justify behaviour of f of Z… Let me use f of W because so that I am consistent with my notation, behaviour of f of W at W equal to infinity is the same as that of f of 1 by W at W equal 0 okay.

So this is if you if you look if you have gone through the 1st course in complex analysis and behaviour at infinity was covered then you would see people saying that f has a has a… The nature of singularity of f at infinity is the same as the nature of singularity of f of 1 by Z at 0 okay and why is that true? Is true in the light of following argument which is based on what we have been saying, you take this map from C star let me not use C star, take this map from C union infinity which is extended complex plane to C union infinity. This is extended complex plane to the extended complex plane okay, so now we are making use of the point at infinity and we are also making use of the topology of the extended complex plane.

So and you know and what is the map, the map is just Z going to 1 by Z you take this map, so this is my map fee, so here my phi of Z is 1 by Z, so W is phi of Z which is 1 by Z so W is 1 by Z this is my map and this is the well-defined map you see that the point is that you have to the you have to just send infinity to 0 and 0 to infinity this is the obvious thing that you will do and so you know so let me write that down, you send 0 to the point at infinity and you said infinity the point at 0 okay and mind you when you send, when you make these definitions it feel continuous to be a homeomorphism okay. See limit Z tends to infinity phi of Z is what?

Limit Z tends to infinity of phi of Z is just limit Z tends to infinity of 1 by Z which is 0 okay and what this will tell you? so limit Z tends to infinity phi of Z is 0 that is exactly phi of infinity as per our definition because we have sent infinity to 0 and what does this tells you? This tells you phi is continuous at 0 at infinity and the same kind of argument will tell you that phi is also continuous at 0, so if you take limit Z tends to 0 phi of Z is limit Z tends to 0 of 1 by Z and this is infinity which is phi of 0 okay and mind you we have defined what limit Z tends to infinity means we have define what when limit is in finite, we are using all those definitions, we are using the fact that (())(26:12) of infinity actually is a point okay and we are using and we are also thinking of infinity as a value okay do not a finite value.

So the fact is that if you look at this map now from extended complex plane to extended complex plane Z going to 1 over Z this is actually homeomorphism this is the important point is a homeomorphism and if you throw away both the origin and the point at infinity okay you get C star which is punctured complex plane C minus 0 okay and C minus 0 goes to C minus 0 and if you take if you restrict this map to C minus 0 it is an analytic, it is a holomorphic isomorphism that is the point. This map is a holomorphic isomorphism which extends to infinity okay, so…in a continuous way okay so let me write that down.

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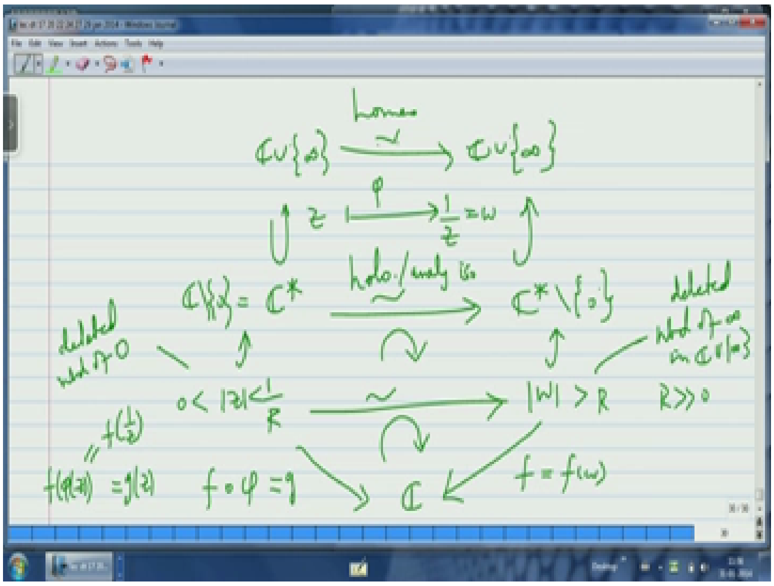

So let me draw the diagram again, so I have the C union infinity here and I have this homeomorphism to C union infinity and this is the map phi which is sending Z to 1 over Z okay which is W right and what is sitting inside this is C star which is C minus 0 with the punctured plane and on this side also I have C star that is the image of C star because if Z is a nonzero complex number and 1 by Z is also a nonzero complex number and this is correspondences is an Z going to 1 by Z is analytic if Z is not 0 because it has the derivative minus 1 over Z square you know that pretty well Z equal to 0 is a is a pole of a is a simple pole of (())(28:01) 1 by Z okay so to the point is that if you restrict phi to C star what you get here is not just a homeomorphism it is a holomorphic analytic isomorphism.

So this is the this is so let me write that here it is a holomorphic analytic isomorphism, this is what you get and now watch suppose I have a function which is defined in a neighbourhood of infinity okay suppose f is a function which is defined in the neighbourhood of infinity, what is the neighbourhood of infinity? A neighbourhood of infinity is the exterior of a circle of sufficiently large radius, so you know if I have a function defined on mod Z say greater than R, R sufficiently large this water neighbourhood of infinity is. This is this is a neighbourhood so let me so let me write that neighbourhood of infinity in where? This is the neighbourhood of infinity in the extended complex plane mind you that is how the topology on the extended complex plane has been given okay and in in fact it is if you look at it in the extended complex plane then you are including the point at infinity and if you do not include the point at infinity is a deleted neighbourhood okay.

