Advanced Complex Analysis - Part 2: Compactness of Meromorphic Functions in the Spherical Metric, Spherical Derivative, Normality, Theorems of Marty -Zalcman-Montel-Picard-Royden-Schottky

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Lecture No 5

Neighborhood of Infinity, Limit at Infinity and Infinity as an Isolated Singularity

Okay so let us again recall what we were doing, we were looking at the Casorati Weierstrass theorem as 1st approximation to the great Picard theorem and we also wanted to get a good idea about dealing with the point at Infinity okay, so the idea is you want to talk about when an analytic function is you know for example analytic at Infinity okay or more generally Infinity when can you talk about an analytic function having a singularity at Infinity okay and then classify that kind of singularity and what does it mean to say that the singularities of a certain type let us say that it is the function is having an essential singularity or a removable singularity or a pole at Infinity okay we need to understand this.

So in order to study the behaviour at Infinity you 1st must be able to think of Infinity as a point okay because you always your use to looking at things sufficiently close a point so you basically when you want to study the behaviour of a function at a point you need an open neighbourhood of that point where the function is defined and for that matter even if the point is an isolated singularity you need a deleted neighbourhood of the point where the function is defined okay and then you want to study how the function behaves as you approach that point in various ways okay and you know that is how the classification of singularity goes, so for example if you take a point in the finite complex plane okay by the time in the usual complex plane then if a function an analytic function is defined in a deleted neighbourhood of that point then you know that the way the function behaves as you approach that point and tell you what kind of singularity that this.

So as you approach that point if the function goes to infinity is the modulus of the function goes to infinity then that point has to be a pole as you approach that point if the function goes to a limit, a finite limit namely you get a complex number in the limit then the function has a removable singularity at that point and on the other hand if neither of these happens namely that you do not get a limit then it is an essential singularity, so in the same way I want to be able to say I want to be able to treat the point at infinity okay, so the question is how do you think of the point at Infinity? And to key to that is Riemann's stereographic projection okay which tells you that you can think of the point that Infinity as the North pole on the Riemann sphere.

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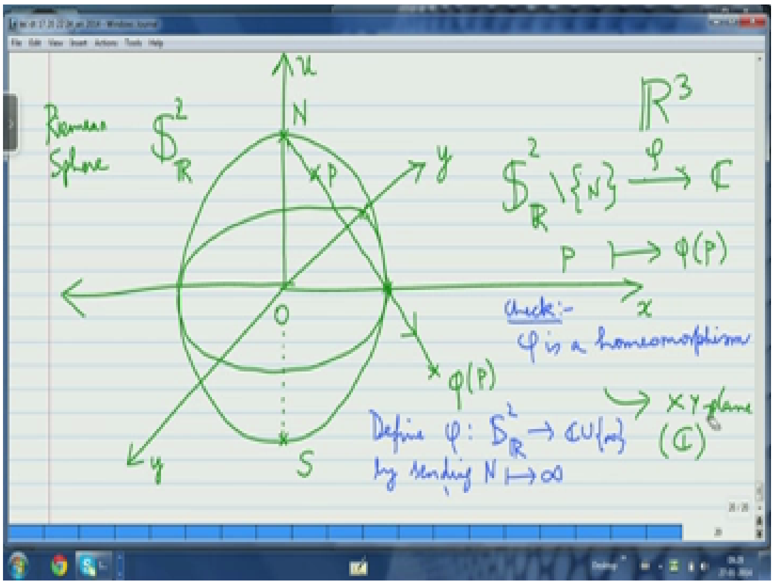

So let me draw attention to this diagram that I was drawing towards the end of the last class last lecture, so here is the so you can see here is the three-dimensional real space R 3 and you have the x y plane which is being thought of as the complex plane okay and then instead of the instead of the usual Z axis I am calling it the u axis which is the axis perpendicular to the x y plane and the reason is obvious because I would not use reserve the symbol Z for x plus I y okay and then what I do is that I take this sphere centred at the origin radius one unit okay and that sphere minus the North pole okay which is the point with coordinates 0, 0, 1.

