Advanced Complex Analysis-Part 2.
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Department of Mathematics.
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Lecture-43.
Schottky's Theorem - Uniform Boundedness from a Point to a neighbourhood.

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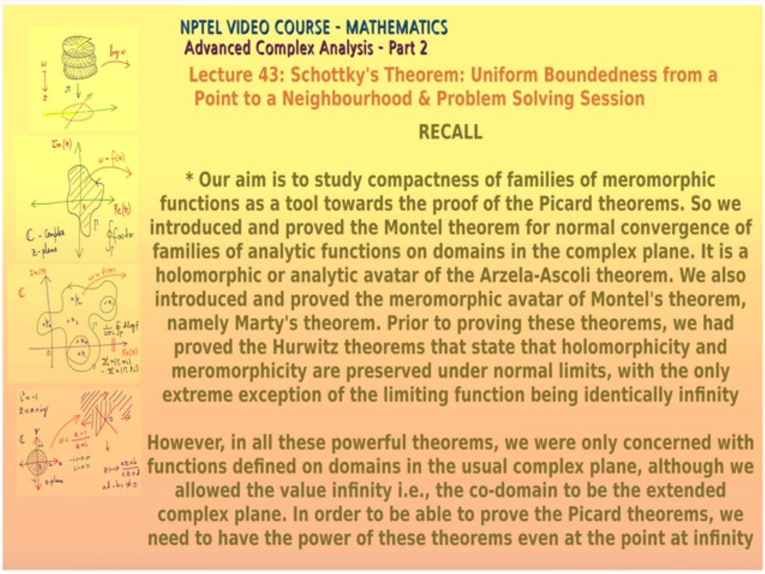
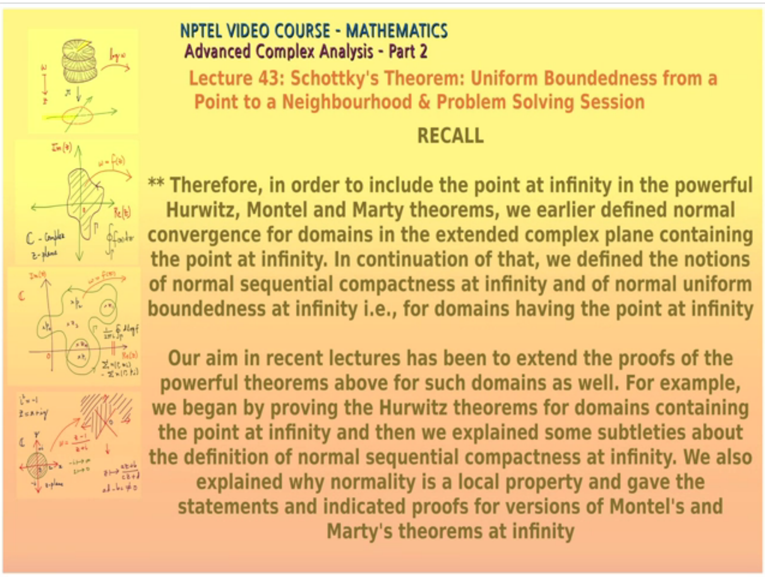
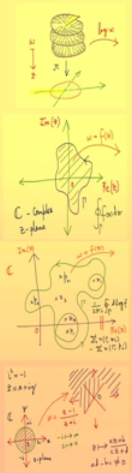






NPTEL VIDEO COURSE - MATHEMATICS

Advanced Complex Analysis - Part 2

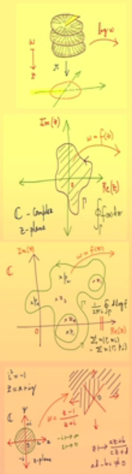
Lecture 43: Schottky's Theorem: Uniform Boundedness from a Point to a Neighbourhood & Problem Solving Session

RECALL

*** We defined the notion of normality at a point as a local property, and tried to analyze what it means for a family to be normal at a point. In order to understand the local behaviour of a normal family very close to a point, we explained a process called the zooming process which produces functions that are zoomed versions of the functions in the given family. The zooming can be made ultra-zoom i.e., we may increase the magnification of the zooming to infinity. This helps getting the behaviour very close to the point

By applying Marty's theorem to the zoomed family, we were able to arrive at a condition on which we expanded by introducing two further levels of intricacy, namely choosing ever-increasing zooming factors and choosing a convergent sequence of centers for the zooming process converging to the point being analyzed for normality. The condition we got is that we will always end up with a limit function that is constant

The whole point about this discussion is that this condition characterizes normality at a point and its negation is the essence of the important theorem called Zalcman's Lemma which is our key tool to understand non-normality at a point for a given family of meromorphic functions, leading eventually to a proof of the Picard theorems. With this motivation, we proved Zalcman's Lemma and its converse



