Advanced Complex Analysis - Part 2: Compactness of Meromorphic Functions in the Spherical Metric, Spherical Derivative, Normality, Theorems of Marty -Zalcman-Montel-Picard-Royden-Schottky

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Lecture No 6

Studying Infinity: Formulating Epsilon-Delta Definitions for Infinite Limits and Limits at Infinity

So you see there are few things which I want you to notice at this point, the 1st thing is that you know we have we have the extended complex plane which consist of a complex plane plus the point at infinity okay and the point at infinity, how it is thought of concretely as a point on a space is by identifying it with the north pole on the Riemann sphere via the stereographic projection which gives you a homeomorphism between the Riemann sphere okay which is the unit sphere in 3 space Centre at the origin okay and the complex plane along with a point at infinity that is extended complex plane therefore see therefore you are able to think of you are able to think of the point at infinity as the as the North pole under this correspondence which is given by the stereographic projection.

Now so you know if you just imagine only the complex plane the point at infinity seems to be an invisible point okay it seems to be far away and out of you know out of sight if you want it is something that you cannot visualise okay but then the way you should really think about it is you must remember that the point at infinity is an interior point okay of the extended complex plane mind you off course for that matter any point in a topological space is an interior point because at point is always contained in the whole topological space which by definition is an open set.

A point in a topological space is called an interior point of a set if it contains there is an open set which contains that point and which is contained in the given set okay, so in that sense of course every point you know in any topological space is certainly an interior point of the topological space but what I want you to understand is that in general and interior point should be able to you should be able talk about a deleted neighbourhood of an interior point and the fact that if you take the extended complex plane you take the point at infinity if you throw infinity what you get is a complex plane and the complex plane itself is a deleted neighbourhood of infinity that is what I want you to understand.

So when you are looking at a complex plane you are looking at a deleted neighbourhood of infinity that is what you must understand okay and of course if you are looking at as we defined just now if you look at the exterior of a circle then you are actually getting a smaller neighbourhood of infinity and the neighbourhood becomes smaller and smaller as the circle becomes larger and larger okay that is how it goes and so you must get used to thinking of infinity as an interior point with a deleted neighbourhood of infinity being thought of as the exterior of a circle in the complex plane okay and all this is all this is not just imagination it is concretely correct okay because this is what you really see when you translate it in terms of the stereographic projection and look at what is happening on the Riemann sphere.

The point at infinity appears on the Riemann sphere as a honest point it is a North pole okay and a neighbourhood of infinity is going to be an exterior of a circle on the complex plane and that appears honestly as an open neighbourhood of the North pole on the Riemann sphere and for that matter if you take a whole complex plane under a stereographic projection will go to the whole Riemann sphere minus the North pole so saying that the whole complex plane is a deleted neighbourhood of the point at infinity amounts to saying in terms of the stereographic projection that the whole sphere minus the North pole is a deleted neighbourhood of the North pole which is correct okay so what I am trying to tell you is that the stereographic projection allows you to really think concretely okay so and this is very important because you see if I want to think of infinity as a point okay then where is that point how do I see it?

So the idea is that you see it as a North pole on the stereographic projection okay and then there is one more thing normally when you are studying properties of function, when you do analysis, normally what you do is that you study the properties of function at an interior point and the reason is because you want to basically what you do in analysis that you study limits and to study limits you will have to take limits in all possible directions so if you want to study the behaviour of function at a point you should be able to study the limit of the function as you approach that point in all possible actions and namely you have to look at functions value that point in all possible directions around that point so you basically you want the function to be binding in a deleted neighbourhood of that point okay otherwise you cannot do this okay and that is the reason why you need an interior point if you want to really do analysis okay and of course taking limit starts with continuity and then even derivatives is the limit and so on and so forth okay.

Now you see what about the point at infinity? You can ask is it also true or false the point at infinity that you are you are able to approach from all directions and the answer is yes you see you take though if you look at the complex plane the point that infinity is not visible okay but you must always think in terms of Riemann sphere and the North pole as representing the point at infinity under this stereographic projection, so you know how do you approach how is it correct to say that you know you are able to approach infinity from all directions?

