Advanced Complex Analysis-Part 2.
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Lecture-42.
Royden's theorem on Normality Based on Growth of Derivatives.

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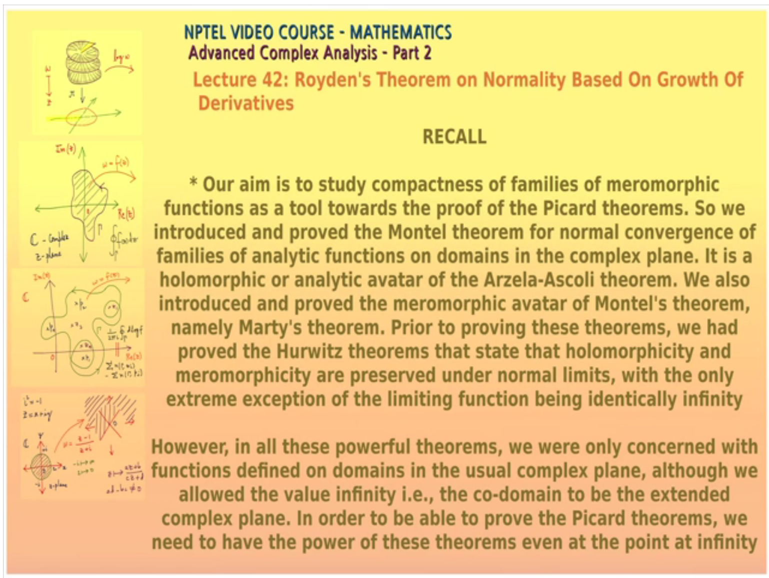
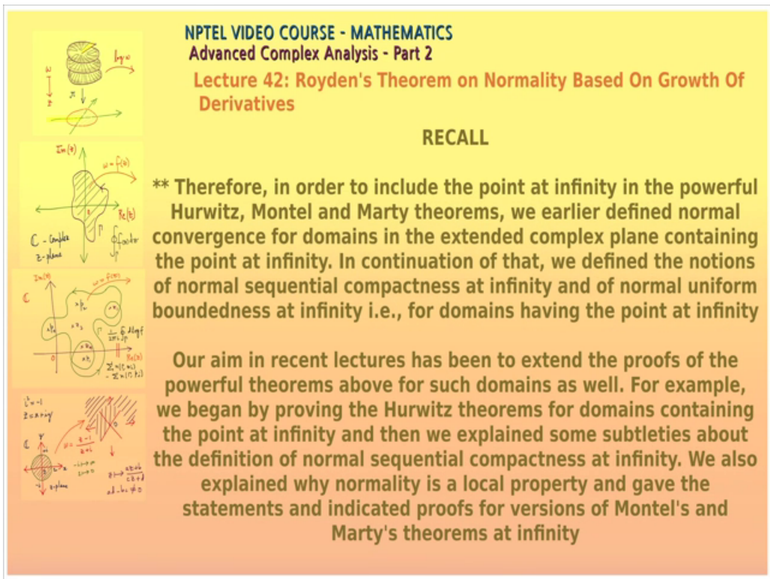
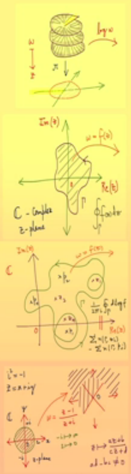






NPTEL VIDEO COURSE - MATHEMATICS

Advanced Complex Analysis - Part 2

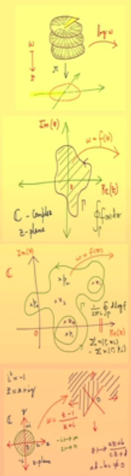
Lecture 42: Royden's Theorem on Normality Based On Growth Of Derivatives

RECALL

*** We defined the notion of normality at a point as a local property, and tried to analyze what it means for a family to be normal at a point. In order to understand the local behaviour of a normal family very close to a point, we explained a process called the zooming process which produces functions that are zoomed versions of the functions in the given family. The zooming can be made ultra-zoom i.e., we may increase the magnification of the zooming to infinity. This helps getting the behaviour very close to the point

By applying Marty's theorem to the zoomed family, we were able to arrive at a condition on which we expanded by introducing two further levels of intricacy, namely choosing ever-increasing zooming factors and choosing a convergent sequence of centers for the zooming process converging to the point being analyzed for normality. The condition we got is that we will always end up with a limit function that is constant

The whole point about this discussion is that this condition characterizes normality at a point and its negation is the essence of the important theorem called Zalcman's Lemma which is our key tool to understand non-normality at a point for a given family of meromorphic functions, leading eventually to a proof of the Picard theorems. With this motivation, we proved Zalcman's Lemma and its converse



NPTEL VIDEO COURSE - MATHEMATICS

Advanced Complex Analysis - Part 2

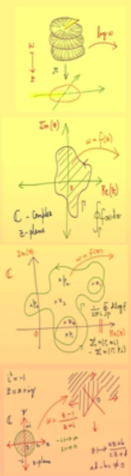
Lecture 42: Royden's Theorem on Normality Based On Growth Of Derivatives

