Advanced Complex Analysis-Part 2.
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Lecture-41.
The great Picard and Little Picard Theorems-Proof using Montel's Criterion and the Zooming Process.

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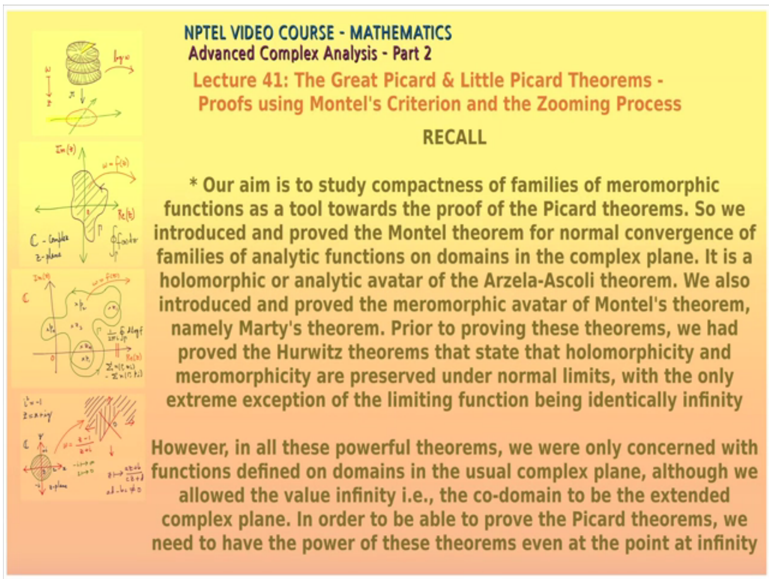
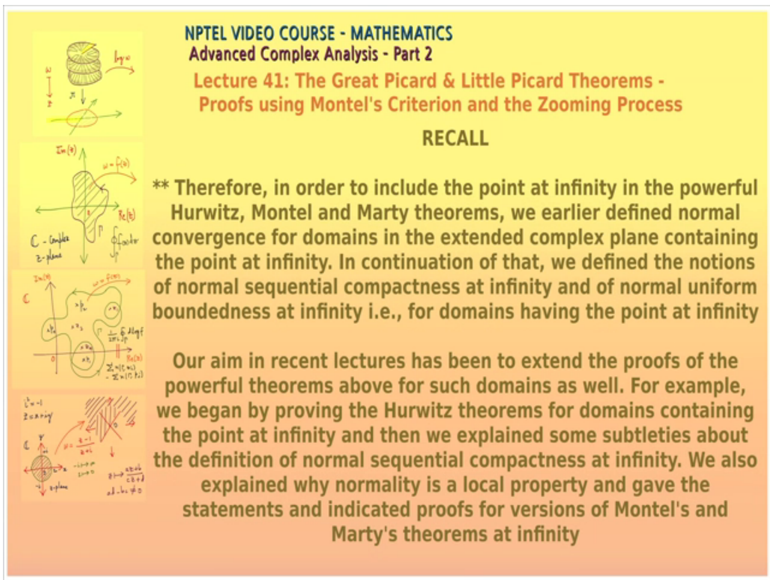
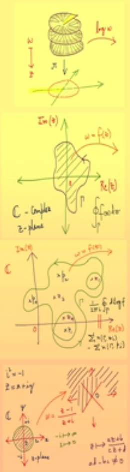






NPTEL VIDEO COURSE - MATHEMATICS

Advanced Complex Analysis - Part 2

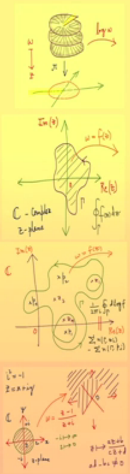
Lecture 41: The Great Picard & Little Picard Theorems - Proofs using Montel's Criterion and the Zooming Process

RECALL

*** We defined the notion of normality at a point as a local property, and tried to analyze what it means for a family to be normal at a point. In order to understand the local behaviour of a normal family very close to a point, we explained a process called the zooming process which produces functions that are zoomed versions of the functions in the given family. The zooming can be made ultra-zoom i.e., we may increase the magnification of the zooming to infinity. This helps getting the behaviour very close to the point

By applying Marty's theorem to the zoomed family, we were able to arrive at a condition on which we expanded by introducing two further levels of intricacy, namely choosing ever-increasing zooming factors and choosing a convergent sequence of centers for the zooming process converging to the point being analyzed for normality. The condition we got is that we will always end up with a limit function that is constant

The whole point about this discussion is that this condition characterizes normality at a point and its negation is the essence of the important theorem called Zalcman's Lemma which is our key tool to understand non-normality at a point for a given family of meromorphic functions, leading eventually to a proof of the Picard theorems. With this motivation, we proved Zalcman's Lemma and its converse in the lecture before the previous lecture



NPTEL VIDEO COURSE - MATHEMATICS

Advanced Complex Analysis - Part 2

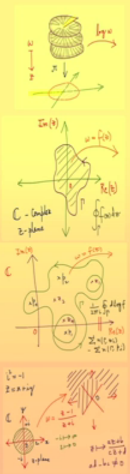
Lecture 41: The Great Picard & Little Picard Theorems - Proofs using Montel's Criterion and the Zooming Process

GOALS

*** * In the previous lecture we proved Montel's Fundamental Normality Criterion, which we may assert as the deepest theorem in the present course of lectures. As we shall see, the Picard Theorems will follow as consequences. The proof involves all the powerful theorems we have proved so far: e.g., those of Hurwitz, Montel, Marty, Zalcman and the Open Mapping Theorem