NPTEL VIDEO COURSE - MATHEMATICS

Advanced Complex Analysis - Part 2

Lecture 43: Schottky's Theorem: Uniform Boundedness from a Point to a Neighbourhood & Problem Solving Session

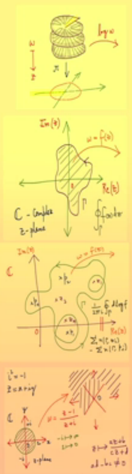
GOALS

*** * We then proved Montel's Fundamental Normality Criterion, which we may assert as the deepest theorem in the present course of lectures. The proof involves all the powerful theorems we have proved so far: e.g., those of Hurwitz, Montel, Marty, Zalcman and the Open Mapping Theorem. Montel's beautiful normality criterion is based on omission of values, and says that any family of meromorphic functions that omits three distinct values in the extended complex plane is normal, which implies that any family of analytic functions that omits two distinct complex values is normal

In the lecture before the previous, using Montel's criterion and the Zooming Process at an essential singular point we proved the Great Picard Theorem and deduced the Little Picard Theorem as a corollary

In the previous lecture, we proved Royden's theorem that asserts normality of a family based on growth conditions of its derivatives. The condition is that the derivatives should grow (in modulus) at most like an increasing function of the (modulus of the) original functions

In the present lecture, we give a simple proof of a classical theorem of Schottky to illustrate the powerful nature of the Montel theorems and end with a problem solving session



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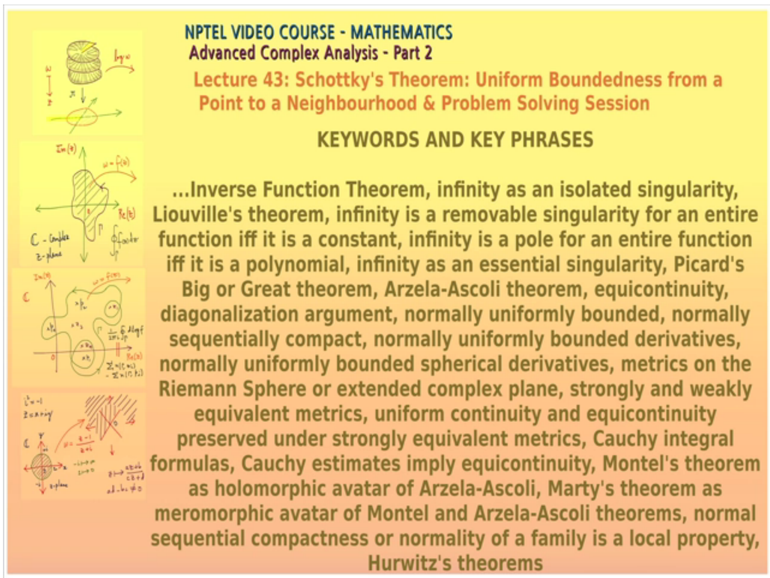
Advanced Complex Analysis - Part 2

Lecture 43: Schottky's Theorem: Uniform Boundedness from a Point to a Neighbourhood & Problem Solving Session

KEYWORDS AND KEY PHRASES

Schottky's theorem, family of analytic functions bounded uniformly at a point, family of analytic functions bounded uniformly on a neighborhood, Montel's Criterion for Normality, Fundamental Normality Criterion, Fundamental Normality Test, normality criterion based on omission of values, any family of meromorphic functions that omits three distinct values in the extended complex plane is normal, any family of analytic functions that omits two distinct complex values is normal, bilinear or Moebius or linear fractional transformation, deleted neighborhood of an isolated singularity, Riemann's Removable Singularities Theorem, analytic same as bounded same as continuous at a removable singularity, automorphism of the Riemann Sphere, automorphism of the extended complex plane, automorphisms or analytic isomorphisms or holomorphic isomorphisms or biholomorphic self maps of the complex plane...



So the next thing that i want to discuss is schottky's theorem, which is very easy to prove, okay.

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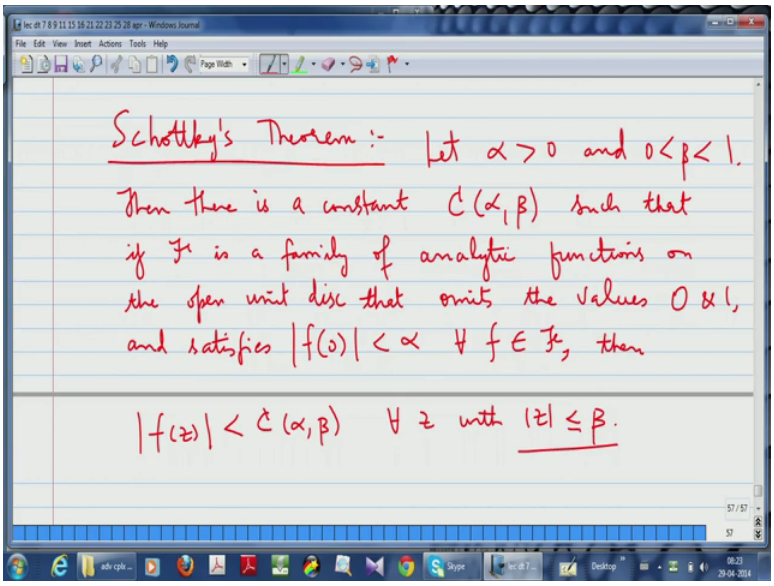