If you throw that point out the rest of the sphere is mapped homeomorphically on the complex plane okay by this map phi which is called the stereographic projection and then what you do is that I asked you to check that this map phi is a homeomorphism okay. Now what you want to do is that what is missing in the complex plane is a point that corresponds to infinity, so basically you want to look at C union Infinity, you want to take this set which is C union Infinity and C union Infinity is called the extended complex plane okay and your idea is to send this Infinity to the North pole because that is the C union Infinity contains C as a subset okay and this C the complex plane is by the stereographic projection homeomorphic to the Riemann sphere minus the North pole okay, so what is missing on the Riemann's sphere side is the North pole, what is missing on the extended complex plane side is the point at Infinity.

So what you do is that you send the North pole to Infinity under the stereographic projection okay this is the definition you make and once you do that you get Bijective map from the Riemann sphere to the extended complex plane okay and now the point is that this Bijective map if you take this map outside the North pole it is a homeomorphism okay. Now what you do is you do the following thing, you put a topology on the extended plane C union infinity in such a way that this extended map from the full Riemann sphere to the extent that plane which is C union infinity that becomes homeomorphism okay, so you see what you must understand here is a technical point okay, see I have a topological space and I am seeking to add a point at infinity to it okay this is done in topology for what are called as locally compact hausdorff space okay.

So if we have studied this in the 1st course in topology which you should have and if you have not it is not very difficult to read it up from a standard textbook, so the idea is that you want to compactify a space, you want to make a space compact and what you should do is that you should start with locally compact hausdorff space okay and then you add a point at infinity okay and just adding a point at Infinity will only give you a set okay but you have to topologies it and the method and the topology that you give is the following, what you do is that for this for this set along with the point at Infinity which will be the 1 point compactification okay the open sets are going to be not only the usual open sets of your topological space but to that you also add compliments of compact subsets of your topological space along with that point at infinity okay and these are supposed to give you neighbourhood of the point at infinity okay.

So this is how you do the 1 point compactification and that is exactly what is happening here okay so, so let me Wright here so topologies topologies the extended plane C union infinity, so that phi becomes homeomorphism this is what you do okay, so again let me tell you the idea, the idea is you want to think of the point at infinity okay and of course so it has to be a new point, no point on the plane can be thought of as a point at infinity okay. You have to add a new point and we give it the symbol infinity okay and then now you have a set you have the complex plane which is a nice topological space (())(8:53) it is R 2 basically okay but then you added one extra point and once you add this extra point you also have to tell me what is going to be the topology and the idea is that you make this topology in such a way so that this complex plane along with this extended point at infinity that is the same by same I mean homeomorphic to the Riemann sphere okay.

So topologically it looks like a sphere okay and at 1st sight this looks a little complicated because you know the complex plane is unbounded okay, the complex plane is unbounded and what you have done this by adding this point at infinity you have made it compact because you know the extended complex plane is now…you are trying to put a topology on the extended plane which is a plane plus the point at infinity in such a way that it is homeomorphic to the Riemann sphere but you know the Riemann sphere is compact because it is a closed and bounded subset of Euclidean space it is compact okay and therefore and you know if your topological space is isomorphic, topologically isomorphic that is homeomorphic to another topological space which is compact then the original space is also compact any space homeomorphic to a compact space is a compact because the image of compact set is compact under continuous map therefore what you have done is you have added this point at infinity and you have put a topology in such a way that the whole thing becomes compact just adding one point you are making it compact, mind you if you remove that point then the space is even unbounded okay so the plane as this is unbounded.

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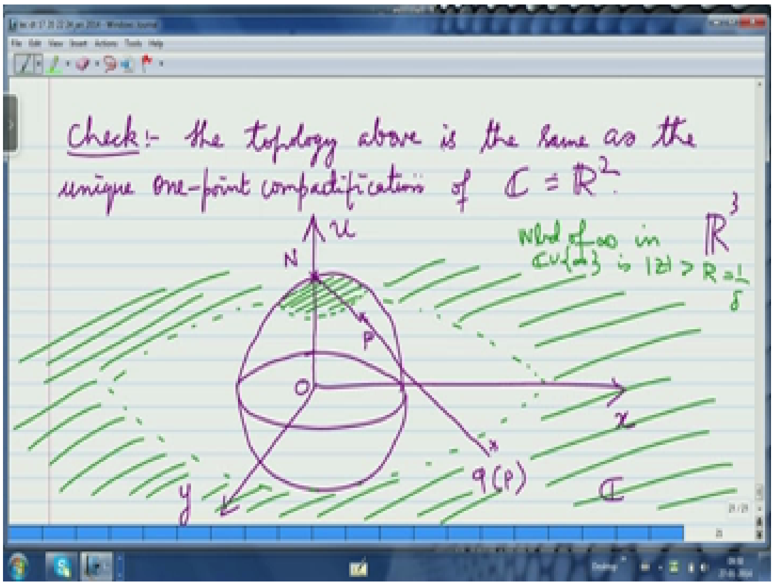