So you can think of approaching infinity by going along the curve on the complex plane which is going to an unbounded part of the curve okay and for example we can play straight line passing through the origin and then as you move away from the origin on the line you can thing that you are approaching infinity and you can think that you know as you change the lines you are approaching infinity in different directions okay so the truth is that all is well agree if you translate it in terms of the stereographic projection okay so if you look at it in terms of stereographic projection you will see that you are actually approaching the North pole you know from various directions okay so if you want just to help you think about it so let me draw this again.

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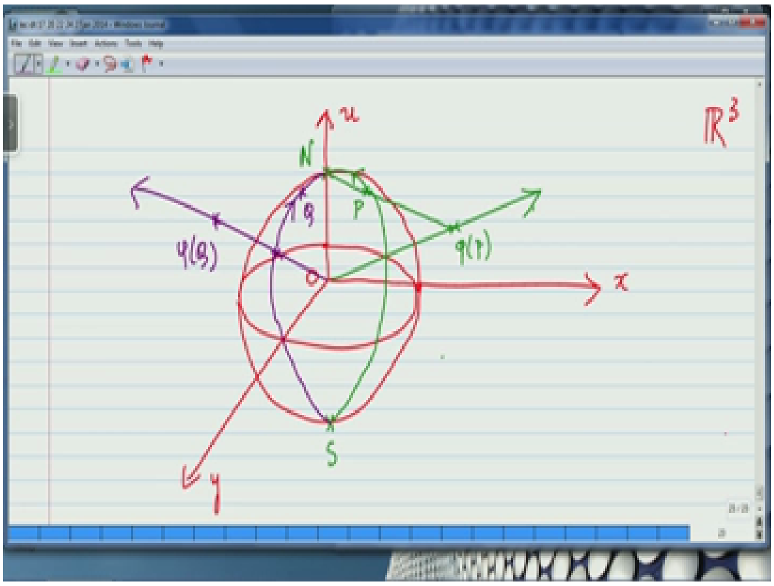

So you know basically you have you have the complex plane here this is the x and y axis and this is the complex plane thought of as x y plane in R 3 and of course as we have done before we are not going to call the 3rd axis as Z we are calling it as u because Z is supposed to stand for x plus iy okay and the stereographic projection is basically you draw the circle I mean you draw the sphere centred at the origin radius one unit and look at the surface of the sphere and so basically I have here this is my circle on the complex plane this is unit circle on the complex plane centred 0 centred at 0 radius one unit and then I have this this sphere which is

Riemann sphere well in fact this is supposed to meet on the x axis, so let me see where I can change that okay, fine.

So here is how it is now you see yes on the complex plane if I want to oppose the point at infinity what I do is that I can I can go along the way okay and move away that moves away from the origin okay and I live move further and further along the way I am supposed to be going to infinity okay and well and the direction of the ray is supposed to give one of the directions in which I can approach infinity okay, so and why is this correct why is this correct? This is because you see if you take the image of this ray onto the stereographic projection okay what you will actually get is that you will basically get, you will basically get this under see mind you so let me remind you see you have the North pole here okay the stereographic projection is done in the following way you give me the give me any point on the sphere okay then that point is mapped by the stereographic projection to its image phi of P and that is done by simply joining the North pole to that point and looking at the point where this line hits the plane okay.

So here is so phi is the stereographic projection okay and now you can see that under a stereographic projection the origin goes the South pole of the sphere goes to the origin okay, so the origin corresponds to the South pole. The South pole on this sphere corresponds to what point on the plane it corresponds to the origin because it is supposed to correspond to the point on the plane which is gotten by joining the South pole to the North pole by a align and looking at where that line hits the complex plane and so the South pole corresponds to the origin okay and now you can see that you know the image of this ray okay is going to be is going to be this this circular arc like this is what I am going to get okay on the you know that is that is what I am going to get on the Riemann sphere okay and mind you if I move this phi of P further and further towards infinity the point P moves further and further along that great circle to the North pole okay.

So you see basically what is happening is that you are going like this so this is how it is going okay and the fact is that if you if you had taken if you had taken another direction okay then you are you will see that you are that does correspond to going approaching the North pole on the sphere in another direction okay, so for example if so let me let me draw that you can easily imagine it but it is good to draw few diagrams now and then, so you know if I if I for example were to take a line like this suppose I were and mind you this this green line and this violate line that I have drawn they are in fact supposed to lie on the x y plane mind you there are lines on the plane they raise on the plane and I am trying drawing to indicate the point is moving towards infinity okay.