So let me write it down, schottky's theorem, so let alpha be a positive constant and beta be a fraction, positive real number between 0 and 1, okay, then there is a constant c, alpha, beta such that if script f is a family of analytic functions on the open unit disc that omits the values 0 and 1 and satisfies f of 0 is bounded above alpha for all functions small f in the family f, then, the mod fz is less than c alpha, beta for all z with mod z less than beta, okay. This is schottky's theorem and the point is that it is constant c alpha, beta, it depends only on alpha and beta and it does not have anything to do with what the family is, it works for any family.

And, such theorems when they were 1st proved, they were pretty, they were considered pretty difficult but then because of, because we have montel's, you know theorem on normality and montel's test on normality, it is easy to deduce this theorem. Okay, so let me tell you the proof, you can see immediately that the, you, you want the family to be arbitrary, therefore you consider the biggest possible family, namely you take all analytic functions on the unit disc, satisfying the condition that the, you know value at 0 is bounded by alpha okay.

So you apply it to the largest possible family that you can think of, okay. And, and you know, see the moment you given that the, these functions omit the values 0 and 1, it means that the family is normal. See in fact, you see what is, what is montel's theorem on normality, otherwise it is called the fundamental normality test. See if you want to decide a family of meromorphic functions on a domain is normal, then you need to know that it omits 3 values. But the values in the extended complex plane, so one of them could be infinity, right.

But then if you are working only with analytic functions, you already know infinity is not going to be taken, okay, so you have to only ensure that for a family to be normal, to be able to apply the normality test, you have to only ensure that the family does not, every function in the family does not take 2 values. So here it is given that the sum all the functions, they do not take the values 0 and 1. So you know if you apply the fundamental normality test, that is montel's theorem, it will follow that if you take the, if you take the family of all analytic functions on the unit disc, which omit the values 0 and 1, that will be normal, okay.

To this is montel's theorem on normality, right. But then we also saw another care of montel, okay, which was translation or improvement of the arzela ascoli theorem, which said that for a family of analytic functions to be normal on a domain, you need that the family is normal uniformly bounded, that is it is uniformly bounded on the compact subsets. So if you see mod z less then, so you know, so let me write, let me consider mod z less than or equal to beta, okay, if you look at mod z less than or equal to beta, so i will change this here to mod z less than or equal to beta which is what i meant to but i did not.

But if you take mod z less than or equal to beta, that is a compact subset of the unit disc because it is closed and bounded. And therefore by the other montel theorem, which is improved version of the arzela ascoli theorem, the normality of the family will tell you that the family is going to be, is going to be normal uniformly bounded, so it is uniformly bounded on any compact subset. So on this compact subset sum all the functions should have a bound and call that bound as c alpha, beta, it is as simple as that, okay.

So what you must remember is that we have applied 2 montel's theorems, one montel theorem which is, which equates the normality of a family, that is a normal sequential compactness of family with uniform boundedness of the original functions as a family on compact subsets, normal uniform boundedness. And that is mind you, that is an improved version of the arzela ascoli theorem and in fact it used arzela ascoli theorem plus the diagonalization argument, okay. And then you apply the more serious montel's theorem on normality, the fundamental normality test or fundamental criterion for normality which is a very deep theorem.

Mind you that was the key to proving picard's theorem, okay. That the moment of family of metamorphic functions omits 3 values which is normal, the moment a family of analytic functions omits 2 values, it is normal, okay. So you apply those 2 theorems, then schottky's theorem is simple corollary. So it happens that there is a paper of zaltzmann in the building of the american mathematical society where several, where he explains how several problems in functions very have easy solutions by use of the zaltzmann lemma.

So in fact there is what is called the , there is a very deep theorem called block's theorem and it involves, roughly it is trying to estimate the size of, the largest size of the disc under the image of univalent or one-to-one analytic function. You take a one-to-one analytic function, okay and then you know you try to estimate, you take the image and then you try to see what is the largest disc , radius of the largest disc that is contained in the image, okay. There are theorems of this type and there is a particular theorem called block's theorem which is very very deep, okay.

And this can be proved by using the so-called block zaltzmann principle, which is also called the block principle, okay. And the whole point is that zaltzmann lemma is very powerful, it gives you proofs of, easy proofs of very deep results, right. So it is not a surprise that you get schottky's theorem, okay, as a simple corollary.