So now what I want you to…so let me write this down so I want you to check the topology above is the same as the unique one point compactification of C is equal to R 2 okay. So this is something I want you to check alright and well in fact let me even illustrate it for a moment because I want you to understand what is that is going on, so you have this so let me again redraw this diagram so I have this this three-dimensional space R 3 and I have this x y plane which is for me the complex plane okay and then I have this this third axis which I do not call Z, I call it as u okay and this is the origin, so I draw this I draw this sphere of radius one unit and look at the surface of the sphere, so here is the circle there is unit circle in the complex plane and that is the equator for this for the further sphere that came out to be little…okay.

So this is Riemann sphere okay and what this stereographic projection does is that there is this North pole and you give me any point on the sphere on the surface of the sphere, it is mapped onto the point in the complex plane that has gotten by joining the North pole to that point and by a straight line and looking at the point where the line intersects the plane okay so this is the stereographic projection, so it is p going to phi or p, so this is the stereographic projection. Now what I want you to understand is that what is and this you see the point at infinity on the complex plane is not visible in the picture, the point at infinity is the point is really at infinity you cannot really see it but it is inverse image under the stereographic projection is visible.

It is the North pole on the Riemann sphere okay and now you see since the topology on the extended complex plane is a homeomorphism with the Riemann sphere and neighbourhood of the point at infinity in the extended complex plane must correspond under this homeomorphism to a neighbourhood of the North pole on Riemann sphere and what is the neighbourhood the North pole on the Riemann sphere after all the topology on the Riemann sphere is just the induced topology, the Riemann sphere is just the sphere in surface of a sphere in three-dimensional space R 3 and R 3 has a standard topology and this inherits the subspaces topology okay.

So what is the neighbourhood of at the North Pole going to look like, it is going to be a set at contains disk the intersection of three-dimensional open disk centred at the centred at the North pole which is what an open disk in 3 spaces with the surface of the sphere, so what you are going to get is that you are going to get something like this, so let me let me again change color to something else, so you see basically a neighbourhood of the North pole is going to look like this. This is how a neighbourhood of the North pole is going to look like on the Riemann sphere okay and this by our definition is the its image should give you a neighbourhood of the point at infinity on the extended complex plane, so that helped us to think of a neighbourhood of infinity on the complex plane okay, on the extended complex plane. So what is the image of this?

So you know if you draw the image of this see this bounding well of course I am I should not include this bounding circle okay because then it will become a compact neighbourhood okay but I but do not worry about the boundary okay so if you look at the boundary circle okay so if you want we can I do not know if the whole of it will go away, yes it does so let me put dotted lines like this okay let me tell you that it is actually an open set okay, now this boundary dotted lines which is a circle, the image of that under stereographic projection is going to give me a huge circle in the complex plane okay, so if I draw this the image of this that circle I am going to get this, I am going to get something like this, this is what the image of that dotted circle on the Riemann sphere this after you do the stereographic projection on the plane okay and what about smaller and as a circle if you make this daughter and circle on the Riemann sphere smaller this circle on the plane which is the image of that circle under the stereographic projection becomes larger okay.

So the image of this shaded region is going to be the exterior of this this dotted circle on the complex plane, so you know what I am to going to get is I am going to get you know so I am going to get something like this and I am going to get all this everything outside, I am going to get…this is what I am going to get on the complex plane. This is the image of that disk that open neighbourhood of the North pole and the Riemann sphere, its image in the complex plane is the exterior of a circle okay, so what this tells you is it tells you the following thing, it tells you that the neighbourhood of the point at infinity on the extended plane should be thought of as the exterior of a circle in the complex plane okay.