GOALS

*** * In the lecture before the previous lecture we proved Montel's Fundamental Normality Criterion, which we may assert as the deepest theorem in the present course of lectures. The proof involves all the powerful theorems we have proved so far: e.g., those of Hurwitz, Montel, Marty, Zalcman and the Open Mapping Theorem. Montel's beautiful normality criterion is based on omission of values, and says that any family of meromorphic functions that omits three distinct values in the extended complex plane is normal, which implies that any family of analytic functions that omits two distinct complex values is normal as well

In the previous lecture, using Montel's criterion and the Zooming Process at an essential singular point, we proved the Great Picard Theorem and deduced the Little Picard Theorem as a corollary

In the present lecture, we prove Royden's theorem that asserts normality of a family based on growth conditions of its derivatives. The condition is that the derivatives should grow (in modulus) at most like an increasing function of the (modulus of the) original functions



NPTEL VIDEO COURSE - MATHEMATICS

Advanced Complex Analysis - Part 2

Lecture 42: Royden's Theorem on Normality Based On Growth Of Derivatives

KEYWORDS AND KEY PHRASES

Royden's Theorem, deciding normality of a family based on growth conditions of its derivatives, Lipschitz type condition, derivatives grow at most like an increasing function of the original functions, Arzela-Ascoli theorem, equicontinuity, diagonalization argument, normally uniformly bounded, normally sequentially compact, normally uniformly bounded derivatives, normally uniformly bounded spherical derivatives, metrics on the Riemann Sphere or extended complex plane, strongly and weakly equivalent metrics, uniform continuity and equicontinuity preserved under strongly equivalent metrics, Cauchy integral formulas, Cauchy estimates imply equicontinuity, Montel's theorem as holomorphic avatar of Arzela-Ascoli, Marty's theorem as meromorphic avatar of Montel and Arzela-Ascoli theorems, normal sequential compactness or normality of a family is a local property, Hurwitz's theorems

So what I want to do is you know try to tell you about some other important theorems which are connected with normal families, okay. So the theorem that we are going to look at today is called Royden's theorem, okay. And the whole point about the theorem is that you know whether you can decide the normality of a family if you have a growth condition of the derivatives, of the functions in the family, okay. So remember that normality is connected with the derivatives, okay. Montel's theorem will tell you that you know is for family of analytic functions, normality is the same as saying that the, you know the, the original functions are themselves normally uniformly bounded, okay.

And the 1st, the Caushy's integral formula will then tell you that the derivatives will also be you normally uniformly bounded. And the normal uniform boundedness of the derivatives will give rise to equi-continuity, okay and then you are in Arzella Ascoli kind of situation and you will get normal sequential compactness, okay. And then the same kind of thing, that the same kind of philosophy with Marty's, with Marty's theorem as well because in that case you are looking at meromorphic functions. And then the theorem says that the normality of a family of meromorphic functions is directly the same as the the normal boundedness of the spherical derivative, okay.

So you have to take derivatives with respect to the spherical metric, all right. And then, so but mind you normally whenever you have a family of functions that whose derivatives for example you have a family of functions which satisfies liveshe's condition, okay. That is a difference in the function values are bounded by a constant times the difference in the variable values, okay. This is a kind of condition that you will get if for example the derivatives are bounded, okay. So basically if you have liveshe's that condition, which is how you must think of condition vary derivatives are bounded.

Whenever you have liveshe's that condition, then you are actually getting equi-continuity and then you can apply Arzella Ascoli theorem to get normal sequential compactness, okay. So here is, so what we are going to look at today is Royden's theorem which says that if we have a family functions, okay and assume that the, the derivatives of the family, so I am looking at, either again I can look at analytics functions or I can look at meromorphic functions, only thing is that if you look at analytics functions, I mean I must consider the derivatives well-defined. So if they are meromorphic functions, I should not worry about the derivatives at the poles, I am not, in principle I am not looking at the spherical derivative, okay.

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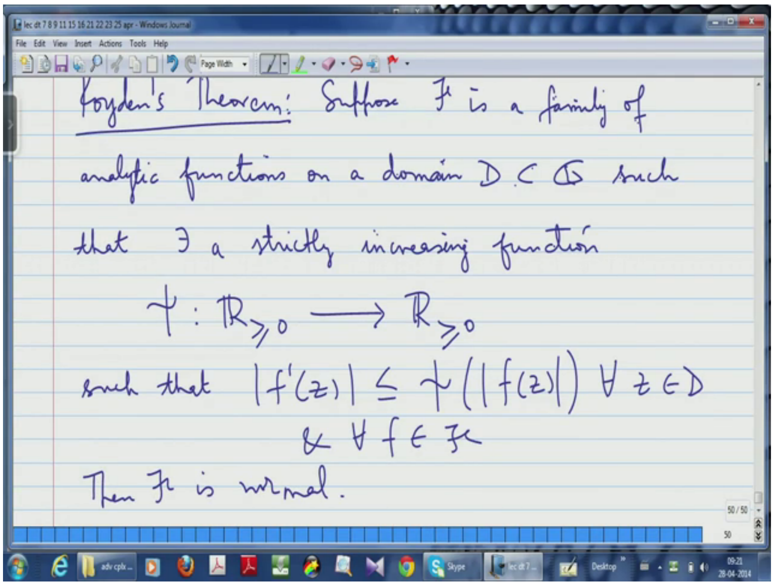