Montel's beautiful normality criterion is based on omission of values, and says that any family of meromorphic functions that omits three distinct values in the extended complex plane is normal, which implies that any family of analytic functions that omits two distinct complex values is normal as well

In the present lecture, using Montel's criterion and the Zooming Process at an essential singular point, we prove the Great Picard Theorem and deduce the Little Picard Theorem as a corollary



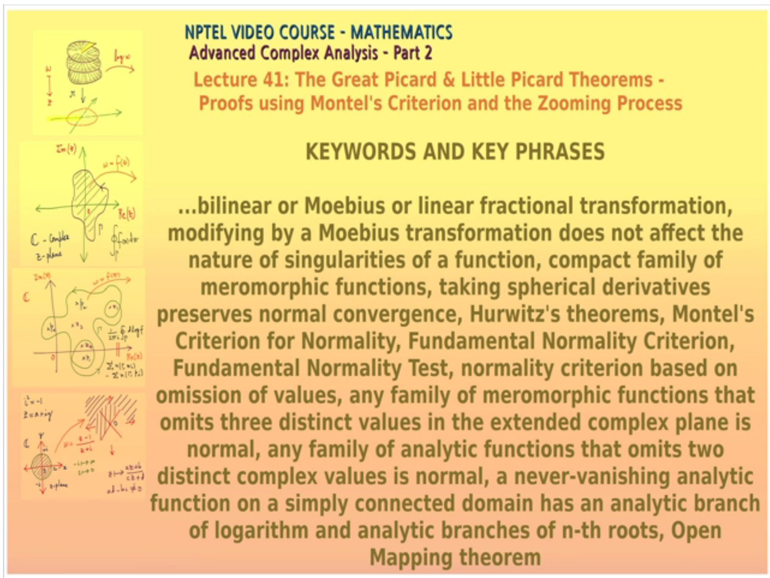
NPTEL VIDEO COURSE - MATHEMATICS

Advanced Complex Analysis - Part 2

Lecture 41: The Great Picard & Little Picard Theorems - Proofs using Montel's Criterion and the Zooming Process

KEYWORDS AND KEY PHRASES

Picard's Great or Big Theorem, isolated essential singularity, Picard's Little or Small Theorem, entire function, infinity as a pole, infinity as an essential singularity, infinity as a removable singularity, Liouville's theorem, Fundamental Theorem of Algebra, domains in the extended complex plane containing the point at infinity, normal convergence in a neighborhood of infinity, infinity as an isolated singularity, Riemann's Removable Singularities Theorem for the point at infinity, analytic at infinity same as bounded at infinity same as continuity at infinity, behaviour at infinity is the same as behaviour at zero with respect to the inverted variable, meromorphic at infinity, normal family at infinity, Montel's theorem at infinity, Marty's theorem at infinity, normal sequential compactness or normality of a family is a local property, local analysis of normality of a family at a point, zooming process, zoomed function, motivation for Zalcman's Lemma, translation, scaling or magnification...



Okay, so we have seen in the last lecture the famous theorem of montel or normality of families and what it says is that suppose we have a family of meromorphic functions defined on a domain, the domain can be extended complex plane. To decide that the family is normal, all you have to do is to ensure that all the functions in the family do not take 3 distinct values in the extended plane, okay. And you know because you are working with meromorphic functions, you have to allow the value infinity because that is the value that a meromorphic function at a pole will take, okay.

And but of course you are looking at a family of analytic functions, okay, then the condition is much more simpler, you have to just find 2 complex values which the functions in the family do not take and if that is true then the family is normal, okay, there is this great theorem of montel. And you see it is the key to proving the picard's theorem which we will do, right.

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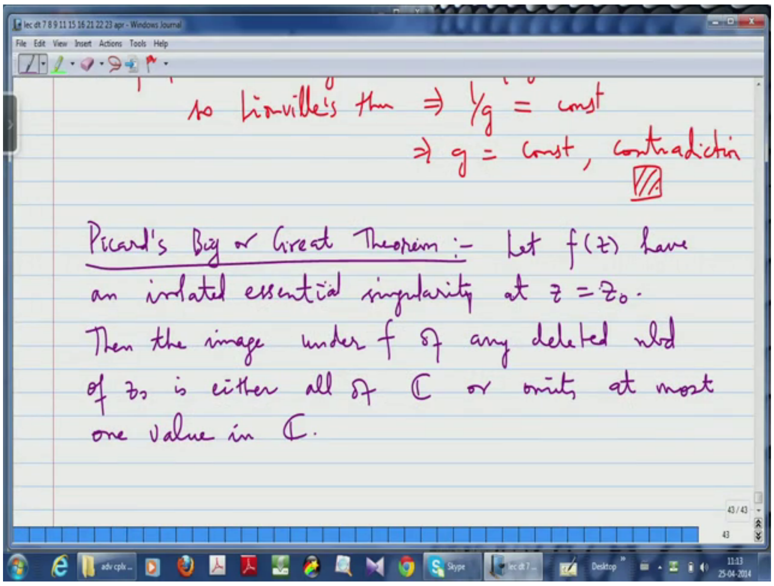

So, so, so here is picard's big or great theorem, okay. So you know what the theorem is. The theorem is that if you take a function which has an isolated essential singularity, then the image of any neighbourhood of that singularity is either the whole complex plane or it is a complex plane minus the single value, okay. That means, it means, it means that it can at most omit one complex value, all right. And what is the restatement, the restatement is that if it omits more than one value, if it omits 2 values, that is something cannot be, that is not possible, okay.