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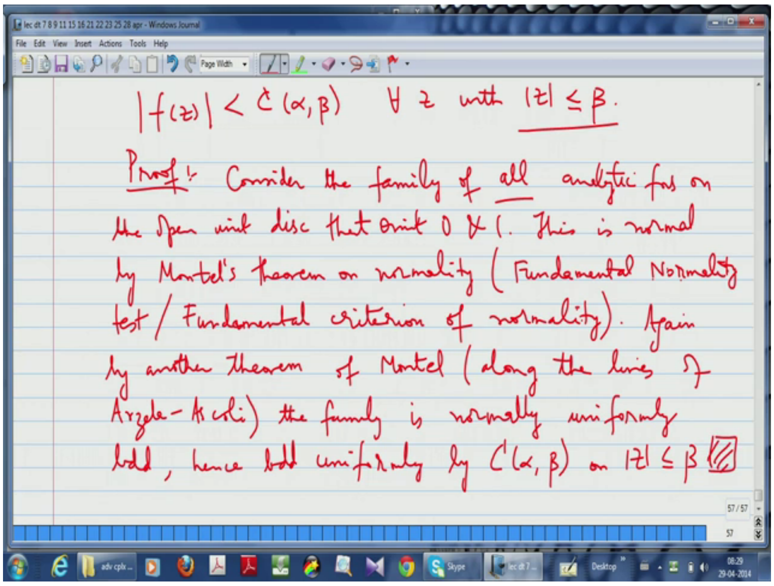

So let me write down, proof is, so let us, so consider the family of all analytic functions on the open unit disc that omit 0 and 1, this is normal , this is normal by montel's theorem on normality, otherwise it is called the fundamental normality test, sometimes it is also called fundamental normality criterion again by another theorem of montel along the lines of arzela ascoli the family is normally uniformly bounded, hence bounded, hence bounded uniformly by c alpha, beta on mod z less than equal to beta, okay.

See the point is i did not even use the fact that the functions at the origin are bounded by alpha, okay. I just i know that there is a, there is a uniform bound, all right and i simply call that uniform bound c alpha, beta. Actually i need not put that alpha there but i can call it c alpha, beta. The point is that i have to put in beta because i am looking at the bound on less than or equal to beta which is sub disc of the unit disc, close sub disc of the open unit disc, okay. Fine. So what you must understand is that this easy proof is because you have the strong montel's theorem on normality, which is a fundamental normality test, okay.

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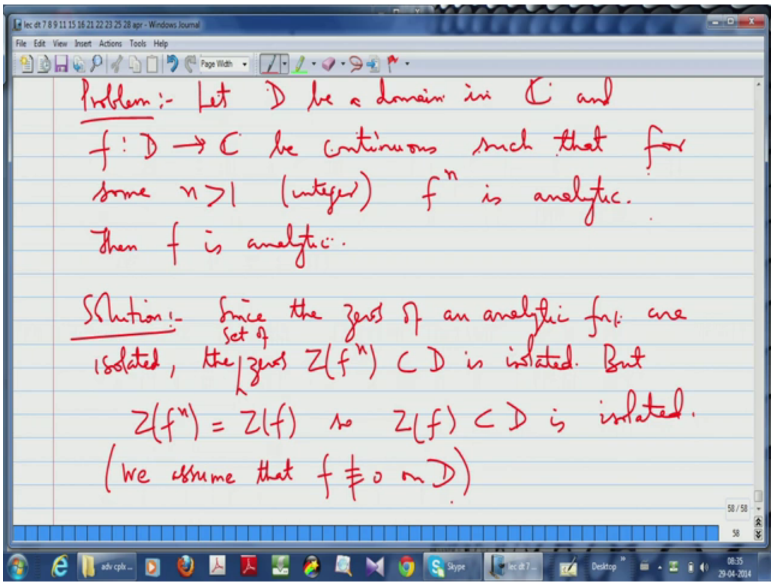

So, all right, so this is one thing, then i would like to, i would like to discuss the sum i would like to discuss this relation to 1st assignment that i gave, okay. So here is the problem that i gave earlier let d be a domain in the complex plane and f from d to c be continuous, such that for some positive integer n, f power n is analytic, okay, then f is analytic. Of course you know i need to take n greater than 1 because otherwise it is trivial, okay. Because if we put n equal to1, f power n is just, f power n is just f power 1 which is f, okay. And what is the, what is the solution to this?

Well, so the 1st thing is that , so f power n, 1st of all f power n means f of z whole power n, which means f of z into f of z multiplied n times, okay. The 1st thing i want to tell you is that we use the fact that the zeros of an analytic function are isolated. So since f power n is analytic, the zeros of f power n are isolated but then the zeros of f power n are the same as the zeros of f, therefore the zeros of f power n are isolated, okay. And therefore what we will do is we will 1st throw away the zeros and look at the complement of zeros in the domain, which is a sub domain, okay.