I am looking at ordinary derivative that all points other than poles, okay. So if the derivatives grow at most likely an increasing function of the original functions, okay, then the family is normal, this is Royden's theorem, okay. So let me write this down, so Royden's theorem, suppose script F is a family of analytic functions on a domain D in the complex plane, such that there exists a strictly increasing function , so this is a function psi from nonnegative real numbers are nonnegative real numbers. So it is strictly increasing function, such that the modulus of the derivatives of the functions of the family are bounded by psi of the modulus of the function. Okay.

So this is the, this is the condition in Royden's theorem, the condition is that the derivative of your, the functions in your family the way they grow is utmost like an increasing function of the growth of the original functions. So what is, what is there on the right side is psi of mod F, okay, mod F indicates the growth of F, okay and modulus. And psi of mod F is the psi of the growth of, it says that psi of mod F will be an increasing function of mod F because psi is an increasing function. So what you are saying is that the, the derivatives grow as an increasing function of the original functions, okay.

Then, then the family F is normal, okay. And the significance of this is that you know because of Montel's theorem, this will tell you that the family is going to be normal uniformly bounded, okay, it also tells you that the derivative will also be normal uniformly bounded, okay. It is a, it is a very powerful condition all right. The point is that when you look at this conditions at looks as if, you know the derivatives are growing pretty fast, okay. It looks as if the derivatives are going see what you want is the derivatives to be bounded on compact subsets, okay.

You want derivatives to be normally uniformly bounded, that means you want them to be uniformly bounded on compact subsets, which means on a compact subset you want a uniform bound, okay. But the point is that what this says is that the derivatives are growing but the growth is at most like an increasing function of the modulus of the original functions, okay. So it looks, directly does not look as if this is going to lead to normality but the theorem is that it does lead to normality. And the reason is that this is also a kind of boundedness of derivative with respect to different kind of metric on the Riemann sphere which can be defined using psi, so that is the whole idea.

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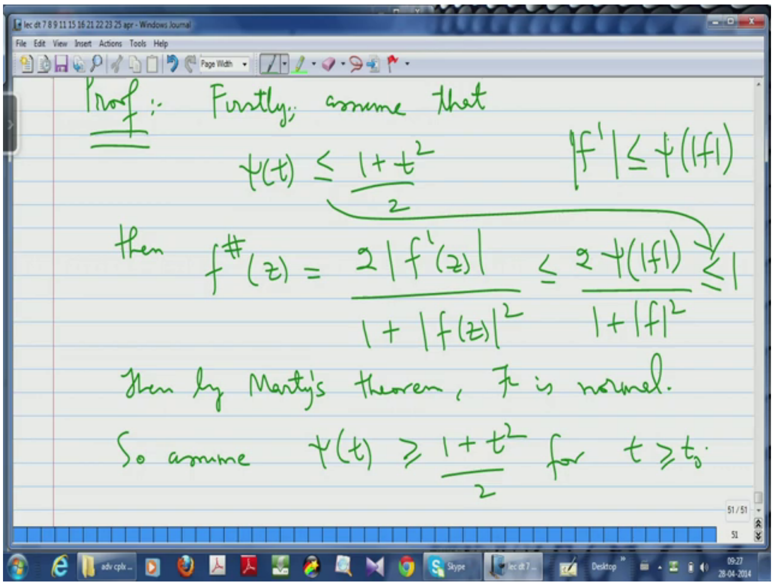

So, so let me write this, let me write the proof, so this is the proof. Firstly, assume that psi of t is you know less than or equal to 1+ t square by 2, okay, suppose you assume this. Because I want to then consider the case when it is better than or equal to 1+ t square by 2, suppose I assume this, all right. Then you see if you can split the spherical derivative of any function in the family, what I will get is F hash of Z is what, it is going to be by definition 2 times mod f dash of Z divided by mod F of Z the whole square. This is what it is, all right, this is a spherical derivative.

Mind you these are all analytic functions but I can also consider them as meromorphic functions and spherical derivatives are defined, okay, even for meromorphic functions. So now you see condition is that mod F dash is supposed to be less than equal to psi of mod F, okay, this is the, this is the condition of the theorem. I can write this as less than or equal to 2 psi of mod F divided by 1+ mod F the whole square, all right. But then 2 psi of mod F by 1+ mod f the whole square is less than equal to1, that is because of this, okay. So if I, if I assume psi of t is 1+ t square by 2, then all the spherical derivatives are bounded on the whole domain.