So what we will do is we will assume that it omits 2 values and then use the montel criterion that the resulting family of zoomed functions is normal, okay. And examine the limit of the zoomed functions and that will give you the proof, okay. So let me state it, let f of z, let f be, let f have an isolated, let f of z have an isolated singularity at z equal to z0 , then the image under f of any deleted neighbourhood of z0 is either all of complex plane or omits at most one value in c. So this is the picard's theorem, okay.

And of course this is valid for every neighbourhood, it means that you know it will take except for one value which it might omits, all other complex values it will take infinitely many times because you, if you find the point where it will take that value, then you can find a smaller neighbourhood, you can take a smaller neighbourhood, deleted neighbourhood of z0 and in that neighbourhood also again it has to take that value and can go on like this, therefore it will take every value except one value infinitely many times, okay. That is amazing behaviour of function, analytics function around an isolated essential singularity, okay.

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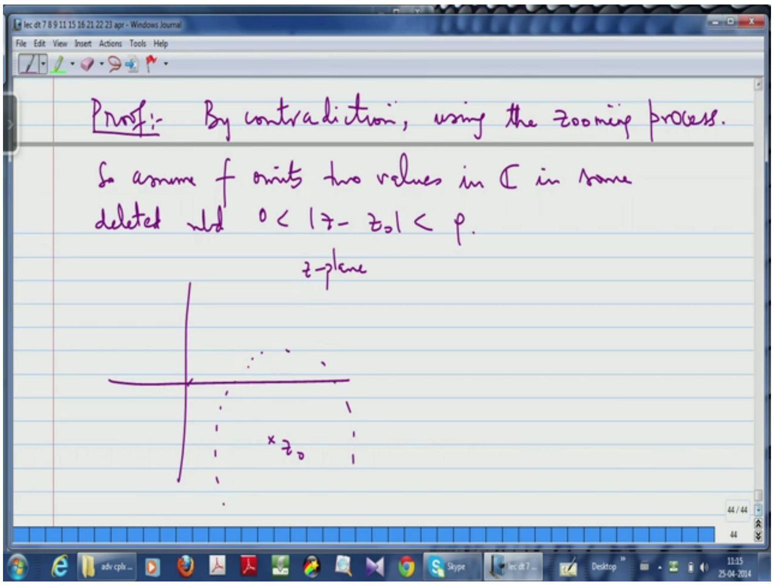

So this is the theorem and, so what is the proof? The proof is, by contradiction, by contradiction, by contradiction using the zooming process. So you know the zooming process has been, we had been using it right from zaltzmann lemma, okay and you know what is the main idea behind the zooming process? The idea behind the zooming process is, as you zoom into a normal point, then all those zoomed functions will converge normally to a constant function. And if you zoom to a non-normal point, then all the zoomed functions will converge to a nonconstant meromorphic function. Okay that is the, basically the principle, all right.

So what we will do is, so assume f omits 2 values in f omits 2 values in c in some deleted neighbourhood 0 less than mod z minus z0 less than say rho, okay. So you have to show that f takes either take all values or it will take all values except 1. So if you want to contradict that, you have to assume that it omits at least 2 values, okay. So let us assume that, all right. So you, what you do is, so here is my diagram. So i have, so this is the, this is the complex plane, this is z plane and i have this point z0. And you know i, there is this small disc surrounding z0, radius rho, this disc with radius rho and on this disc f does not take 2 values, 2 complex values.

That means there are 2 distinct complex values which f will take, it may not take many more values also but at least 2 values of it misses, okay. And what am i going to do, i am going to construct a sequence of zoomed functions. So what is the sequence of zoomed functions, what you do is, well take any, take any, so before that let me write it ideologically. Zoom into the function f at z0 itself, okay. Mind you the function in, z0 is an isolated singular point, it is an essential single point, therefore the deleted neighbourhood of z0 in this deleted disc, this punctured disc centred at z0 and radius rho, you throw out the point z0.

In the punctured disc it is the function is analytic, mind you, okay. And what i am going to do, i am actually going to zoom in to z0, okay, i am just going to zoom in to z0 and how do i zoom in to z0, by taking smaller and smaller, discs of smaller and smaller radii which are centred at z0. And of course i have to exclude z0 because z0 is a point of singularity of f. So what you do is, zoom in to z0 and so let me say that is take a sequence epsilon n tending to 0, 0 less than epsilon n lesser than rho, okay. So you take a sequence of smaller and smaller radii, okay.

And let gn of zeta be the zooming of f the same function f centred at z0 , the scaling factor is 1 by epsilon n and the variable is that, okay. So this means that you are just taking f of z0 plus epsilon n times theta, this is a function. So this is what my gn is, so i am using that single function and constructing a family of functions. Using the single function f i am constructing a family of functions gn, okay. And where are these gns defined? So you see, this is the z plane and then if you look at correspondingly, we have the, you have, this is a complex which is the z plane, then i also have complex thing which is zeta plane, okay.