And what we will do is on that subdomain we will 1st prove that f is analytic, all right. And then we will have to worry only about these points where f becomes 0, all right. But then we can apply riemann's removable singularity theorem because each of these points will be isolated points in a neighbourhood of which f is continuous. Therefore they will be analytic even at those points and that is the proof, okay. So let me write this down, since the zeros of an analytic function are isolated, the zeros z of f power n, the set of zeros is isolated but the set of zeros of f power n is same as the set of zeros of f, so the set of zeros of f in d is isolated.

Of course you know, when you want to say the zeros of an analytic function are isolated, you must make sure that the analytic function is not identically you know constant. So the only case where this will fail even the analytic function is identically 0. If the analytic function is identically 0, then the 0 that is the whole domain, okay that is the only extreme case. But of course if f power n is identically, is f is 0, then f power n is identically 0 and if f power n is identically 0 , f is 0, so let us assume that f power n is not 0. I assume that f is not 0, okay, so there is no need to prove if f power n is identically 0, okay.

So there has to be, we assume that f is not radically 0 on d, right. So that is the only thing that we will have to worry about. When you, whenever, whenever you want to apply this result that zeros of an analytic function are isolated, you better make sure that the function is not identically 0, okay. And usually we are not interested in that function, right.

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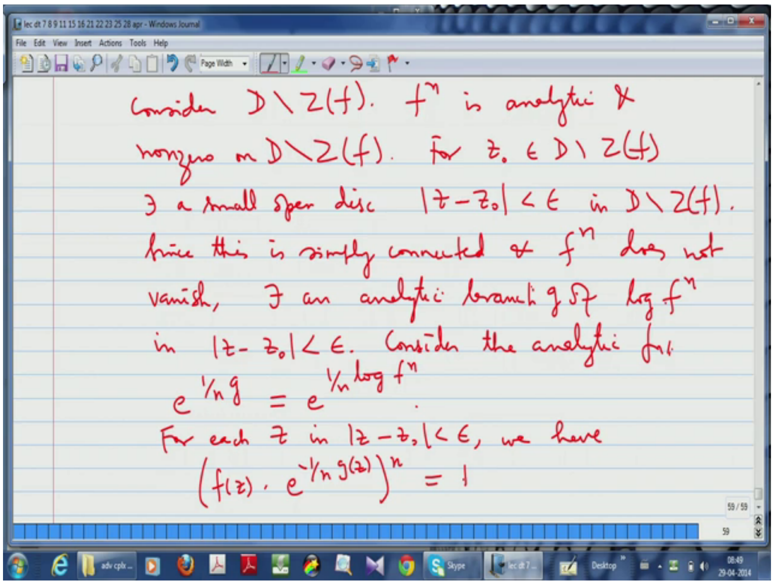

Fine, so now what you do is, in any case you look at consider d minus z f, okay, throwaway the zeros of f, it is an isolated set of points, so that is and again. So d minus zf is also a domain, right, and it will still be open, okay, it will still be an open set and because you are throwing away some isolated subset and it will also be, it cannot get disconnected, okay. So because you are just throwing isolated points away, it cannot get disconnected so d minus zf is also a domain, okay. And now we are going to look at the domain, the advantage with this domain is that f power n does not vanish, f does not vanish because the zeros have been thrown away.

So f power n does not vanish and f power n is an analytic function, okay. So you have a non-vanishing analytic function on a domain, now you know if you take any point in the domain, if you take a, there is a sufficiently small disc around that point which is inside domain, okay. And the point is that if you have a non-vanishing analytic function on a simply connected region, simply connected domain, then you can find an analytic branch of the logarithm of that function and in particular you can find nth roots of the function for any n.

The point is that you can find nth roots which are analytic, that is the whole point. So if you want to find an nth of the function which is analytic, okay, then the functions should not vanish and the region, the set on which you want to find it must be simply connected, all right. So the point is that if you take f power n which is analytic and f power n does not vanish on d minus zf, so if you take any point in d minus zf and you take a small disc surrounding that point, in that small disc i can find nth root of f power n. And what you expect it to be, it has to be f, okay.

But this nth root is supposed to be analytic, therefore it will prove that f is analytic, okay. But little bit of, little bit more has to be written down, so let us do that. F power n is analytic and non-zero on d minus zf for z0 in d minus z f, there exists a small disc, small open disc mod z minus z0 lesser than epsilon in d minus zf, since this is simply connected and f power n does not vanish, there exists an analytic branch of log f power n in mod z minus z0 less than epsilon, okay. Consider, consider the analytic function, so let me call this analytic branch as g, okay.