And of course you know, if the spherical derivatives are bounded on the whole domain, then Marty's theorem tells you that this is equivalent to the normality of the family. Okay, so this is just equal to normality of the family, considered that the family of meromorphic functions but that is also the same as the normality of the family, considered as a family of analytic functions, the only thing is that you should allow the possibility that you can have normal convergence to the functions as identically infinity, okay, when you consider the spherical metric.

So, so you know the case when psi of t is less than equal to 1+ t square by 2 is trivial, all right, it is just because of Marty's theorem. Then by Marty's theorem, so let me write this down, F is normal, okay. So assume the other possibility that psi of t is greater than or equal to 1+ t square by 2 for beyond a certain stage t greater than or equal to t0, okay. I can replace the function psi of t by another function for which the same condition holds with t great than or equal to 0, okay. That means I am saying that without loss of generality I can assume t greater than, t0 equal to 0.

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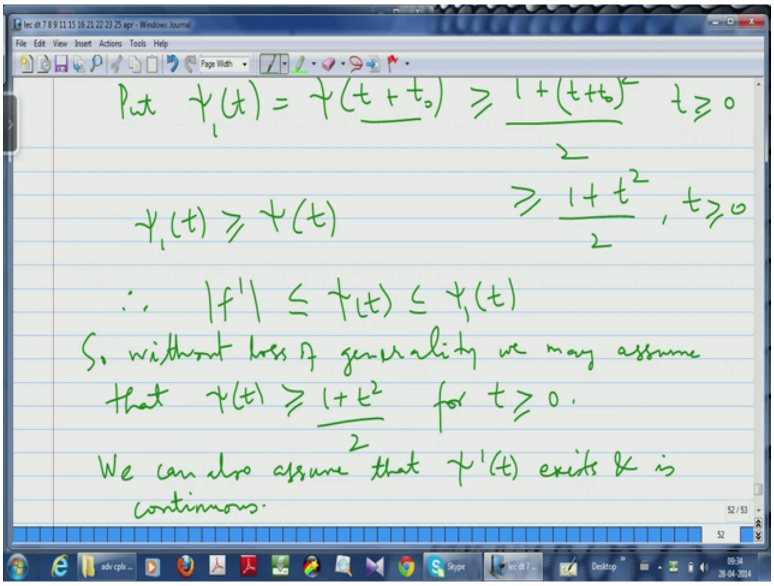

And why is that, so because, you just have to put psi 1 of t to be equal to psi of t plus t0, if you do this, all right. Then you know psi of t plus t 0 is certainly going to be equal to, going to be greater than equal to 1+ t plus t0 the whole squared by 2, that is if t greater than equal to 0. So mind you psi of t is greater than or equal to 1+ t square by 2 provided that t is greater than t0. So I need so I need t plus t0 greater than t0 and that is the same as t greater than equal to 0. I will get this, But then of course this is rather than equal to 1+ t square by 2, okay.

And, so the, so you know if psi of t is greater than equal to 1+ t square by 2 for t greater or equal to t0, I can replace psi of t by psi 1 of t, which psi 1 of t greater than or equal to 1+ by2 for t greater than equal to 0, okay. And the point is that psi 1 of t is certainly greater than psi of t because psi 1 of t is psi of t plus something and psi is increasing. So psi 1 of t will be also greater than equal to psi of t, this will also be true and the point is that , this is of course because psi is increasing. And therefore what will happen is that you would have got mod F dash is going to be less than or equal to psi of t which is less than or equal to psi 1 of t, okay.

So all these considerations tell you that without loss of generality, you can assume psi of t to be greater than or equal to1 plus t square by 2 for t greater than equal to 0, okay. So without loss of generality, we may assume, we may assume that that psi of t is greater than or equal to1 plus t square by 2 for t greater than or equal to 0, okay, we can do that. Now, namely if not you replace psi by psi 1 and call psi 1 as psi if you want, all right again. So then the next deduction I am going to make is that I am going to assume that psi is also continuously differentiable function of t, that is because psi is a monotonic function, okay and I can always approximate it to by a continuously differentiable function, okay.

So we can also assume that psi dash of t exists and it is continuous. This is the same as saying psi is C1, okay, continuously differentiable function. You can do that because after all the can replace psi by any bigger function, okay. And you can replace psi by a bigger function which is smooth, okay, you can always do that. And well, now you know what one is going to do, you are going to, one is going to use the psi to give a metric on the Riemann sphere. Okay. And look at the induced metric on the domain in your complex plane.