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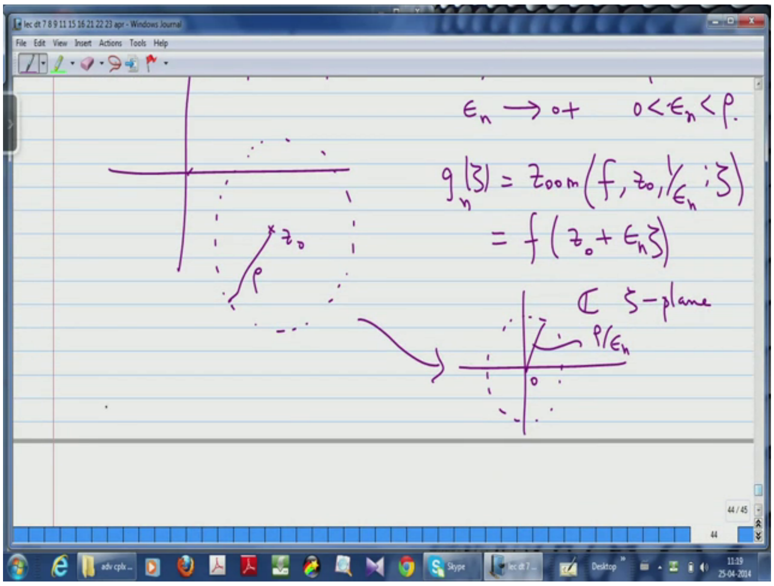

And in the zeta plane what happens is that, you know if i take, if i take this disc centred at the origin and radius, if i take the radius to be rho by epsilon n, okay, that is for mod zeta less than epsilon n, mod, epsilon and theta will be less than rho. And therefore this is the, so this is the disc in which gn is defined by the only thing is it is not defined as the origin because at the origin the origin corresponds to z0. And f at z0 is not defined because z0 is a singularity of f, okay.

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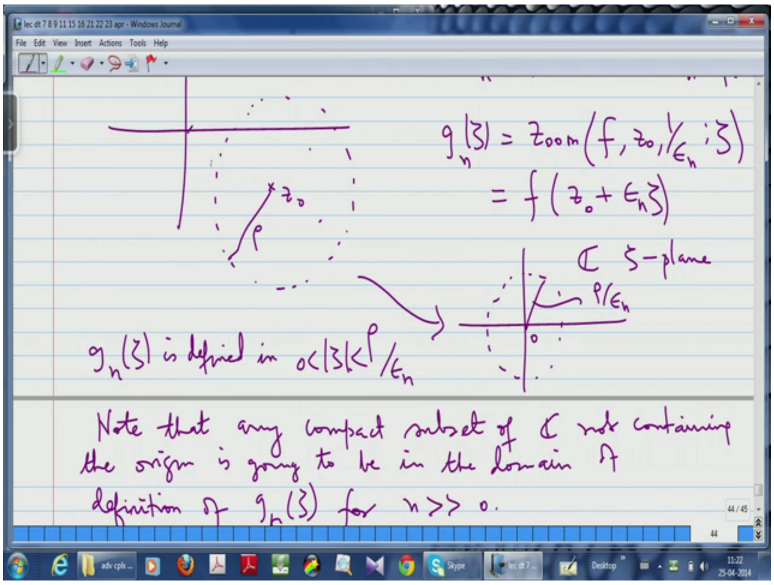

So let us write that down, gn is defined, so let me write this here, gn of zeta is defined in a 0 less than mod zeta less than rho by epsilon n. And of course you know the point about this whole business is that this epsilon n tends to 0+, rho by epsilon n tends to infinity and therefore gn zeta is going to be, you can talk about convergence, normal convergence of gn zeta on the whole punctured plane, okay, except that, the whole plane except the origin. Because eventually any compact subset of the punctured plane other than which does not contain the origin is going to be, is going to be contained in the domain of gn zeta for n sufficiently large, okay.

So, so let me write this down, note that any compact subset of c not containing the origin, the origin is going to be in the domain of definition of gn zeta for n sufficiently large, okay. So now, now i want you to just watch. See after all, you know the, the values of gn in this punctured disc centred at the origin o and radius rho by epsilon n correspondence to exactly the values of f in the punctured disc centred at z0, the values of f inside this whole disc, punctured disc centred at z0 radius rho, they are exactly the values of gn, in the punctured this centred at the origin, radius row by epsilon n, because this is just a scaling and data solution, okay.

Now you see now you know f, but what is, what have we assumed about f, we assumed that f omits 2 complex values, f is analytic function and it omits at least 2 complex values in this disc, punctured disc centred at z0 radius rho. Therefore each of the gns will also omit those 2 values in their domains, okay. And of course each of the gns are also analytic functions because they differ from f only by bilinear transformation consisting of scaling under and incarceration. But now you know we are in good shape because what we have done you know, we have been able to get a family of, we have been able to get a family of functions gns which are analytic and which omit 2 values.

Now immediately montel's great theorem will tell you that there has to be a normal and you get a convergence of sequence and then you have to examine the limit, okay. And the limit will give you contradiction, all right. So basically the contradiction will be set the limit function at the origin will, examining the limit function at the origin will tell you that the origin has to be either pole or removable singularity for f which is not true. I mean analysis of limit function will tell you that the, you take this limit function, this limit function will also be defined on the punctured plane, okay.

Because all the original gns are all defined outside 0, okay, so if you analyse the limit function. See the limit function is like zooming into f at z0 infinitely many times, okay. So behaviour of the limit function at the origin which is an isolated singularity will reflect upon the behaviour of f at z0. And by analysis we will show that if you analyse the limit function, there are only 2 possibilities z0 should either be removable singularity or it has to be a pole and both of these contradictions because i have assumed z0 at f to be a essential singularity, okay. And that is how the proof goes, so it becomes as simple as that.