So that is the reason that when you try to write a neighbourhood of infinity in the complex plane you write it in the form mod Z is greater than 1 by delta okay you have to write it the exterior of the circle centred at the origin okay equation for a circle centred at the origin as mod Z equal to R where R is the radius and you want R to be sufficiently large you want to look at points outside the circle so that means mod Z should be greater than R okay and instead of taking R large you can replace R by 1 by delta where delta is small okay therefore the… What this tells you, the Riemann's stereographic projection as you that in the extended complex plane a neighbourhood of infinity is just the exterior of a circle okay and what is it that corresponds to smaller and smaller neighbourhoods of infinity, it corresponds to the exterior of larger and larger circles, the larger your circle is and you take the exterior of that, that means that you are getting closer to infinity okay.

A larger circle means if you take the exterior of a larger circle means you are going to a smaller neighbourhood of infinity and how do you actually which visualise this? You visualise this by looking at the body translate to in terms of this stereographic projection on the Riemann sphere. On the Riemann sphere you get smaller and smaller and smaller disks close to the North pole, so it does make sense okay. So the moral of the story is that so let me write it here is very important, neighbourhood of infinity in the extended complex plane C union infinity as mod Z greater than R. If you want you can write it as 1 by delta, R sufficiently large delta sufficiently small okay and the larger you take R you are going to a smaller neighbourhood of infinity okay of the point at infinity okay and because of this stereographic projection this is very easy to now visualise okay. Okay now given this we can now we are in a good position to define the limit of the limit as Z tends to point at infinity okay, so I think let me see whether I have done this. No I have not done it before so let me do this.

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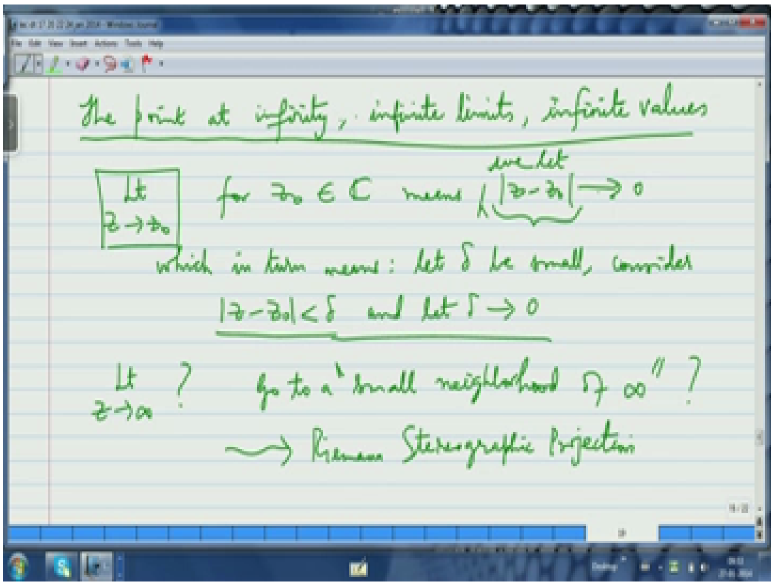

So let me anyway recall you see so let us go back to this definition what is the meaning of limit as Z tends to Z naught of Z naught an ordinary complex number, it means that your… mind you Z naught is fixed it is a fixed complex number Z is a variable and you are letting the variable Z tends to Z naught and this is happening on a plane okay and what does that mean, it means at your making the distance between Z and Z naught which is mod Z minus Z naught you are letting that to go to 0 okay and which in terms of delta notation is that you are looking at this mod Z minus Z naught less then delta which is the sufficiently small open disk centred at Z naught it is delta interior of that this okay and you are making delta is small enough okay that is what it means.

Now what I want to do is that, I want to translate this definition and or rather give a version of this definition for which Z naught is actually the point at infinity okay because I want to make sense of limit as Z goes to infinity okay and how does one do that? So one uses this one uses this… the intuition that comes in the last line of the definition namely you letting Z to tends to Z naught supposed to mean that you are looking at you are allowing Z to vary in a small neighbourhood of Z naught okay that is what limit Z tends to Z naught means alright. In the same way if you now think letting Z tends to infinity should means that you should allow Z to be in a small neighbourhood of infinity okay so that is what it should mean so but now you know how to think of a small neighbourhood of infinity. A small neighbourhood of infinity is the exterior of a large circle on the complex plane centred at the origin, so now you have a very easy way to give a definition as to what it means when Z tends to infinity so let me write that down so let me write it here.

