Advanced Complex Analysis-Part 2.
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Lecture-40.
The Fundamental Criterion for normality or Fundamental normality Test Based on Omission of Values.

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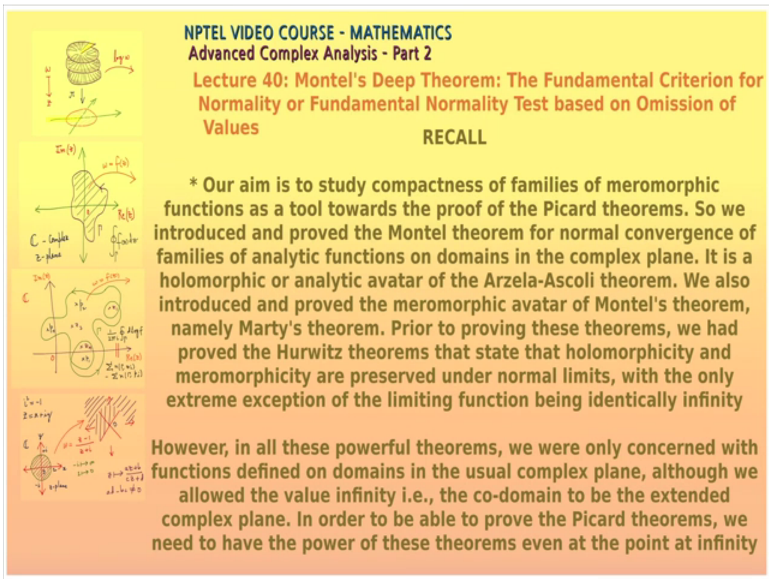
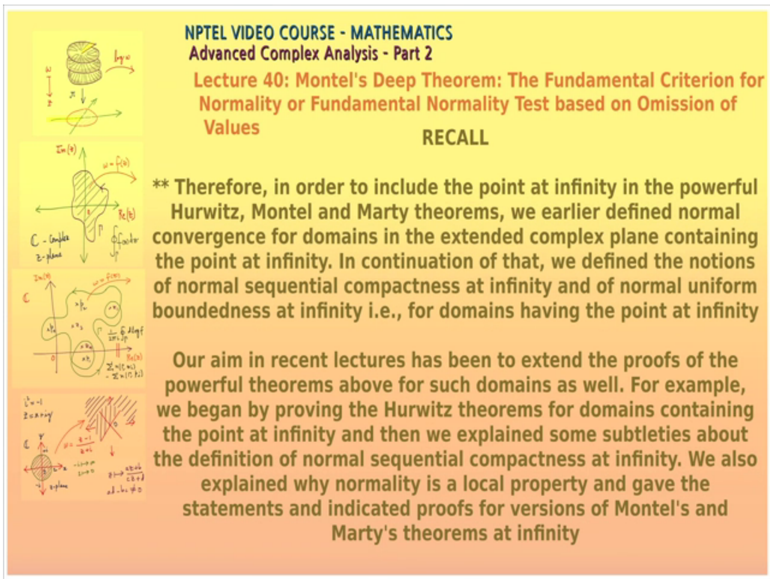
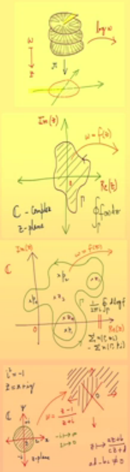






NPTEL VIDEO COURSE - MATHEMATICS

Advanced Complex Analysis - Part 2

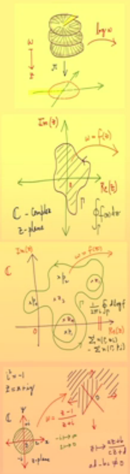
Lecture 40: Montel's Deep Theorem: The Fundamental Criterion for Normality or Fundamental Normality Test based on Omission of Values

RECALL

*** We defined the notion of normality at a point as a local property, and tried to analyze what it means for a family to be normal at a point. In order to understand the local behaviour of a normal family very close to a point, we explained a process called the zooming process which produces functions that are zoomed versions of the functions in the given family. The zooming can be made ultra-zoom i.e., we may increase the magnification of the zooming to infinity. This helps getting the behaviour very close to the point