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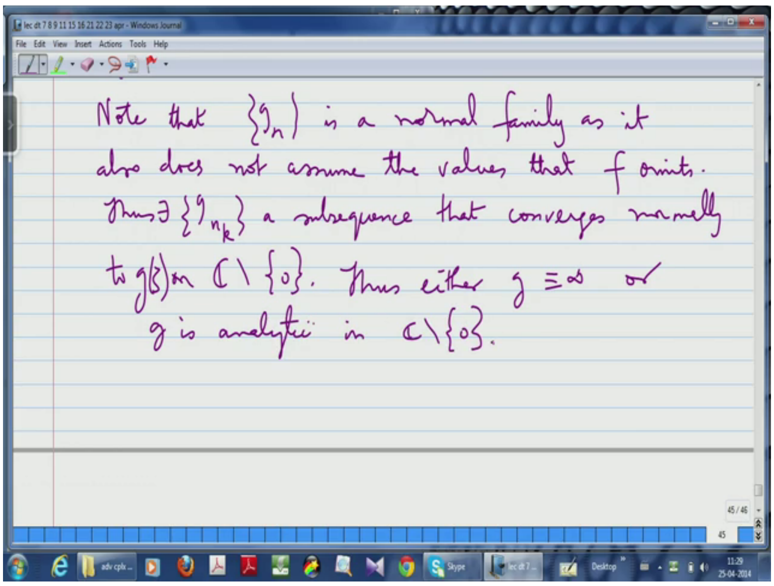

So let me write this down note that gn is a normal family as it also does not assume the values that f omits, okay. So, so what does it mean, it is a normal form means, it means that you know, and it is a normal family and mind you for gn because it is a zoomed function, whose domains are becoming bigger and bigger and bigger, you can think of them as a normal family you know in with a limit in the punctured plane, all right. So what you must understand is that if you take, for example if you take the punctured unit disc, okay, to take the punctured unit disc then for n sufficiently large, all the gns are going to be defined, their domains are going to become bigger and bigger and they are going to contain the punctured unit disc, okay.

So you can see that if you want to take the, if you want to talk about the domain of the gns, okay, you can assume that for n, p on, for n sufficiently large the domain contains unit disc if you want. Or for that matter, any finite disc with of course the origin omitted, okay. And when you take the limit function that is because gn is a normal, is normal, if you take the limit function, the limit function will be defined on the whole punctured plane because it will make sense, because you are covering every point in the plane literally. Because for every point in the plane if you take epsilon n sufficiently small, rho by epsilon n becomes sufficiently large and gns beyond a certain state will be defined with that point.

And therefore the limit of all the, if you take convergence of sequence of gns, the limit will also be defined with that point, okay. So let me write this, thus gn k, thus there exists gn k case of sequence that converges normally as g on, g of zeta if you want on c minus the origin, okay. So this happens, because what is the meaning of normal family? Normal means that, normally sequentially compact, that is give me any sequence, there is a normal convergence subsequence. So when gn itself is a normal family, it is already a sequence, so it will have a normally convergence of sequence. So you have a sub sequence gn k which converges normally to g on c minus origin, okay.

But the point is that it is not, it is not, what is important is that it is g lives in a neighbourhood of 0, okay and now, now try to understand each gn is analytic on, on the punctured disc, on the punctured disc centred at the origin, okay. Therefore this limit function g is a normal limit of analytic function. We have already seen such a normal limit can have only 2 possibilities, either the, either the normal limit can completely be analytics or it can be identically infinity, these are the only 2 possibilities. So let us, so let us write that down. Thus g is identically infinity or g is analytic in c -0, okay.

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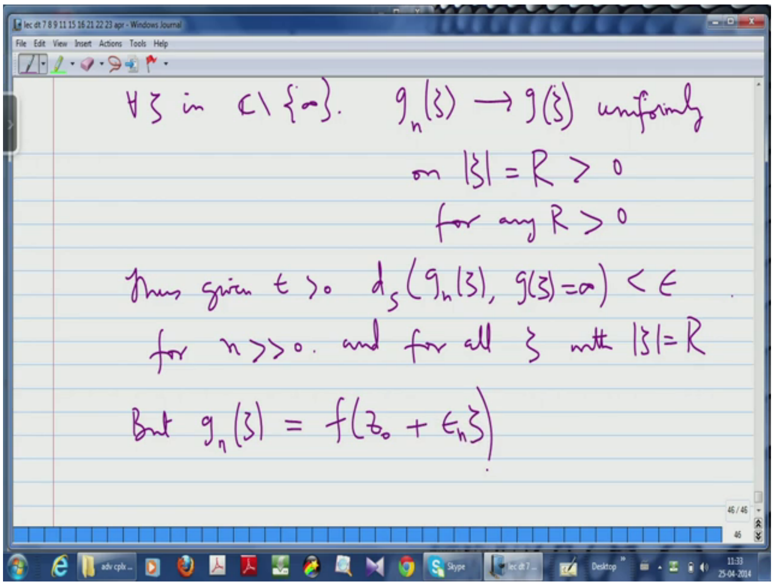

Now let us look at both of these cases. Suppose, g is identically infinity, okay, suppose g is identically infinity, so what does it mean, what this will mean, see, think of it heuristically, g is f zoomed you know infinitely at z0 and if g is identically infinity, what you are actually saying is that f is infinity in the neighbourhood of, f is going to be infinity in the neighbourhood of z0, right, because the values of g are just limits of values of gns and the values of gns are just values of f in smaller and smaller neighbourhood. So if g is identically infinity, okay, that means that the value, the gns are getting larger and larger in modulus, okay.