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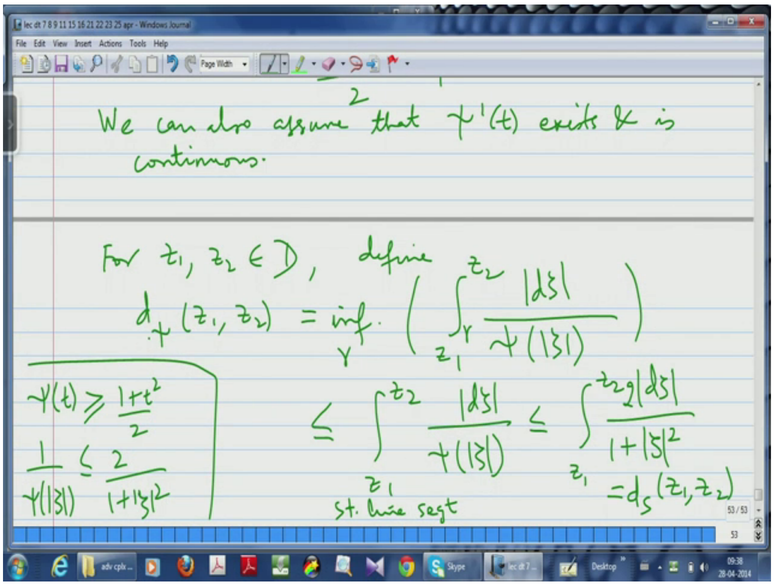

So what you do is, for Z1, Z2 in the domain, defined d psi of Z1, Z2 to be equal to , you need, you take integral from Z1 to Z2, okay, along any smooth path along any piecewise smooth path even contour, okay. And what you integrate is that, this is integration, you integrate mod d Zeta by psi of mod Zeta, okay. See notice and of course you will take the, you could take the infimum of all this. So this is, this integral is over gamma, so infimum over all gamma, that gamma is a smooth or piecewise smooth path from Z1 to Z2. You take all possible paths, contours from Z1 to Z2, integrate along that, this, this quantity, mod D Zeta by psi of mod Zeta okay.

And notice that, you know the point is psi, please try to understand, psi is greater than, psi of t is greater than 1+ t square by 2. So 1 by psi will be bounded by 2 by 1+ t square, 1 by psi will be less or equal to 2 by 1+ t square, okay. And 2 by 1+ t square is 0, that is the, that is a form of the integrand that you have to put very what is spherical metric, okay. 2 by 1+ t square is bounded as t goes to infinity if you want. So the integral is always nicely defined, the. So you and you take this infimum, all right, then the point is that this is a, this gives a metric on the, on d sub psi the way I have written it, gives a metric on the on the Riemann sphere, okay, or you can think of it also has a metric on D where D is identified with its image on the Riemann sphere of you want.

But the point is that this gives a metric, right and you know in particular and infimum of a set of quantities is always less than or equal to each of those quantities. So if for gamma I had taken the straight-line path from Z1 to Z2 this is also less than or equal to integral from Z1 to Z2, okay, straight-line path, straight-line segments from Z1 to Z2 of this quantity mod D Zeta by psi of mod Zeta, okay. And the point is that , see but the point is that this is bounded, see 1 by psi zeta is bounded by 2 by 1+ modern zeta the whole square because that is exactly, that is this condition, right.

We have assumed psi of t is greater than or equal to 1+ t square by2, okay. So 1 by psi of Zeta, psi of mod Zeta is going to be , this is going to be bounded by 2 by 1+ mod of Zeta the whole square, okay. You have this, so this is bounded by integral from Z1 to Z2 mod d Zeta, 2 mod d Zeta by 1+ mod Zeta the whole square. But what is this, this is actually a spherical length from Z1 to Z2, so this is just d spherical from Z1 to Z2. So what you are proved is that d sub psi is a metric which is bounded above by the spherical metric, okay.

But the point is that, for the spherical metric the Riemann sphere itself is compact, okay. It is compact with respect to the spherical metric because you know the spherical distance, the maximum spherical distance is, there is a maximum to it, it is, you know it is the it is going to be just the half the circumference of the sphere of the radius 1, okay. It is going to be namely it is going to be just pie, that is the maximum distance you can get on the Riemann sphere. The Riemann sphere is a sphere of radius 1, okay. Then any great circle on it we have radius 2 pie, and we have circumference 2 pie, okay.

And the maximum distance you can get is from the, for example from a point to its anti-podal point, for example from the North Pole to the South Pole, North Pole representing infinity, South Pole representing 0. At the maximum distance you get is pie. So it is a space with finite diameter, I mean it is compact, okay. And the point is that in a for a compact space okay, if one metric is bounded above by another metric, then these 2 metrics are uniformly equivalent, okay.

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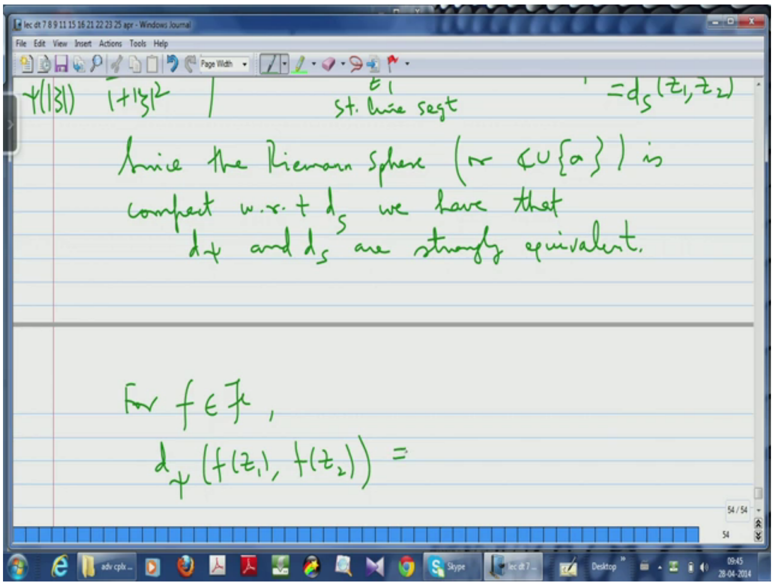