So if you look at this this violate line that I have drawn okay that is going to the left of the diagram then what is its image under the stereographic projection that corresponds to again you know a circle like this a great circle like this is going like this okay and that is again another so you know if I if I take a point Q here under this stereographic projection it will go to phi of Q so it will go to phi of Q and you know as phi of Q move away to away and away from the origin which is supposed to be thought of as going to infinity then Q on the Riemann sphere is going to move closer and closer the North pole okay.

So you see that if you look at points on race starting from the origin and moving towards moving towards the point at infinity namely moving away from the origin they do correspond to different directions of approaching the North pole on the Riemann sphere okay under the stereographic projection therefore what I am trying to say is that I am trying to say that the point at infinity can be approached in all directions okay. This is the kind of justification you can give thanks to the stereographic projection okay, so if you if you let the variable Z to go to infinity along and arc or a path or ray or even a curve which is unbounded in going to infinity then you are going to infinity okay in a certain direction and you can do this in many ways and that amounts to really approaching infinity from all directions and the way you see it is that you see on the Riemann sphere okay using the stereographic projection.

So you must understand concretely that you know thinking of infinity as concrete point, thinking for neighbourhood of infinity, thinking of small neighbourhoods of infinity okay and then thinking of being thinking of being able to approach infinity from different directions all this is concretely possible and it is because of the stereographic projection that is the important thing okay, fine. So now with all of this let me again continue with this is the string of definition which I wanted to give, so 1 definition was to tell you that that I have already done, I have defined what limit as Z tends to infinity of f of Z means okay, so I have defined what it means to take a limit at infinity okay which is what I will read last time.

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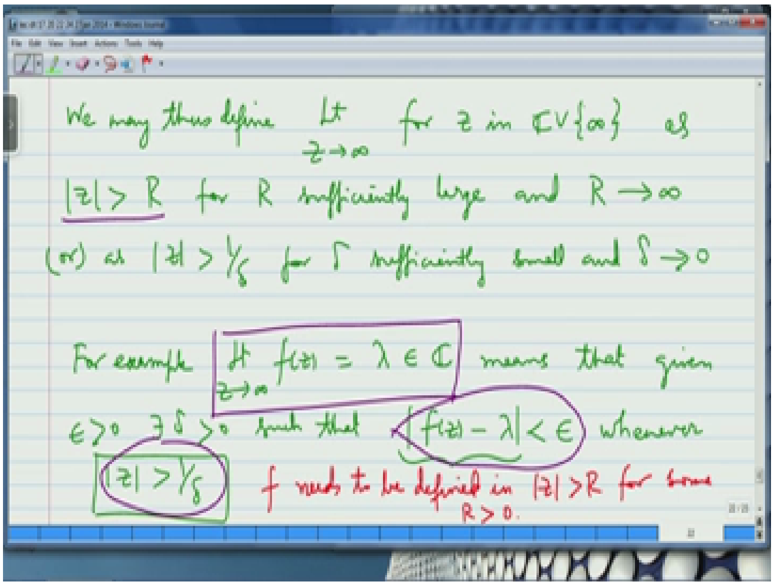

So if you want if we look at it so here is the definition we define limit Z tends to infinity for Z in the extended complex plane as going to Z as allowing the modulus of Z to become arbitrarily large okay which is written by this condition or Z greater than R and that is essentially you know it is the exterior of a circle and that you know as R becomes large and larger corresponds to a smaller and smaller neighbourhood of infinity if you look at it by this stereographic projection and therefore you know we define limit Z tends to infinity f of Z equal to lambda where lambda is the complex number to means the obvious thing that as Z approaches infinity f of Z should approach lambda so which means that distance between f Z and lambda namely this quantity can be made as small as you want provided you make Z close to infinity sufficiently close to infinity and that is reflected by choosing this delta okay and the hypothesis is that there is a delta even an Epsilon okay.

So now this is how you define limit Z tends to infinity, so this corresponds to getting a finite limit at infinity okay. You are getting a limit at infinity but the limit is finite okay, now I am going to tell you the other thing the other 2 possibilities. I am going to look at the situation when you can get an in finite limit at a finite point of the complex plane that is one more case and then the 3rd cases is when you get an infinite limit at the point at infinity so there are 3 definitions and all of them are important so let me make that definition.