And that means that the values of f are getting closer and closer to infinity as you approach z0. But that means z0 is a pole, but that is not possible because z0 is an essential singularity, so this is not possible, so you ruled out this case, okay. So let me write this down. This means that g of zeta is infinity for all zeta in c minus infinity. I will have to make use of the fact that you know gn converges to g, okay, g is not just, it is not simply limit of gns. Of course it is point wise limit of gns but it is not, it is more than that, it is not just a point wise limit, it is just a normal limit.

So the convergence is uniform complexion since, okay. So you know gn of zeta converges to g of zeta uniformly on mod zeta is equal to say r for any r greater than 0. Because you see mod theta equal to r is a circle in the case of plane centred at the origin, radius r and that is certainly a compact circuit, it is closed and bounded. And g is a normal limit, therefore the convergence gn to g should be uniform on any compact subset, so it has to be uniform on mod theta equal to r, okay. But, but of course what is, but what is g zeta? G zeta is infinity, if g is identically infinity. So what this and what does uniform convergence mean, it means that all the gns, they will come to within an epsilon of g zeta, okay, if you take n sufficiently large irrespective of zeta, right, therefore that means, okay.

But of course you want to come to an epsilon of infinity so you will have to be careful and you have to use a spherical metric. So let me write this down, thus given epsilon greater than 0, the spherical distance between gn zeta and g of the zeta which is actually infinity can be made less than epsilon for n sufficiently large. And for all theta with mod theta equal to r, okay but now, but what is gn zeta? You see gn zeta, our definition is just f of z0 plus epsilon n theta, this is what it is. Okay.

And as you know, you see even if mod zeta is r, if your, your epsilon ns are becoming smaller, so you are covering smaller and smaller disc, you are covering smaller and smaller circles centred at z0, okay. And therefore what you are saying is that the function values of f on smaller and smaller circles centred at z0 are getting close to infinity, okay. And that is enough to tell you that f, the limit of f let as z tends to z0 is actually infinity, which means that z0 has to be a pole. But that is a contradiction to our assumption that z0 is actually an isolated essential singularity, okay.

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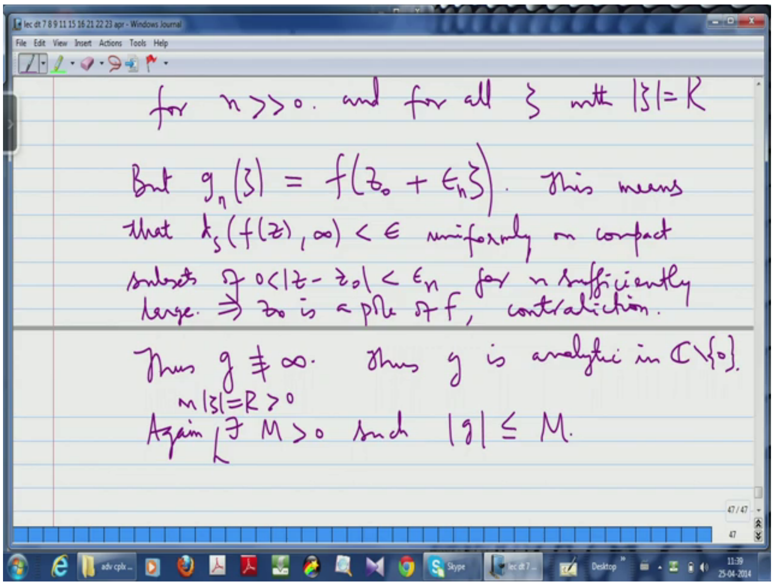

So let me write this down, this means that the spherical distance between f of z and infinity can be made less than epsilon uniformly on compact subsets of mod z minus z0 lesser than epsilon n for n sufficiently large, okay. So, you know because, what you must understand is that this, this r is, this capital r is at our disposal. You can make this capital r as small as you want, close to 0, you can make it as large as you want, okay. So you can, you can literally cover all the , you can cover all the circles centred at z0 of fixed radius, okay, below a certain positive value.

So in some sense you are, therefore you are able to cover complete deleted neighbourhood of z0, okay, that is the whole point. So, well, so , so this implies z0 is a pole of f, which is a contradiction. Because you will assume z0 to be an essential singularity, okay. So does, you know you, the limit, the zoomed limit function g cannot be identically infinity, okay. Therefore what is the other possibility, it has to be only be an analytic function in the, with, in the punctured plane, punctured r, okay. Thus g is analytic in the punctured plane, okay. And what does that mean, it means of course the origin is a singularity, origin is an isolated singularity for g, okay. And now you can ask what kind of singularity it is.

But you know the point is that again you should not try to study the singularity of g at the origin, if not do that. Because after all g at the origin is going to reflect f at z0, okay. So, mind you the gns are all zoomings of f at z0 and their limit is g, okay. So g is some kind of infinite zooming of f at z0. G at the origin is infinite zooming of f at z0, okay. Therefore you should not study g at the origin but you must make use of the fact that g is analytic in the outside the origin. So if you again take this mod theta equal to r, it would take circles centred at the origin, this is the plane, radius r.