So the point is that since the Riemann sphere, so the Riemann sphere, sphere or you can just consider C union infinity or the external complex plane is compact with respect to the spherical metric, okay, we have that , these 2 metrics d psi and ds are strongly equivalent. So at this point let me very quickly tell you about the strong equivalence. See if we have 2 metrics on a space, we generally say that these 2 metrics are equivalent if they induce the same topology, okay. This is called weak equivalence. Now what is strong equivalence, strong equivalence is a condition that each metric is bounded by the other metric by up to a constant, an absolute constant.

So if you have 2 metrics D1 and D2 on a space X, these 2, to say that, to say that D1 and D2 are equivalent for X, means that the topology induced by D1 is the same as the topology induced by D2. One of the sufficient conditions is that every ball in D1 contains a ball in D2 and every ball in D2 contains a ball in D1, okay, the nesting of balls condition as it is called, okay. So this is just to say that, the 2 metrics are topologically equivalent. But there is something called strong equivalence, strong equivalence is that, the 2 metrics, each metric is bounded above by the other metric up to multiplication by an absolute constant.

So if the 2 metrics are D1 and D2, you should get D1 less than or equal to Lambda D2 and you should get D2 less than or equal to mu D1, you should be able to find such absolute constant. If you are able to find such absolute constant, these metrics are said to be strongly equivalent. Now, of course strongly equivalent means equivalent but what is the beauty about the strong relevance is the following. See, if you on a space, suppose you are considering functions, continuous functions, okay, then it will change metric to an equivalent metric, that is you changed it up to weak equivalence, that is a change metric by another metric which gives you the same topology, continuity will not be affected, because after all you have not changed is the topology, continuity, it just depends on topology.

But the problem is uniform continuity will become a problem, okay. If you have a uniformly continuous function, okay on a subset, suppose you have a function which is continuous, which is uniformly continuous on a subset with respect to one metric, if you replace that metric by an equivalent metric, the uniform continuity may not be preserved. If you want also the uniform continuity to be preserved, you should replace the metric necessarily by a strongly equivalent metric, not as by any other metrics which gives the same topology.

So what will happen is that if you change metric by just another equivalent metric, namely you do not, you are only worried about the topology, what will happen is that a function which is continuous uniformly with respect to one metric may fail to be uniformly continuous with respect to other metric. But you want to preserve uniform continuity, you have to replace the metric only by a strongly equivalent metric, okay. And the point is that you see if you have 2 metrics on a space, such that one metric bounced, is an upper bound for the other metric and the boundary metric with respect to the boundary metric the space is compact, then it also means the, this some multiples of the, of the smaller metric, some multiples of the smaller metrics also point the larger metric, that can be grouped, okay, because of compactness.

Therefore the bounding of one metric by another on a compact space, where the bigger metric, while the space is compact with respect to the bigger metric, will tell you that they are strongly equivalent, okay. Now you know why I am saying all this, I am just saying all this to tell you that, you know if you are looking at a family of functions, you know to decide uniform continuity with respect to the spherical metric, I can as well decide uniform continuity with respect to the metric, any metric that is strongly equivalent to spherical metric, okay.

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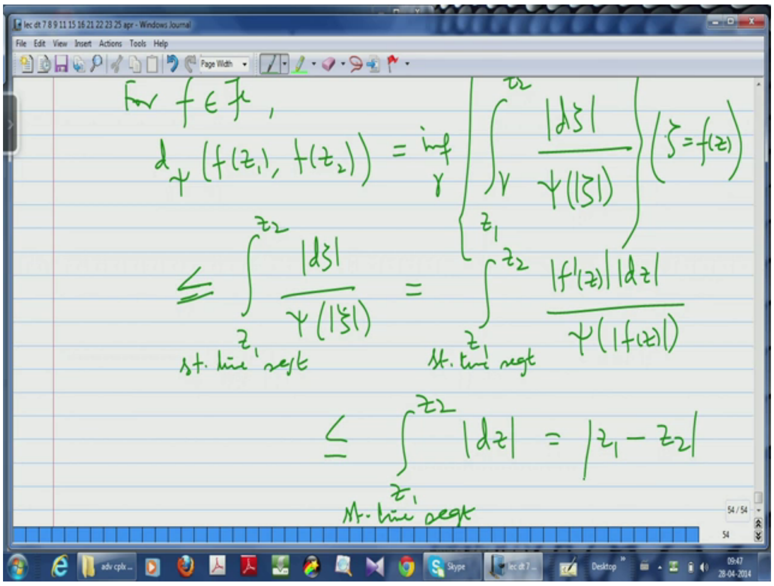

So this applies to uniform continuity, it applies to equi-continuity and things like that, right. So we are now more or less done, see, now what you do is that, now you look at the following thing. If you calculate for small f and script F what happens is that if I calculate the under the new metric d sub psi if I calculate f Z1 f Z2, what will I get, this is going to be infimum over all contours from Z1 to Z2, integral from Z1 to Z2, long that contours gamma, mod d zeta by psi of mod zeta, this is what this, with you know this solution zeta equal to f Z, okay, this is the definition.