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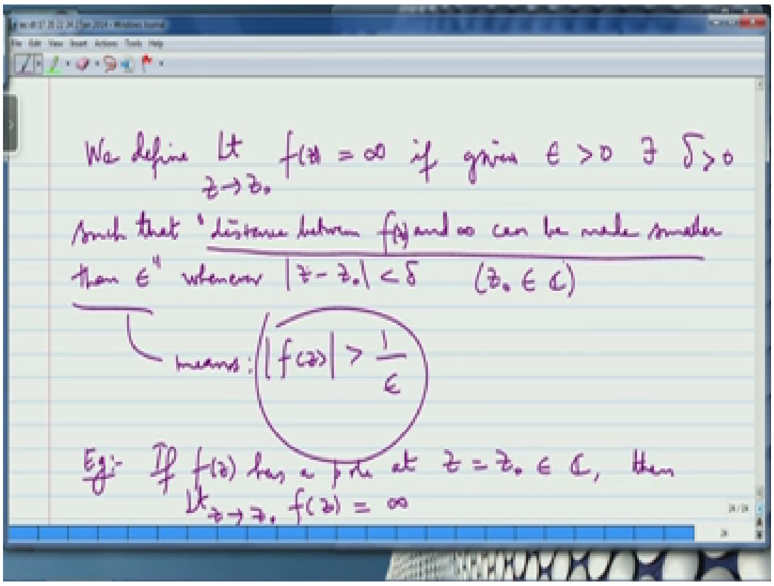

We define limit Z tends to Z naught f of Z equals to infinity okay. If given an Epsilon greater than 0 that exist at delta greater than 0 that exist at delta positive such that distance between f and infinity can be made less than Epsilon provided if I make the distance between Z and Z naught less than delta so let us translate that so let me write it in quotes distance between f and infinity f of Z and infinity can be made smaller than Epsilon whenever mod Z minus Z naught is less than delta okay and now you know it is a little bit I am just trying to say that f the values of f namely f Z they get closer to infinity and I want to say that they are closer to infinity by Epsilon but then you know if you want to really talk about distance you must have some notion of distance between the point at infinity and a point on the plane okay only then you can talk about distance between infinity and finite point okay.

Now it happens that it can be done okay what you can do is you can define a distance between a point at infinity and the point in the complex plane in the following very clever way and of course the key is the stereographic projection gives you everything, so what you do is you take the point at infinity and you take any other point, take the images on the stereographic projection you will get the North pole and some other point of the sphere then you take the distance and again for the distance you can you have 2 choices one is either you can take the distance in R 3 of those 2 points on the surface of the sphere or you can take the you can take the geodesic which is supposed to be the it is going to be the circular arc of shortest length that connects those 2 on the sphere okay.

So this respectively will give you the R 3 the usual metric and R 3 and the other one is called the usual matrix in R 3 is called the cordial metric because you are taking 2 points on the sphere then you join them by a straight line segment that is going to be a chord and you just measuring the length of the cord and the other thing is called the spherical metric. The spherical metric is you are you are moving on the surface of the sphere and you are taking the shortest distance and that shortest distance is going to be the length of the circular arc along the circle the great circle at passes through these 2 points okay.

So and then what you can do is you can take that as your definition of distance of 2 points in the extended complex plane and the beautiful thing that happens that with this the extended complex plane in fact even becomes a metric space okay and whether you put the cordial metric or whether you put the spherical metric you get isomorphic metric spaces okay and in fact the topology induced by these metrics is the topology on the extended complex plane given by the one-point compactification which we saw last time, so these are all facts that you can really write down and verify okay because you are in 3 dimensions you can even write down the stereographic projection in terms of x y and u coordinates okay you can write down the stereographic projection you can verify all these statements but you know here is where we not be so concerned about distance okay we will interpret this fact at the distance between f and infinity can be made sufficiently small by simply saying that f is a sufficient a small neighbourhood of infinity okay.

So you know this is where you try to escape from having to do more competent job of trying to give a distance between a point at infinity and a point on the complex plane. What you do is that you say that the you say that f of Z values of f are closer to infinity are as close as you want and now that means that you modified this statement to say that f of Z lies in a very small neighbourhood of the point at infinity and what is a very small neighbourhood of the point at infinity is given by of course you know the exterior of a circle of sufficiently large radius and how large well you can make it to be inversely proportional to Epsilon okay because Epsilon is supposed to be sufficiently small if you want you can make one by Epsilon to be the large radius, so you know so we will replace this you know to mean the following thing, so we will make it mean that mod f Z can be made greater than 1 by Epsilon, okay.