So since I am looking at it in C star minus 0 is a deleted neighbourhood of the point at infinity okay so this is a deleted neighbourhood, a deleted neighbourhood of infinity where infinity being deleted okay and you know under this map Z going to 1 over Z this should correspond to a deleted neighbourhood of the origin which is mod Z less than 1 by R in no R sufficiently large so one by are sufficiently small, mod Z is less than 1 by R is the…so here probably since my target variable is W I should not have use Z let me correct that this should have been W, so I have reserved W of the target variable and W is…so if you plug in their W equal to 1 over Z I will get 1 by mod Z is greater than R which is same as saying mod Z is less than 1 by R that is the…

So this corresponds to mod Z less than 1 by R which is and you know of course this is the this is the neighbourhood of the origin but if I but since I have not included the point at infinity on the right-hand side I am not the thing that I got get on the left-side is going to not include the origin because I am already you know say I am considering this as a subset here, so infinity is not included and I am considering this as a subset here 0 is not included, so this is a deleted neighbourhood of the origin, so this is so let me write this deleted neighbourhood of this okays origin and mind you Z going to 1 over Z is still a holomorphic isomorphism of this small punctured disk centred at the origin radius one by our open disk with the exterior of the disk with the radius R alright, so now suppose I have a function defined in a neighbourhood of infinity okay that means I have a function f here, I have my function f here okay and it is taking complex values alright then if I use.

So you know this so you know this diagram commutes basically this is just this diagram commutes means that I am just restricting the map phi that is all okay, so this map from this punctured disk surrounding the origin to exterior of the disk with radius R is just the holomorphic isomorphism Z going to 1 over Z which is (())(31:45) as phi and you compose that with f you get as before you get g, so g is just first apply phi then apply f as before but what is that, so you know f is f of W okay and W and so you see that if I calculate g of Z, g of Z will then be f of phi of Z okay but what is phi of Z? Phi of Z is one over Z, so g of Z is nothing but f over 1 over Z okay, so what you see that if you look at the map Z going to 1 over Z, the pullback of f of W becomes f of 1 over Z okay and now you go back to this philosophy that whenever you have an isomorphism of punctured domain and you have an analytic function on the target domain and you pull it back to an analytic function on the source domain than the nature of the singularity of the function at the target on the target spaces is the same as the nature of the singularity in the source space.

So you know if you apply that philosophy you can see that f is at the nature of singularity of f of W at W equal to infinity is must correspond must be the same as the nature of singularity of g of Z at Z equal to 0 but what is g of Z? g of Z is f of 1 by Z okay so you know this justifies the statement that nature of singularity of f of W at W equal to infinity is the same as nature of the singularity of f of 1 by Z at Z equals to 0 okay, so you must so you should understand what is going on, why this is a very natural thing to do? Okay fine so the moral of the story is that we have a justification as to why studying f at infinity f of W at infinity is the same as studying f of 1 by Z at Z equals to 0 okay, now let us go and try to look at what we are going to get okay and so we will we will get 3 cases as usual because we are trying to classify the singularity of f at infinity mind you in all these things will talk about to be able to talk about these kinds of things the function f should be defined in neighbourhood of infinity which means the function should be defined the exterior of the circle okay for all values of the variable in the exterior of a circle of sufficiently large radius okay which is what a neighbourhood of infinity is.

So what is the 1st what is the 1st case, the 1st case is and will you say that f has infinity as a removable singularity okay or more you know removable singularity is the same as a point where the function is analytic okay that is exactly what the Riemann's removable singularity theorem says. It says that if you take a function which has an isolated singularity at a point in the complex plane then it has a removable singularity at that point if and only if it can be extended to an analytic function at that point and of course the weakest condition is that it is even bounded in the neighbourhood of that point that is the strongest that is the strongest part of the Riemann's removable singularity theorem, so I would like to ask when will f be analytic at infinity okay.

Now you know you must be careful when you think about the point at infinity because there are issues, so for example you should not be attempted to say that f normally what is the definition of (())(35:28) at a point in the complex plane the definition the simplest definition of analyticity is that the function is differentiable at that point and in every neighbourhood in in some neighbourhood of that point, at every point in some neighbourhood of that point.

Now you cannot adapt this definition at infinity you cannot say a function f is analytic at infinity if it is differentiable at infinity and it is also differentiable in neighbourhood of infinity okay that the function is already differentiable in neighbourhood of infinity is given because it is already given to me because infinity is an isolated singularity what trying to say that the function is differentiable at infinity will not make sense because the derivative at infinity does not make sense.