By applying Marty's theorem to the zoomed family, we were able to arrive at a condition on which we expanded in the lecture before the previous lecture by introducing two further levels of intricacy, namely choosing ever-increasing zooming factors and choosing a convergent sequence of centers for the zooming process converging to the point being analyzed for normality. The condition we arrived at is that we will always end up with a limit function that is constant

The whole point about this discussion is that this condition characterizes normality at a point and its negation is the essence of the important theorem called Zalcman's Lemma which is our key tool to understand non-normality at a point for a given family of meromorphic functions, leading eventually to a proof of the Picard theorems. With this motivation, we proved Zalcman's Lemma and its converse in the previous lecture



NPTEL VIDEO COURSE - MATHEMATICS

Advanced Complex Analysis - Part 2

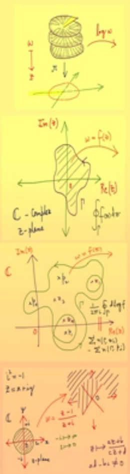
Lecture 40: Montel's Deep Theorem: The Fundamental Criterion for Normality or Fundamental Normality Test based on Omission of Values

GOALS

*** * In the present lecture we prove Montel's Fundamental Normality Criterion, which we may assert as the deepest theorem in the present course of lectures. As we shall see, the Picard Theorems will follow as consequences

The proof involves all the powerful theorems we have proved so far: e.g., those of Hurwitz, Montel, Marty, Zalcman and the Open Mapping Theorem

Montel's beautiful normality criterion is based on omission of values, and says that any family of meromorphic functions that omits three distinct values in the extended complex plane is normal, which implies that any family of analytic functions that omits two distinct complex values is normal as well



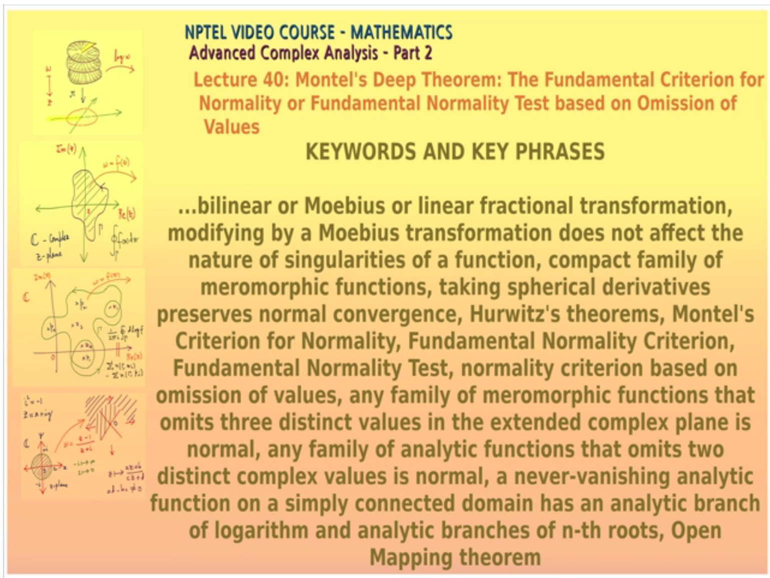
NPTEL VIDEO COURSE - MATHEMATICS

Advanced Complex Analysis - Part 2

Lecture 40: Montel's Deep Theorem: The Fundamental Criterion for Normality or Fundamental Normality Test based on Omission of Values

KEYWORDS AND KEY PHRASES

domains in the extended complex plane containing the point at infinity, normal convergence in a neighborhood of infinity, infinity as an isolated singularity, Riemann's Removable Singularities Theorem for the point at infinity, analytic at infinity same as bounded at infinity same as continuity at infinity, behaviour at infinity is the same as behaviour at zero with respect to the inverted variable, meromorphic at infinity, normal family at infinity, Montel's theorem at infinity, Marty's theorem at infinity, normal sequential compactness or normality of a family is a local property, local analysis of normality of a family at a point, zooming process, zoomed function, motivation for Zalcman's Lemma, translation, scaling or magnification...