So you know this is the replacement I am trying to this the replacement for the definition of trying to actually have a distance between a point the value f Z and the point at infinity and having that made less than Epsilon okay so this is where you have to pay attention as to how the definition and this is intuitively correct because you see if you if the Epsilon is sufficiently small okay then you know 1 by Epsilon is pretty large okay and then mod f Z greater than 1 by Epsilon means that you are going to sufficiently small neighbourhood of the point at infinity this is what we have to understand okay.

So here that the fact is that you know Epsilon is actually not the distance okay we should not think of it you know verbatim like that you must think of it as going to going outside a circle of very large radius and you if you want you can make that radius to be inversely proportional to Epsilon okay so see this is what it means for us okay so with so you know with this definition you know what it means to say that so here of course I should say that Z naught is a complex number it is a finite complex number.

So this is how you define function taking the value infinity at a finite point Z naught okay and this is something that you already seen for example whenever a function has an has an isolated singularity which is a pole than the limit of the function as you approach the poll is infinity I mean this is what we always right and we interpret it to say to saying that the modulus of the function approaches to infinity and what is another way of saying is that the modulus of the function approaches infinity it is just saying this namely that the modulus of the function becomes arbitrarily large okay and that is what mod f Z greater than 1 by Epsilon says okay.

So here is an example if you know f of Z as a pole at Z equal to Z naught in the complex plane then limit as Z tends to Z naught of f of Z is actually infinity okay and now you see you can really think of infinity as a point and you are saying that the function takes the limit the limiting value is that point okay. So this is how you get an infinite limit at a finite point okay then there is 3rd case which is when you get an infinite limit at the point at infinity okay that is the 3rd case, so and why is that important that is important because I want to think of infinity as a pole for a function okay you know and if you want to think of a point as a pole then you know the function should tends to infinity as we approach that point, so if I want to think of infinity as a pole than the function should tends to infinity as I approach infinity, so I want an in finite limit at infinity okay, so that is the motivation for the and the necessity for the next definition.

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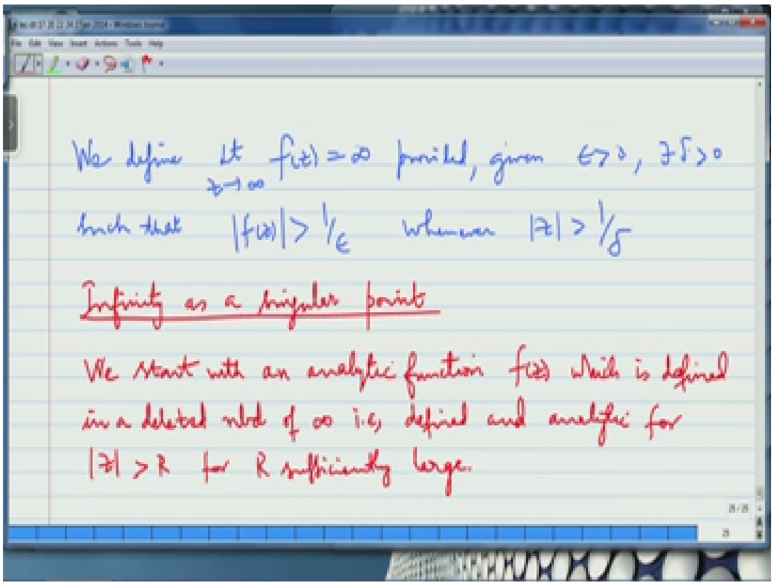

So let me use a different color okay so just a moment we define limit Z tends to infinity f of Z is equal to infinity, now how do you define this, so again the I mean value know if you really try to use Epsilon delta definition you will say that I want the values of f of Z to come to with an Epsilon distance of infinity given any Epsilon however small for Z sufficiently close to infinity okay, so here again you know you have to so the moment you say distance from infinity mind you the way out is that there is no distance from infinity as it is either you will have to do the hard work of define distance from infinity to a point by going to the stereographic projection and taking either the cordial metric or the spherical metric and instead of doing all that we can be very topological and we can say that whenever you want to say you are going close to infinity you are going to neighbourhood of infinity and that is very easy to define is the exterior of a circle of sufficiently large space.