And you, well what is this going to give me? See notice that this is, being an infimum, this is certainly less than or equal to you now the integral over the straight line segment. So if I take this integral over Z1 to Z2 and I take the straight line segment, what I am going to get is mod D zeta by psi of mod zeta and of course psi, zeta equal to Fz, so if I do that, I am going to get integral from Z1 to Z2 along the straight line segment of some I will get mod f dash of Z into mod d Z because that is what D zeta will be.

And I will get mod, divided by psi of mod f Z, okay. But then what is my, what is, what is the hypothesis in the theorem? The hypothesis in Royden’s theorem is that the numerator mod, the numerator of the integral mod f dash of Z is bounded by psi of mod fz. So it means that this integrand is less than or equal to 1. So you know what it means that this is less than or equal to integral from Z1 to Z2 over the straight line segment of you know mod DZ. But you know integrating mod DZ will give you just you Euclidean metric, okay. So this will be simply mod Z1 minus Z2, this is all I am going to get, okay.

So, all right, so what have we proved, we proved therefore, that what is there on the left side, on the left side I have D psi of f Z1, f Z2, you prove that this D side of f Z1, f Z2 is bounded by, is mod Z1 minor that too. That is the Livshe's condition, that is liveshe's condition on the f with respect to the new metric D psi on the Riemann sphere. But the moment you have liveshe's condition, it implies equi-continuity. So it means that the family F is equi-continuous, okay. This family script F is equi-continuous with respect to the metric D psi.

But then D psi is strongly equivalent to D s, therefore the family script S is also equi-continuous with respect to the spherical metric but if it is equi-continuous with respect to the spherical metric, I am in the sum I am, I can use as in the prove of Marty's theorem, I can use Arzela Ascoli theorem, I can use the diagonalization argument to conclude that the family is normal and that is the proof.

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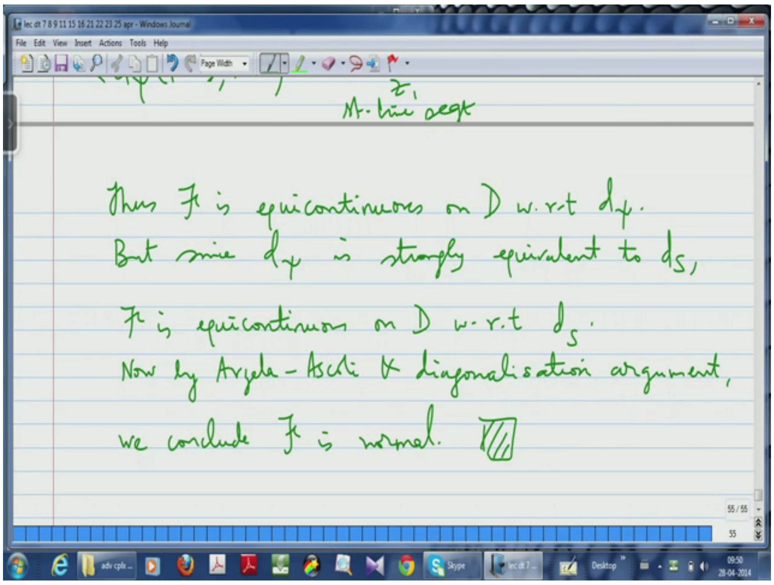

So thus script F is equi-continuous on D with respect to the spherical metric but since the spherical metric, sorry, the script F equi-continuous with respect to D psi but since D psi is strongly equivalent to the spherical metric DS, script F is equi-continuous on D with respect to the spherical metric. Now by the Arzela Ascoli theorem and the diagonalization argument, we conclude script F is normal, it is a normal family, okay. So that is the proof of Royden's theorem. So the whole idea is that, this, this Royden's condition, is actually, Royden's condition translates to a liveshe's condition when you use different metric on the Riemann sphere, okay.