Alright, so we proved the zaltzmann lemma in the last lecture and you know basically it is a theorem and it is a characterisation of normal families and I also remarked, at the end of the lecture I remarked that the converse of zaltzmann lemma is also true, therefore zaltzmann lemma is the characterisation of you know non-normal family. And it is like and you notice like Marty's theorem, and Marty's theorem is like a characterisation of normal families of meromorphic functions. And zaltzmann lemma is a characterisation of nonnormal families, okay. And both of them are very important in the, when you go to the proof of the Picard's theorem which is our main aim, okay.

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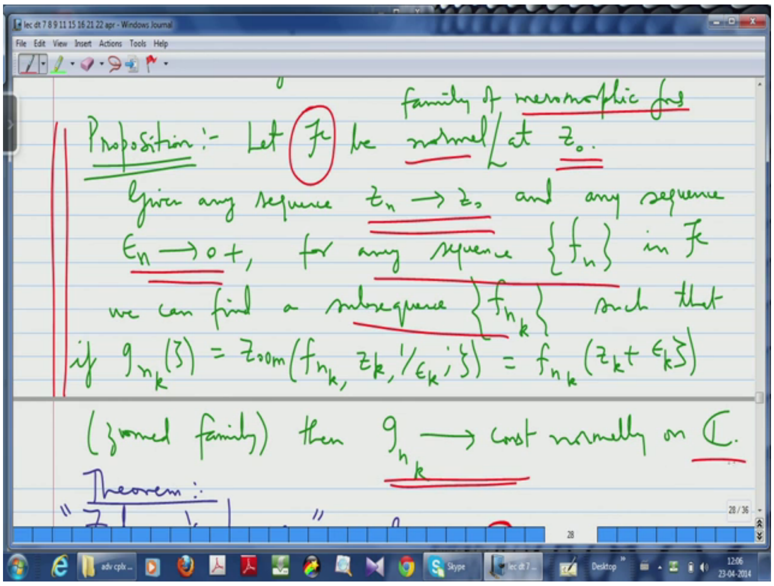

So I just want to begin with the following remark , so recall that, you know when I was giving you the motivation for zaltzmann lemma, we actually proved this result. So if you take the family script F of meromorphic functions defined on a domain in the complex plane or even in the extended plane and you take a point z0 in the domain where the family is normal, okay, then you give, then this normality at a point which is supposed to be normality in some neighbourhood of the point. So normality at the point is defined just like analyticity at a point is defined, okay.

So normality at the point means normality in some open disc surrounding that point, okay. And if you have family which is normal at the point, then it has this property that you know given any sequence of points tending to that point, zn tending to z0 and a sequence of decreasing positive radii, sequence of radii tending to 0, okay, then give me any sequence in the family, I can find a subsequent such that the zoomed functions converge normally to constant function on the plane, okay. This is the, this is what we proved and this was very easy to deduce, okay. And zaltzmann lemma actually tells you that if the family is not normal, you will get the exact opposite.

namely we will be able to find a sequence such that the zoomed functions converge to a nonconstant meromorphic functions, okay, that is the big difference, okay. And what I have to tell you is that converse of this proposition is also true if you apply, it is a simple, it is an exercise which you might for example do, you will have to use diagonalization argument, okay. And you can, you can do this simple exercise and not. Well but not so hard also but you have to, you can use zaltzmann lemma and so that the converse of this proposition is true.

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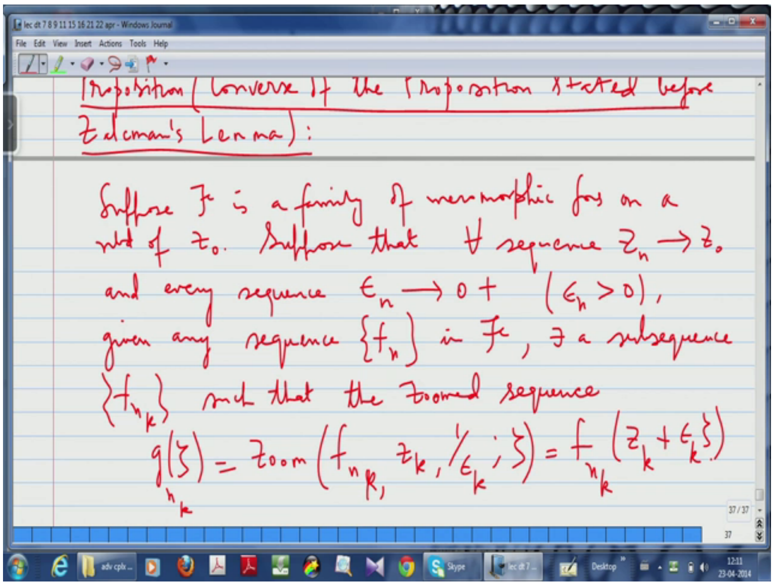