So in that sense what does this mean? This means that you know given you can get the values of f of Z can get as close to infinity as you want provided Z gets as close to infinity, sufficiently close to infinity that is what it means, so you know so how do you write it down? By now you should have got it, so let me write this down provided given Epsilon greater than 0 there exist delta greater than 0 such that mod f Z is greater than 1 by Epsilon okay this is supposed to mean that f is going close to infinity because mind you must think of Epsilon becoming smaller.

So one by Epsilon is becoming larger and more f Z greater than 1 by Epsilon thing that you are in a smaller neighbourhood of the point at infinity okay whenever mod Z is greater than 1 by delta. So this is again you say mod Z greater than 1 by delta mind you that you are trying to find a sufficiently small delta if you find a sufficiently small delta 1 by delta becomes sufficiently large, if one by delta becomes sufficiently large mod Z greater than 1 by delta is a sufficiently small neighbourhood of the point at infinity. This is this is the point you have to be careful about okay, so this is how you define an infinite limit for a function at infinity okay and for example this will help you to define when infinity is a pole for a function okay for example if you take any polynomial right so now we are through with all these 3 definitions okay.

So what we have done us we are able to think of infinity as concrete point, you are able to have infinity in the domain of definition of the function namely you can allow you can write f of infinity okay you can write the value of f at infinity and if you want f to be continuous at infinity it had better b limit Z tends to infinity f of Z okay and you also allow infinity to be a value of the function okay, so the function can take the value infinity okay and the way that is facilitated is because of these definitions okay. Now what we will see next is using this definition, how you can treat infinity as an isolated singularity and classify the kind of singularities namely look at what it means to say for a function to have an essential singularity or a pole or the removable singularity at infinity okay that is what we are going to do next, okay.

So what we will do now is look at infinity as a singularity okay, so let me write that down, infinity as a singular point so you want to look at infinity as a singular point for a function okay and of course mind you we are only worried about isolated singularities okay, so we are not going to the complicated case when infinity is an non-isolated singularity okay, so infinity is an isolated singularity for the function means that whenever you say something is an isolated singularity of function it means that there is a deleted neighbourhood about that point where the function is analytic.

So it means that you function should be analytic in a deleted neighbourhood of infinity and that means that you function should be analytic and you know by definition a deleted neighbourhood of infinity is just something that could should contain the exterior of a sufficiently large circle on the complex plane centred at the origin if you want okay and therefore you function should be 1st of all defined at infinity I mean it should be defined in the sense that it should be defined outside circle of sufficiently large radius okay, so that is prerequisite okay mind you because if you want to study function at a point that point even if it is not good point for the function it may be a singularity for the function. The function should be defined in a deleted neighbourhood of that point I mean the supplies even to very simple things like continuous functions.

See if you want to talk about the continuity of function at a point then you know you need to study the function close to that point and look at what happens to the limit of the function as you approach that point therefore in a deleted neighbourhood of that point the function should be defined in the same way okay the prerequisite for studying infinity as a singular point is that the function should be analytic in a deleted neighbourhood of infinity which means it should be defined on 4 mod Z greater than R for R sufficiently large.

So let me write that down we start with an analytic function, analytic or holomorphic an analytic function f of Z which is defined in a deleted neighbourhood so I am using nbd for neighbourhood of infinity that is defined and analytic for mod Z greater than R for R sufficiently large okay mind you this also includes a case of an entire function all I am saying is that the function should be defined outside circle of sufficiently large radius but I am not saying it need not be defined inside the circle okay, so what I am interested is only the behaviour of the function outside at all points outside a circle of a sufficiently large radius because that for me is what a neighbourhood of infinity is? Okay.

So by definition you know by the definition of singularity infinity becomes a singularity because what is a singularity? A singularity is a point singularity is defined only for an analytic function and our definition of singularity is that it is a point which can be approached by points where the function is analytic so there must be…and we are interested in isolated singularity so you see if you take a deleted neighbourhood of infinity mod Z greater than R is a deleted neighbourhood of infinity if you take only the finite complex plane I mean only the complex numbers and of course if you take mod Z greater than R in the extended complex plane your also including the point at infinity okay which by the way is also an open set with infinity as an interior point on the extended complex plane okay.