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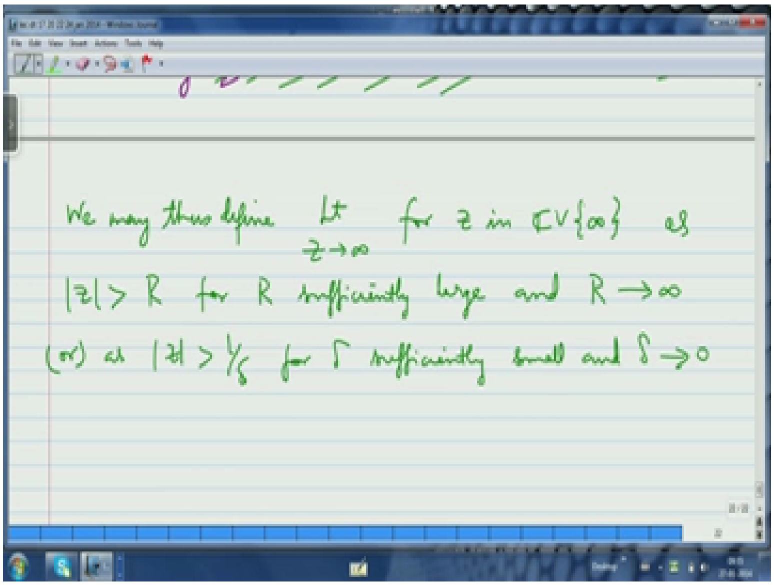

We may define we may thus define limit Z tends to infinity or Z in mind you now Z should be now Z is in the extended complex plane which is… or let me just keep for Z in C union infinity as mod Z greater than R for R sufficiently large and R tending to infinity okay or as mod Z greater than 1 by delta for delta sufficiently small and delta tending to 0 okay. So this is how you can define what limit Z tends to infinity means okay you are allowing Z to come closer and closer to infinity that means you are restricting Z be in a neighbourhood of infinity and you know what an neighbourhood of infinity looks like it is the exterior mod Z greater than R of a circle of a sufficiently large radius and if you want to get more closer to infinity and you must increase R you must take the exterior of larger and larger circles okay. Now well let me go back to something which I completely forgot which I said I would do and I did not.

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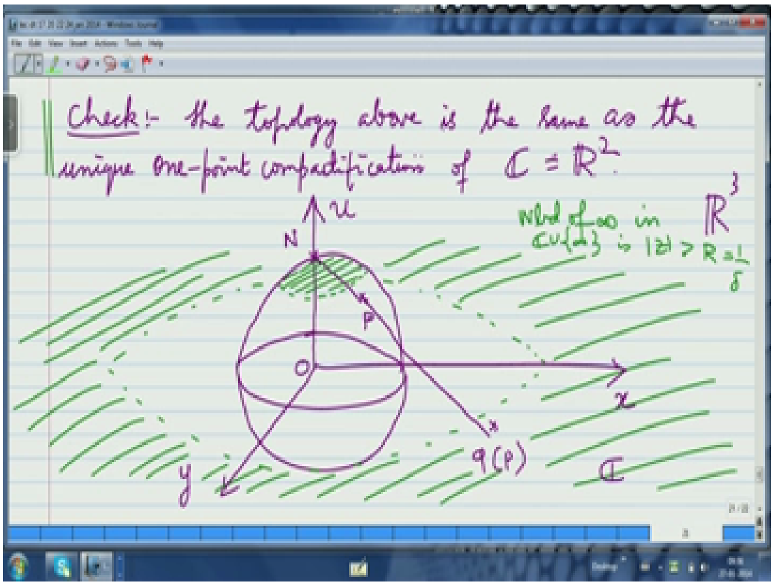

So here in this about this check here, see this check was that the topology that you put on the extended complex plane so that the stereographic projection is a homeomorphism is the same topology that you would give to the 1 point compactification and that is correct because in 1 point compactification the topology is such that the open sets are the usually open sets of the space plus you also a complement of compact subsets of the space and add the point at infinity and if you look at these geographic projection okay, what is an open neighbourhood of infinity?

An open neighbourhood of infinity in the extended complex plane corresponds to an open neighbourhood of the North pole on the Riemann sphere okay and if you take an open neighbourhood of the North pole on the Riemann sphere mind you it is a complement of a compact set because it is complement on the Riemann sphere is a close set and it is a close subset to the Riemann sphere is already compact, so it is already compact so it is a complement of a compact set okay so this argument should tell you that this check is correct this is how you check it okay. So that is what I forgot so I just wanted to go back to an now let us come back to it.