Consider the analytic function e power 1 by ng, okay, which is actually, see it is actually e power 1 by n, g is actually log of f power n, okay and you know this must be f, all right, you should expect this to be equal to f, right. Now you see, we will use the, we will use the following, we will show that this is the, the claim is that e power 1 by ng is actually equal to f, okay. The claim is e power 1 by ng is actually equal to f, once you, was that is true, it means f is analytic because e power 1 by ng is already analytic. And e power 1 by ng is analytic because g is analytic and why is g analytic because g's analytic branch of the logarithm, okay.

So i just have to, we just have to prove that e power 1 by ng is equal to f in this small disc, okay. And this will show f is analytic in a small disc but then the point z0 was arbitrary, so it will show that f is analytic on d minus zf, okay. And then to come at point of zf you can apply riemann's similarity and conclude that f is analytic on the whole of d, all right. So the only issue is that now i will have to show that e power 1 by ng is f, okay. Now what is common to e power 1 by ng and f, they are both nth roots of f power n.

F is nth root of f power n by definition, right and e power 1 by ng is also the nth root of f power n because if i take e power 1 by ng and raise it to the power n, i will get f power n. Okay. I will get e power g, if i take e power 1 by ng and ready to the power of n, i will get e power g but g is log f power n, so i will get e power log f power n which is this f power in, okay. So both e power 1 by ng and f are nth roots of f power n, all right and the point is, you see, you take , if you see, so now we have to use the following property.

If you take the, take any 2 logarithms of a complex number, okay, the table differ by constant, there will differ by constant multiple of 2, constant multiple of 2 pie i. See if you take, if you calculate the logarithm of the complex number, of course it is only defined for a complex number which is different from 0, there is no logarithm for 0, okay. So if you take a nonzero complex number and calculate its logarithm, then you know different logarithms, you know logarithm is a multivalued function, okay and the point is that, the real point is, the, the real logarithm of the modulus of the number which is nonzero, since the number is nonzero and imaginary part is the argument of that number, of the complex number.

And the argument can be, the argument is defined up to a multiple of 2n pie, to a multiple of 2n pie. Therefore the imaginary part of the logarithm can be changed by 2n pie i, i mean by 2n pie, okay. So any 2 logarithms of a number will differ by 2 n pie i, you have to use that, for each z in mod z minus z0 less than epsilon, we have 2 do a little bit of thinking, see you look at the function f of z times e power -1 by ng of z, look at this function, okay. See look at this function, the function, if i raise this the power of n, i will get 1 because you see if i raise this power n, f of z will give me f of z power n and e power minus1 gz, if i raise it to the power of n, i will get e power minus gz.

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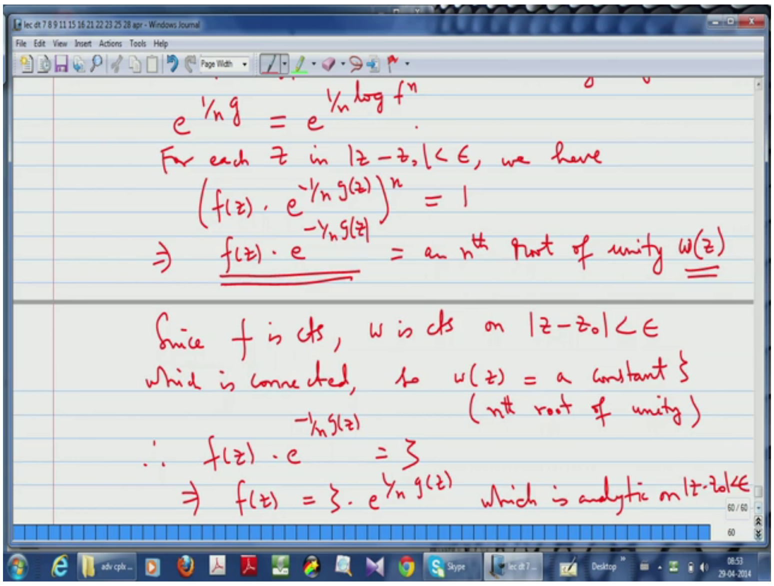

Okay, but e power minus gz is 1 by f power n because g is a log of f power n, right. So this is equal to 1, so this means that f of z into e power minus1 by ng of z is, it is an nth root of unity, okay and this nth roots of unity but the point is an nth root of unity and this nth root of unity in turn will change if you will change the z. I will call this omega z because it depends on z seemingly. For every z if you take f of z times e power 1 by ng of z, it is power n is equal to1, so it is an nth root of unity, so for every point z you are getting nth root of unity, for that function as w of z.

So w of z is that is that function, okay. But you see what is this, so i am just calling this function as w of z. So what is w of z, w of z is just f of z times e power -1 by n gz, okay. But notice, here is where i will use the fact that f is continuous. We have been given that f is continuous, i have to, i, so f is continuous and e power minus1 by ng z is also continuous. So the power is continuous, so w becomes a continuous function, w of omega of z is a continuous function. So it is a continuous function from a disc and what is the image z, it is the nth roots of unity, that is the discrete set, okay.