Now you see you have done this so I am trying to bring your attention back to something that you should have seen the 1st course in complex analysis, normally the philosophies that if you want to study f of Z at infinity you will study f of 1 by Z at 0 and for the obvious reason that as Z tends to 0 1 by Z goes to infinity which you can now make sense of because of our definitions okay. Well you know the question is why is this why is this the right thing to do okay, so you should ask yourself why certain things are defined in a certain way or what the what is the philosophy behind these things. So you can ask this question what is the justification for saying that studying f of Z at Z equal to infinity is the same studying f of 1 by Z at Z equal to 0 you can ask this question.

So this this comes to this this brings us to something interesting it is got to do with this idea that you know when you say to objects are isomorphic okay then properties of the objects should also correspond okay, so this is a very general philosophy in mathematics if you have 2 isomorphic objects okay both of them should have the same type of property okay because an isomorphism supposed to preserve the property preserve all properties okay so for example if you have 2 groups and they are isomorphic you cannot expect one of them to be abelian the other is the other non-abelian okay, you cannot expect such things because an isomorphism carries an abelian group only to an abelian group okay so in the same way this also applies to spaces.

So if you have 2 spaces let us say topological spaces, if 2 topological spaces are isomorphic which means that they are of homeomorphic and all the topological properties of one space should agree with all the corresponding topological properties of the other space or example if 2 topological spaces are homeomorphic if one is connected then the other is also connected, if one is not connected the other cannot be connected, if one is compact the other is compact okay and so on and so forth, so the fact is that there are properties which are supposed to be intrinsic properties, these are called intrinsic properties for an object and they are called intrinsic because they will not change if you change the object up to isomorphism okay.

For example we say that the nature, the abelian nature of a group okay is an intrinsic property because if you replace the group by an isomorphic group then it will happen that a replace group will also have to be abelian because an isomorphism carries an abelian group to an abelian group okay, so you say the property of being abelian is an intrinsic property, so in the same way this also extends not only to properties of object it also extends to properties of functions defined an objects this is the very important thing okay.

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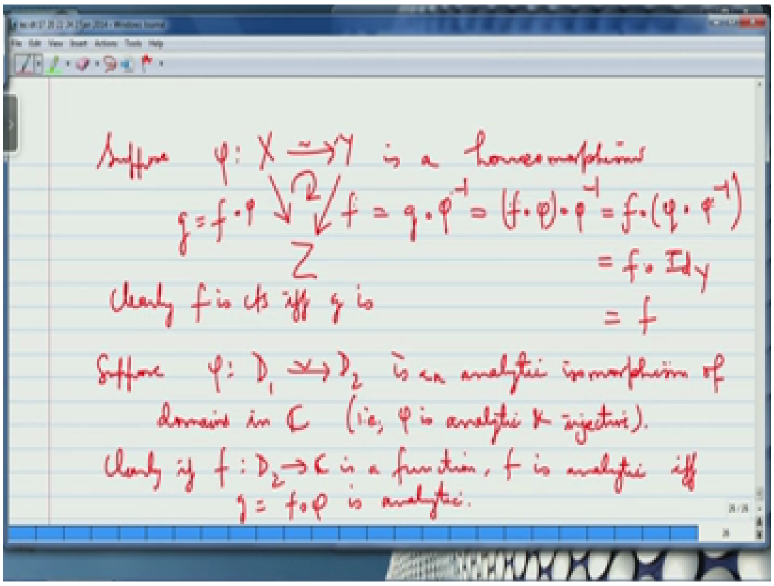

So I will give you an example see so suppose phi from x to y is homeomorphism which means that you know this is a so I am putting tilde above the arrow to signify that. This arrow is actually an isomorphism, a topological isomorphism this otherwise called a homeomorphism okay and of course you know as I just told you all properties of x should correspond to same properties intrinsic properties of y but what I want to say is that this also carries over to functions, so you know see suppose Z is another topological space okay and suppose f from y to z is a function is a set theoretic map okay.

I can ask this question as to when f is continuous okay after all y and Z are topological spaces and f is set theoretic map and I can always ask when a set theoretic map between 2 topological spaces continues okay now the fact is that you know I can complete this diagram by into a triangle by drawing this arrow which is the composition of phi followed by f so it 1st apply phi then apply f and then I put a circular arrow and this will tell you that the circular arrow that I have drawn inside the triangle is supposed to be it is supposed to call be called as commutativity of the diagram okay which is often use in algebra .