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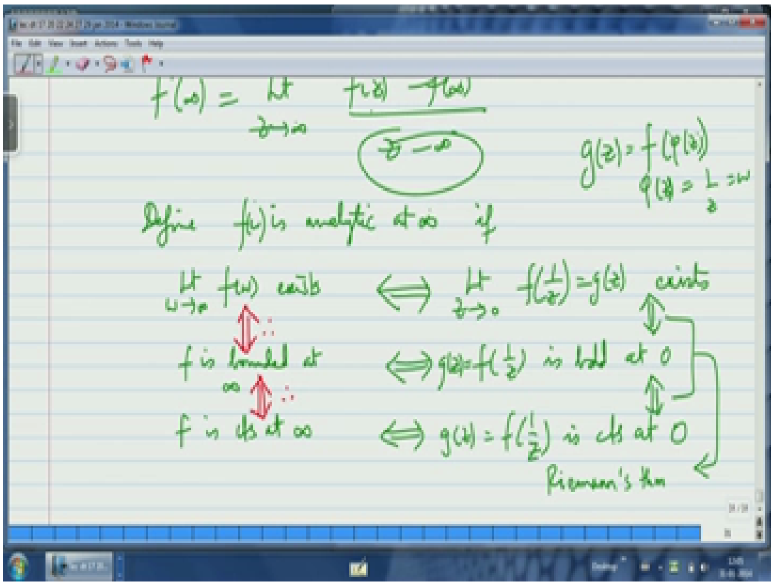

So what is a derivative at infinity if you really try to define f dash of infinity if you want to define it like this (())(36:10) you will write limit Z tends to infinity f of Z minus f of infinity divided by Z minus infinity which really does not make any sense. See f of infinity might make sense because if for example f extends to something continuous at infinity f of infinity could be defined as limit Z tends to infinity f Z okay that is fine but this Z minus infinity is absolutely absurd and trying to let Z tends to infinity so you know this is not trying to make f differentiable at infinity is not going to help because there is no way to do it okay, so how will you do it so the trick is you do it in rather you know indirect way you recall the Riemann's removable singularity theorem which says that if you look at a point in a finite complex plane then the function is analytic there at that point which is an isolated singularity if and only if it is if you want to bounded at that point in a neighbourhood of that point or if it has a limit at that point which means it extends to a continuous function at that point okay.

So you do that so what you do is instead of trying to define the function to be analytic at infinity if it is differentiable at infinity which is wrong because you cannot define the derivative at infinity, what you do is? You say you define the function to be analytic at infinity if either it is bounded in a neighbourhood of infinity or it is it has a limit at infinity namely that is continuous at infinity okay you make this definition and then you are in a very good shape and it will also agree well with… both these definition will agree well with the earlier philosophy that the nature of the singularity at infinity of f of W at infinity is the same as the nature of singularity of f of 1 by Z at 0, so you can see that so here is so defined f is analytic at infinity okay.

So mind you I am so this definition of analytic is very funny okay it is not the definition that the function is differentiable at that point and differentiable in a neighbourhood of that point, it is not the definition okay but it is a definition that, that point is the removable singularity okay it is an indirect way of defining it, so f is analytic at infinity, so let me write f of W if well limit W tends to infinity f of W exist okay or limit let me let me write this here f is bounded at infinity or f is continuous at infinity okay, so and you know all these things all these things are equivalent is the 3 different ways of trying to define the function is analytic at infinity but the philosophy is that you are using you know you are using Riemann's removable singularity theorem and why are they equivalent?

They are equivalent because of the following thing because you see you see this is equivalent to saying that limit Z tends to 0 f of 1 by Z exists which is which we have called as g Z because the map Z going to 1 over Z is a homeomorphism these 2 are equivalent okay and well f is bounded at infinity is the same as saying that f of 1 by Z is bounded at 0 g of Z is bounded which is defined by f of 1 by Z mind you g of Z is by our original notation g of Z is f phi of Z and phi of Z is 1 by Z which is W okay, so g is bounded at 0 okay and the 3rd thing as well g of Z is equal to f of 1 by Z is continuous at 0 and mind you all these 3 are in fact equivalent, all these 3 conditions are equivalent or the function g of Z, why because now I am looking at a function g of Z with 0 as an isolated singularity and all the 3 conditions are equivalent by Riemann's removable singularity theorem to saying that g is actually having a removable singularity at 0 okay, so these 3 are so all these 3 are equivalent any here this is Riemann's theorem.

So I need to make some more space to write down, so let me write it here so this is by Riemann's theorem on removable singularity and because these 3 are equivalent therefore the conditions that I have written on the left side are equivalent that is the point I want you to notice. See this is equivalent to this is equivalent to this, this is therefore this therefore comes from the right side okay, so this will tell you that so you know now we have reconciled all the definitions f has a removable singularity at infinity if f of 1 by Z as removable singularity at 0 is same as saying f of 1 by Z is analytic at 0, f of 1 by Z is continues at 0, f of 1 by Z is bounded in the neighbourhood of 0 and you see we have used to things we have use the fact that Z going to 1 by Z is a homeomorphism and we have also use the fact that we are using the Riemann's removable singularity theorem for a point singularity in the finite plane okay, so we will continue in the next talk.