Therefore the image is to be constant, okay, the image of, the image of disc has to be connected under continuous function. We must get a connected subset of the set of nth roots of unity, it has to be a value, it can be only a constant, okay, it can be only a single term. So that means this omega of z is a constant, it is what, you get the same nth roots of unity for all z, okay, you get the same nth root of unity for z. So that is why you are using the continuity of f, okay. Since f is continuous, w is continuous on mod z minus z0 less than epsilon which is connected, so w of z is equal to a constant, nth root of unity, okay.

So what you get is you get f of z times e power minus1 by ng of z is equal to constant. So which means, which tells you that f of z is equal to the constant times e power 1 by ng of z, but of course right side is analytic, so f is analytic. So which is analytic on mod z minus z0 less than epsilon, okay.

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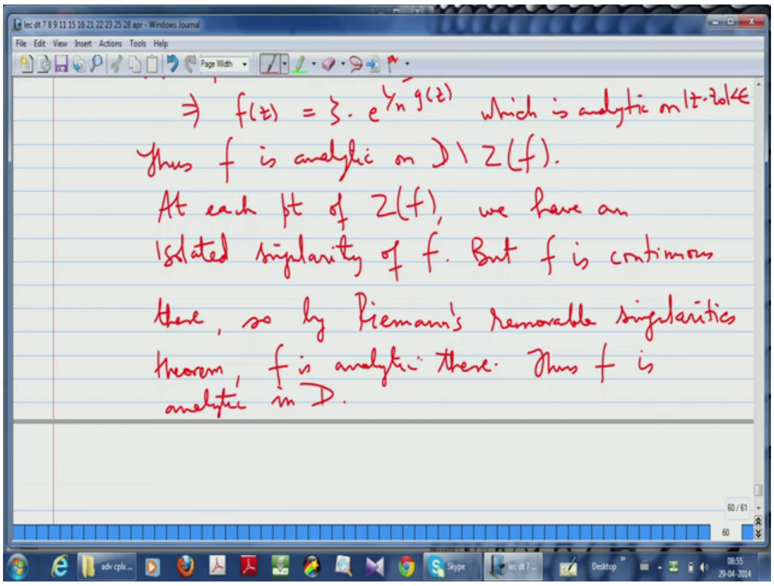

So the moral of the story is that since z0 was arbitrary, you get that f is analytic on d minus z f, okay, on d minus z of f. Now i have to only worry at points of z of f, points at which f becomes 0. You will take a point where f is 0, that is of course an isolated point, we have already seen that. So it is an isolated singularity for f, all right but f is continuous there, therefore by riemann's removable similarities f is analytic at those points as well, therefore f is analytic on all of d, okay. So it is an application riemann's removable singularity theorem.

At each point of zf we have an isolated singularity of f but f is continuous there, so by riemann's removable singularities theorem f is analytic, this f is analytic in d, okay. So for that is the proof that f is analytic, okay. So you should see that, the point is that you are bringing in, you are using isolatedness of zeros of an analytic function, okay, you are using the existence of an analytic branch logarithm, you are using riemann's removable singularity theorem. You have a question?

Look, what we proved this around that point f is analytic you have proved, so it becomes, if around the point the function is analytic, that point is automatically by definition it is a singularity, it is an isolated singularity. And riemann's removable similarities theorem applies, okay. What is a singular point of functions? It is a point which can be approached by the points of the, where the function is analytic. And what is an isolated singularity? It is a point where in a deleted neighbourhood the function is analytic.

So if you take any point of zf, it is an isolated point and you can find a deleted neighbourhood of that point where the function is analytic because i have already, we have already shown that the function is analytic outside the zeros of f. So that point becomes an isolated singularity and then the question is what kind of isolated singularity is it? And you know riemann's removable singularity theorem says that if the function is, as a limit at that point of discontinuous at that point, or bounded in the deleted neighbourhood of that point, all these things are equivalent to the function of being analytic at that point, you can extend the function to that point, you can define, we defined the function value at that point if it is not already defined and make it analytic.

But in our case the function value at the point of zf is 0 by our own definition, okay. And the point is again, it is given, you are again using importantly the hypothesis that the function is continuous even at points of z f, that is importantly used. You have given the function is continuous everywhere, so in particular the function is continuous at each point of zf and you now apply riemann sphere removable singularities theorem, okay. That is one thing and then of course i also wanted to discuss this problem, namely that the only one-one onto maps from the complex plane to the complex plane are the form z going to az plus b where a is not 0, okay.