It just tells you that you know if you go from x to z either via going 1st through phi and then through f or from x to Z by the other map that have it and which is actually f circle phi they are one and the same, so the advantage of this circular arrow is that sometimes I can write g here instead of writing f circle by phi I can simply write g and then if I put the circular arrow means that this g is supposed to mean f circle phi okay so that is what it means, okay. Now the question is that I am trying to say the obvious thing as you can expect you see that f is continuous if and only if g continuous okay the reason is because phi is the isomorphism it is a homeomorphism okay.

So let me write that down clearly f is continuous if and only if g is and the reason is because you can use you can use the fact that a complex (())(41:34) of continuous function is continuous and you can use the fact that phi being a homeomorphism has an inverse its inverses rather actually continuous okay, so they can get from f to g can get from g back to f okay so the way of course the way of going from f to g it is just g is just f circle phi and how do you go from g to f you apply you apply 1st phi inverse and then you apply g you will get f okay, so f is 1st applying phi inverse and then applying g okay.

Well if you write if you write it down remembering that g is equal to f circle phi what you will get is that f it will just simplify to f circle phi circle phi inverse which is since composition of map is associative I can change the bracket I will get f circle, f circle phi circle phi inverse and that is f circle…well phi followed by phi inverse is supposed to give me the identity on x, so this will be f circle identity on x and 1st if I apply identity on x no it is first applying phi inverse and then phi it is not identity on x it should be identity on y and f circle identity y is just f okay so that is I am just trying to write down algebra (())(43:03) that f is just a g circle phi inverse, so f is continuous if and only if g is that is obvious okay.

Now you know now this you can take this over and do it to not only to continuous function you can do it to differentiable function, you can do it to analytic functions and so on and so forth. So for example you look at a different situation suppose phi is an analytic isomorphism okay it is a holomorphic isomorphism of one domain in the complex plane to another domain in the complex plane okay, so then a function on the target domain of complex valued function on the target domain is holomorphic or analytic if and only if the composition with phi is holomorphic on the or analytic on the source domain okay. See if you look at this diagram what it says is that f is a function on the target which is why and f is continuous if and only if its composition with the isomorphism phi is going to give you continuous function on the source okay.

So we often in algebra (())(44:21) we always say that f is actually pulled back to g okay we say f is pull back of f of g and by pull back what do you mean is that you compose f with the isomorphism, so that you get a function on the source and that is why it is called pull by because you are taking a function from the target, a function defined at the target and here from that you are cooking up a function on the source and all you have to do is compose with the isomorphism from the source of the target and for that matter you do not even need an isomorphism even if you have a morphism this will work okay, so let me write that down, suppose phi from D 1 to D 2 is an analytic isomorphism of domains in the complex plane.

So mind you phi it means that D of course you know I am assuming D 1 and D 2 are nonempty they are open connected sets their domains and they are isomorphic by a map phi which is an analytic or holomorphic isomorphism, what it means is that it is basically it is an analytic map which is 1 to 1 okay we have this deep theorem which says that open mapping theorem says that the image of the non-constant analytic map is always open and you also have this theorem there that the an injective analytic map is a holomorphic isomorphism namely its inverse is also going to be the inverses also going to be holomorphic okay so saying that phi is an analytic isomorphism it is the same as saying that phi is analytic and phi is injective okay so let me write that down that is phi is analytic and injective.

Now you can now I can say the same thing if f is a complex value holomorphic function complex value function on D 2 then f is holomorphic or analytic if and only if f circle phi is holomorphic or analytic okay the same argument applies okay so clearly if f from D 2 to C is a function f is analytic if and only if g is equal to f circle phi is analytic okay, so the philosophy is the same okay and now and now I want you to now I want to give you the justification as to why the studying f at infinity, the same as studying f of 1 by Z at 0 that is because the map Z going to 1 by Z that is that is what phi stands for in our particular case that is a isomorphism it is homeomorphism and from the punctured plane to the punctured plane it is holomorphic isomorphism okay and that gives you the justification that studying f at infinity is the same as… f of Z at infinity is the same as studying f of 1 by Z at 0 and the Z going to 1 by Z is the takes the role of phi okay, so I will stop here.