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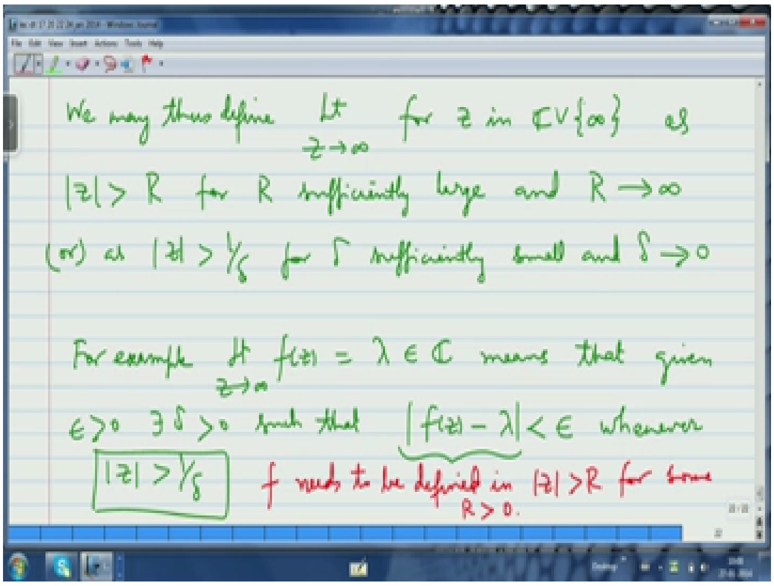

Now let us come back to this business of defining limit as Z tends to infinity, now that you have this you can now for example define the limit of for example say you can take the limit of the function as Z tends to infinity okay. So for example a limit Z tends to infinity f of Z is equal to lambda it is a complex number means that even Epsilon greater than 0 there exists a delta greater than 0 such that the distance between f of Z and lambda can be make smaller than Epsilon whenever the distance between Z and infinity can be made small enough which is the same as saying that mod Z can be made larger than 1 by delta okay.

So this is how you define the limit as Z tends to infinity of a function being a finite value, finite complex number okay and so you must understand limit Z tends to infinity f of Z is lambda means that the function value is f Z their distance from lambda which is mod f Z minus lambda can be made as small as you want, how small? As small as Epsilon which is already given to you that Epsilon could have been any small okay. You can make it as small smaller than Epsilon provided you choose Z sufficiently close to infinity and what is choosing Z sufficiently close to infinity you have to choose Z outside a sufficiently large circle in the complex plane centred at the origin and that is exactly this condition okay that is this condition that mod Z minus mod Z is greater than 1 by delta for delta sufficiently small okay.

So mind you in order to say this it is implicit that the function should be defined in a neighbourhood of infinity that means the function should be defined outside a circle okay. The function should be defined in a neighbourhood of infinity of course the function is not defined in a neighbourhood of a point then you cannot talk about the limit of the function as you approach that point okay because to take a limit you must be able to approach that point and if this is to be a well-defined limit then you should be able to oppose that point in all directions basically you need to allow the way able to approach that point a neighbourhood of the limiting point.

In this case the limiting point is a point at infinity and neighbourhood of that is exterior of circle in the complex plane and so what you must understand as, it is very important, so let me write it in a different color. f needs to be defined in mod Z greater than R for some R greater than 0 otherwise this definition does not make sense okay. At least this is you need to require this of f if you want to study properties of f such as analyticity okay. So now what I want to say is that so suppose you are looking at a function f which is analytic and which is defined outside in a neighbourhood of infinity namely outside a circle in the complex plane okay.

Then you see that infinity becomes a single point it becomes an isolated singularity okay it is because the function has been assumed to be analytic in a neighbourhood of infinity okay therefore infinity if you take a deleted neighbourhood of infinity in the extended complex plane you will simply get the exterior of the exterior of the sufficiently large circle in the complex plane and then the function is analytic therefore infinity becomes a single point and then how you can talk about whether the single point is removable or essential or a pole and the standard way this is done in the 1^{st} course in complex analysis is that you study the behaviour of the function at infinity, you study the behaviour of f of Z at infinity by looking at the behaviour of f of 1 by Z at 0 this is what you do but what I want to tell you is that that is the same as what we are doing now okay. So there are 2 definitions and we need to reconcile them okay and we will try to do that next okay.