Mind you again that is a complex set and g being analytic, if continuous, then on the compact that it will be bounded, okay. And now this bound will apply to f in the neighbourhood of z0, okay. And we will tell you that f is bounded in the neighbourhood of z0 but then riemann's removable singularity will tell you that z0 has to be removable singularity and again that is a contradiction. And that is the proof of the theorem, okay, proof of the theorem is so simple. So let us go to the other case, again there exists an m greater than 0 such that again on mod z, mod zeta equal to r greater than 0, there exists an m greater than 0 such that mod g is less than or equal to m, okay.

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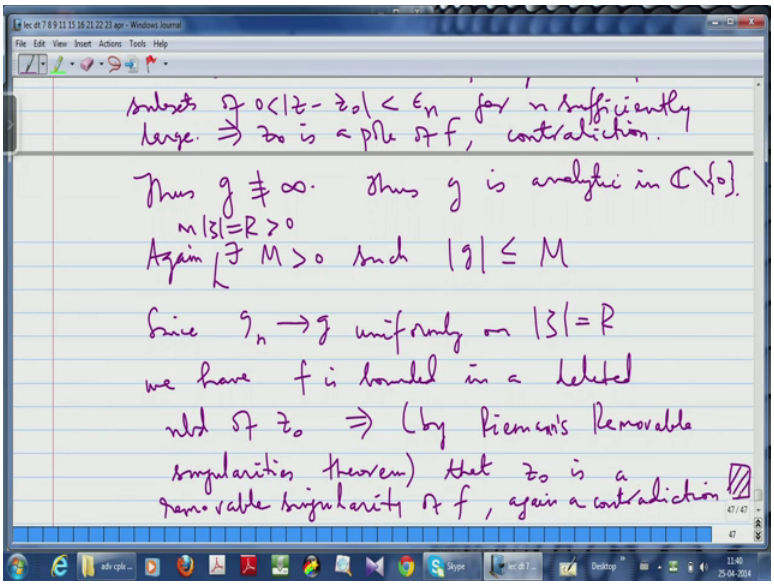

And, but after all, since gn converges to g normally, in fact uniformly on mod zeta equal to r, because, again because it is a compact set we have fn f is bounded in a deleted neighbourhood of z0, this again implies by the riemann's removable singularity theorem that z0 is removable singularity of f, again a contradiction. And you know that finishes the proof. There are only 2 choices for g and both choices lead to contradictions. So that is the of the famous big picard's theorem. And we can, as a corollary we can deduce the little picard's theorem.

What is the little picard's theorem? It tells you that the image of the complex plane under entire function is again the whole plane or the plane minus the point. And now what is the proof, the proof is very simple, taken entire function, of course we should take a nonconstant entire function, okay because if you take a constant entire function, the image of a constant function is always only one point. So you must be careful, i must have been carefully saying that statement. If you take a nonconstant entire function, then you know the image of the complex plane should be either the whole complex plane or complex plane minus a point, it can omit only one value at most.

What is the proof, very simple, take the, take the entire function and look at the point at infinity, okay. The point at infinity becomes the isolated singularity because the complex plane is a deleted neighbourhood of infinity in the external complex plane, okay. So your function f, your entire function f has infinity is an isolated singularity, okay. Now for an isolated singularity what are the possibilities? It can be removable, it can be pole or it can be essential. If it is removable, it means that, if infinity is a removable singularity, it means that f is bounded at infinity and that means that by louisville's theorem, f has to be a constant.

So if we take f to be a nonconstant entire function, okay, infinity cannot be removable singularity, all right. So it can only be a pole or essential singularity. If infinity is a pole, then you have already seen that f has to be polynomial, okay. And you know a polynomial will take all values because of the fundamental theorem of algebra. So if f is a, f has infinity as a pole, it is a polynomial, the image is a complex plane under f is the whole complex plane, using the fundamental theorem of algebra. So the only thing is infinity is an essential singularity.

If infinity is an essential singularity for f, apply the great picard's theorem, okay. F can, any neighbourhood of infinity has to be mapped by f into the whole complex plane or the complex plane minus the point. At the complex plane itself is a deleted neighbourhood of infinity, so f has to map the whole complex plane or the complex plane minus the point, that is it. Okay, so i will just write this down.

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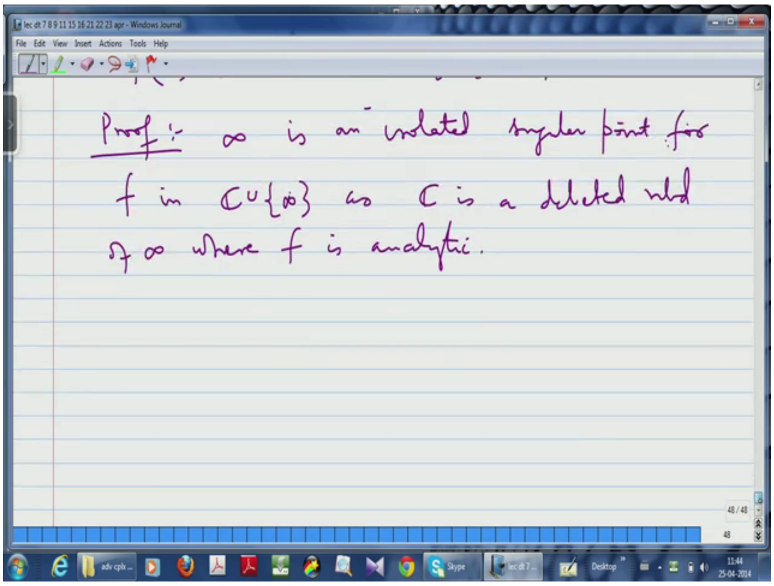