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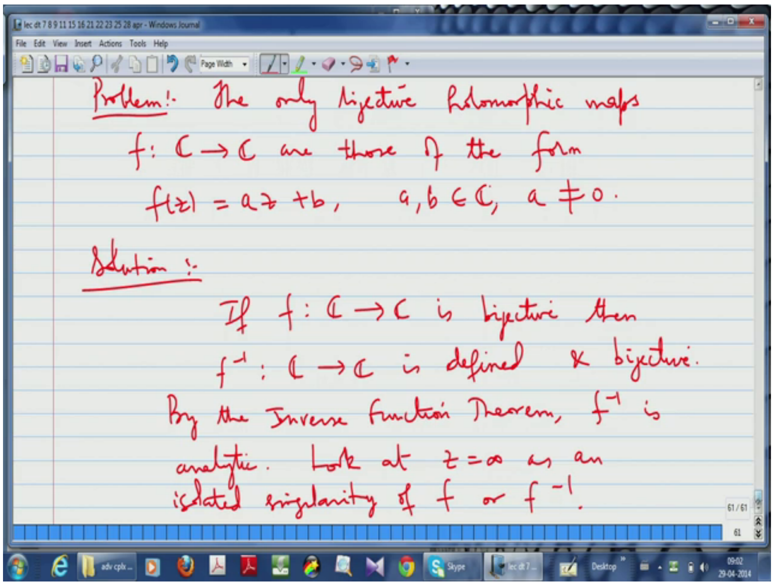

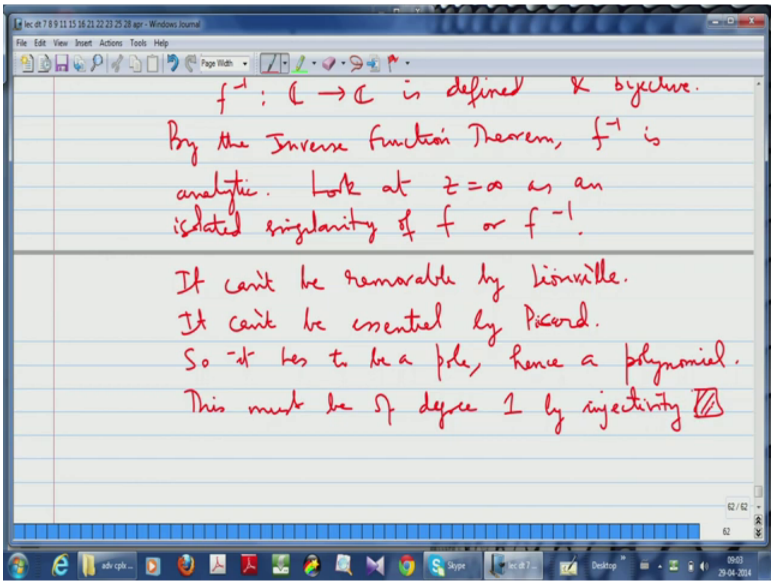

So these are the only automorphisms of the complex plane, okay. So let me do that also, because it is an application of the idea of singularity. So here is another problem, the only bijective holomorphic maps f from c to c are those of the form f of z is equal to az plus b where a and b complex numbers and a is not 0. And what is the solution to this? Well, the point is that if f from c to c is bijective, then f inverse from c to c is defined and bijective, okay. And mind you that see f is, f is analytic and that is the inverse function theorem which will tell you that f inverse will also be analytic because f inverse will be locally analytic, okay.

So by the inverse, by the inverse function theorem, f inverse is analytic, okay. And now we use the following thing, you know, you treat, so the whole point is to treat infinity is an isolated singularity of f, okay. You treat infinity is an isolated singularity of f inverse, okay. So you look at f inverse, okay and look at infinity, all right, infinity is an isolated singularity, okay. Or you can also take f, actually does not matter. Now what kind of singularities is isolated singularity is infinity? It can be either removable or it can be pole or it can be essential, okay.

If it is removable , since f is entire, it will, by louisville's theorem f will become a constant, okay. So certainly f is not a constant function because it is bijective, okay it is surjective, so infinity not a removable singularity. The other possibility is infinity is a pole, if infinity is a pole then f has to be polynomial, okay. But if it has to be bijective, in particular if it has to be injective, it should be polynomial of degree 1, so it has to be of the form az plus b, all right. And the only other possibility is that f is, the infinity is an isolated essential singularity but if infinity is an isolated essential singularity, then in every neighbourhood of infinity f will take every complex value except one, several times, in fact infinitely many times and that will contradict the injectivity of f.

So it cannot be an essential singularity, so here you are using picard's theorem, all right. So so the moral of the story is that because of picard's theorem you are forced to conclude that f is of the form, f of z is of the form az plus b, all right and a cannot be 0. So this is an application of picard's theorem, that is why i wanted to mention it. By, look at z equal to infinity as an isolated singularity of f or f inverse. It cannot be removable by louisville, it cannot be essential by picard, so it has to be a pole, hence a polynomial. This must be of degree 1 by injectivity and that finishes the proof, okay. So i will stop here.