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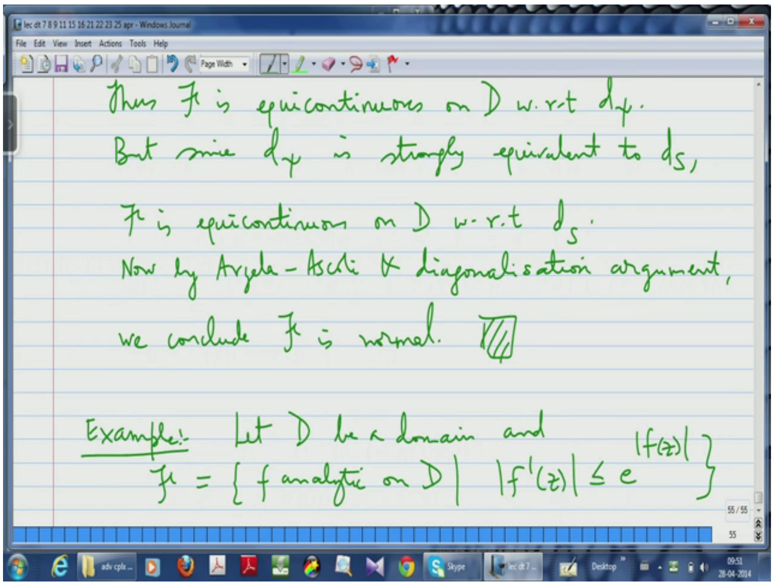

So what is the, what is the advantage of, what is the advantage of this? The advantage of this is we can apply it to an example like this, let D be a domain and script F be the set of all functions f analytic on D, such that mod f dash of Z is bounded by e Power mod f Z, look at, look at functions like this. Mod f dash is bounded by e Power mod f, okay. Now then this family is normal, because here e Power mod F is the function e Power t and E Power t is of course a smooth, it is an increasing function of t for t greater than equal to 0 and you can apply Royden's theorem.

So you know you get this nice condition that if you are looking at a family of analytic functions whose derivatives grow at most as exponent, grow exponentially as the functions from other derivatives are F dash from other derivatives of the function are given by F dash, okay. And their moduli, mod f dash, that is the rate at which F dash grow and mod F dash is bounded by, is at most by e Power mod F, that is the condition. So, so the function involved is e Power t which is increasing function of t, all right.

And you know, mind you this is for any domain, for example if you are taking the domain to be the whole plane, okay, e Power mod F, e Power mod F is a kind of exponential growth and it seems to be, even if you put mod f dash is equal to E power mod F, it looks like that the derivative is growing exponentially as the original function. But, so it looks as if the derivatives, it seems as if the derivatives cannot be normally uniformly bounded. Because whenever there is something growing exponentially, you are worried about boundedness, okay.

So if you take a compact subset of the domain, all right, one is worried whether, you know whether the derivatives will be bounded. But the truth is that they will be and it is not obvious, it is not an obvious result. What you have is the derivatives are growing exponentially, okay, as the original functions and from that try to conclude that the derivatives are bounded uniformly on any compact subset is a great thing. So you see Royden's theorem will now apply, it will tell you that this family is normal but if this family is normal, Montel's theorem will tell you again or you can directly even use Marty's theorem, Marty's theorem will tell you the spherical derivatives are bounded, okay.

And if you want use, you can use Montel's theorem which will tell you that the original functions are themselves normally uniformly bounded. So if you take a compact subset of the domain, then the functions are themselves uniformly bounded and Cauchy's integral formula will then tell you that the derivatives of the functions are also normally uniformly bounded, they are bounded on a very compact subset which is not obvious because the derivatives seem to be growing exponentially as the functions. So that is the significance of this theorem, okay.

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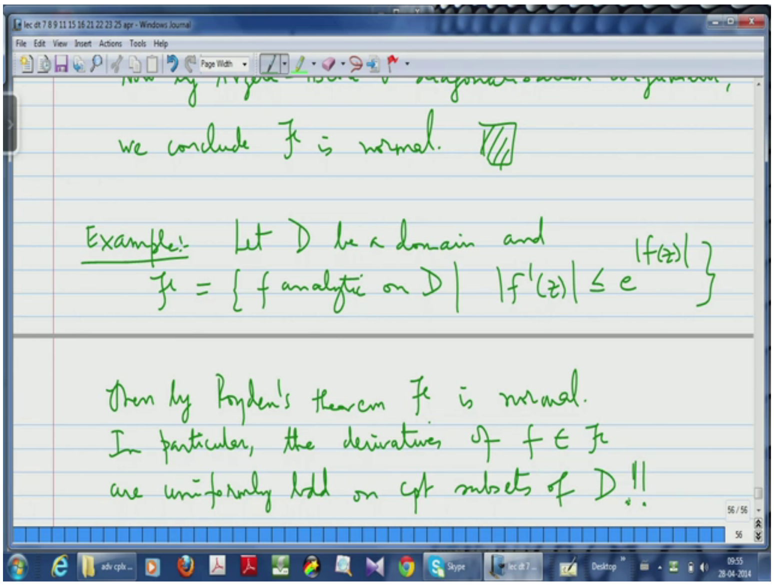

And it has got to do with the study of normal family and that is the reason why I wanted to present it here, okay. So I will stop here. So let me complete this sentence, I will just say that, then by Royden's theorem, script F is normal in particular the derivatives of f belonging to script F are uniformly bounded on compact subsets of D which is, which is quite surprising, given that the original condition was, just that the derivatives are going exponentially with respect to their functions, okay.