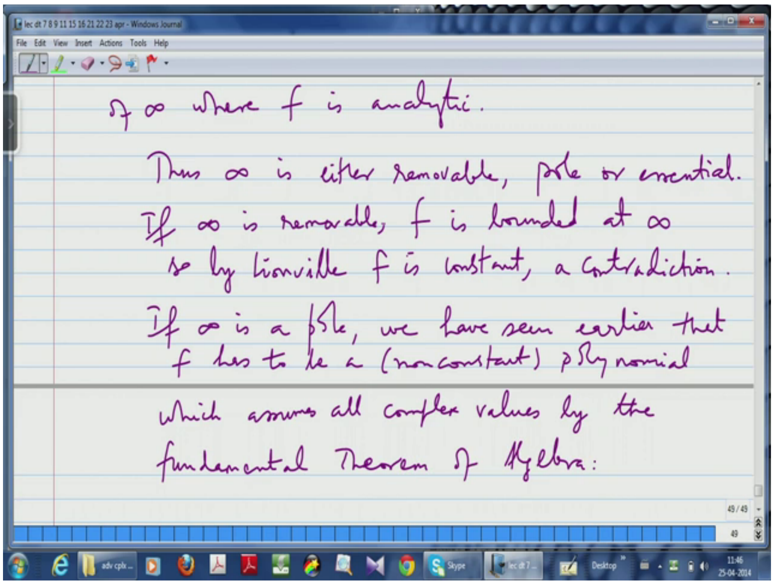

So corollary, picard's little theorem, is f is a nonconstant entire function, then f c is equal to c or c minus z0, okay, for some z0 in c. So proof is infinity is an isolated singular, is an isolated singular point for f in c union infinity as c is a deleted neighbourhood of infinity where f is analytics, okay. Thus infinity is either removable pole or essential. If infinity is it removable, f is bounded at infinity, so by louisville f is constant, a contradiction. Because i am assuming f is a nonconstant fashion, okay, nonconstant entire function. If infinity is a pole, we have seen earlier that f has to be polynomial, be a nonconstant polynomial, which assumes all complex values with the fundamental theorem of algebra, okay.

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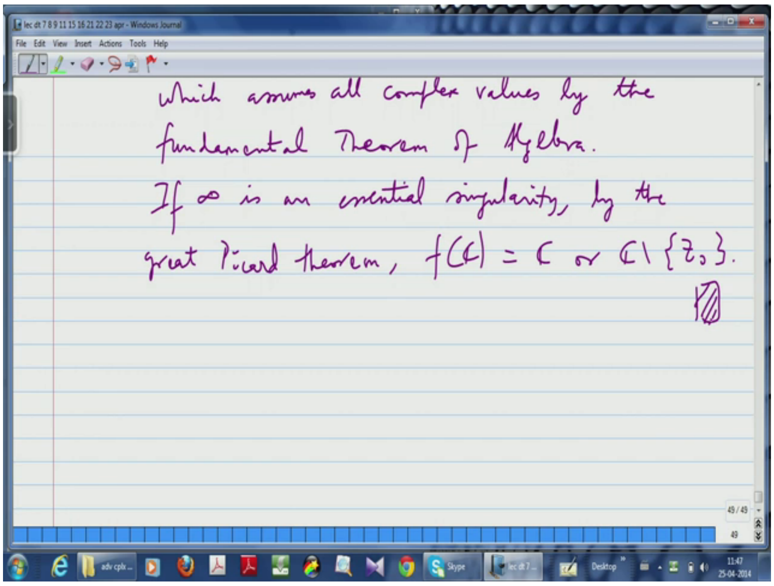

So the only other case is if infinity is a pole, i mean if infinity is an essential singularity, infinity is an essential singularity, plus the big picard's theorem, by the big picard's theorem or the great picard's theorem, f of c is c or c minus point z0, okay. So that finishes the proof of the little picard's theorem. And therefore you see you are able to prove the picard's theorem very easily. And the key to all this is, this really great theorem of montel, it says, this is the criterion for normality of a family, okay.

And it is a very very simple criteria, in the sense that if you know is family functions, if it is a family of meromorphic functions, if you know that it omits 3 values, okay, in the extended plane, then you what is normal. If it is a family of analytics functions, that means it omits 2 complex values, then you know again it is normal. And the advantage of normality is that it is a kind of compactness. Namely it is normal sequential compactness which allows you to extract from any sequence is of sequence which converges normally, that is which converges uniformly at compact subsets, okay.

So that finishes the proof of the picard theorem which was the main aim of this course, okay. What i would like to next do is to tell you that, to tell you how, how powerful zaltzmann's lemma is in several other contexts, okay. Mind you that this reasonably simplified proof of the great picard theorem was possible because of the montel's theorem on common normality, okay. And that in turn was proved by zaltzmann lemma, okay.

So these are all actually all the simplifications are because of zaltzmann lemma, that is the most important thing. But the zaltzmann is, so similarly in the proofs of various other theorems and various other theories of complex functions, zaltzmann lemma provides us with easier proofs of some very deep results and also provides us with new results. And i will try to outline those results in the coming lectures, okay. So i will stop here.