So I will write that down 1st, so there is a theorem, so here is again a proposition. This is converse of the proposition stated before zaltzmann lemma and what is this proposition. Well, it is a criterion for normality , so suppose script F is a family of meromorphic functions on neighbourhood of a point z0, okay. The point z0 could be pointing the extended plane, really, it could even be the point at infinity, it does not matter, okay. Suppose that that for every sequence zn tending to z0 and every sequence Epsilon n tending to 0+, Epsilon n are all positive numbers, okay, so given any sequence fn in the family script F, there exists a sub sequence fnk such that the zoomed , so what is the zoomed sequence, it is gnk of the dice the looming of fnk and zk with the looming factor 1 by Epsilon k and using the variable zeta and this is just fnk of zk plus Epsilon k times zeta, okay.

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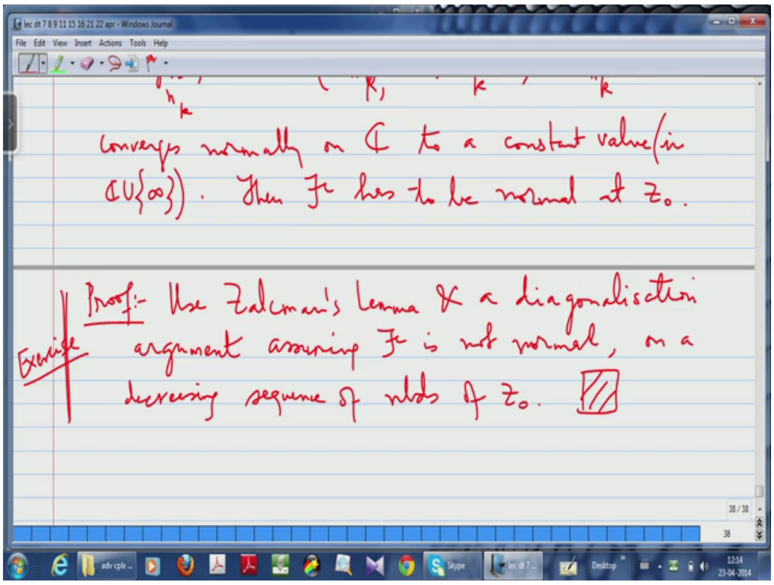

So this zoomed sequence converges normally on the complex plane to a constant value in, so this constant value can also be the point, the value infinity. So it is a constant value in the extended plane, okay. So suppose this, this property satisfied by the family is script F, that whenever you give me a sequence of points converging to z0 and a sequence of radii going to 0, from any sequence fn I can extract a subsequence for the zoomed sequence converges normally to a constant, okay. Then script F has to be normal at z0, okay. Then script F has to be normal at z0.

That means it has to be normal at some open neighbourhood of z0, okay. And I will not write down the detail but I leave it as an exercise to you, proof is, use zaltzmann lemma and diagonalization argument assuming the family script F is not normal on decreasing sequence of neighbourhoods of z0. So I, I will just, this is actually more offered, this is an exercise, okay. This is exercise that I want you to do, so the proof is by contradiction. So you assume, I have to show that family is normal at z0, which means I have to so that the family is normal in some open neighbourhood of z0.

So if that is not true, it means that in every open neighbourhood of z0 the family is not normal. So you take the decreasing sequence of open neighbourhoods of z0, okay, the neighbourhoods becoming smaller and smaller, for example you can take decreasing, decreasing sequence of open disc centred at z0 with radii 1 by n where n goes to infinity, okay. And on each of these discs you can apply zaltzmann lemma because zaltzmann lemma applies to a nonnormal family, okay. And then you will get sequences from the zaltzmann lemma and then you apply the right diagonal, you apply diagonalization argument and then apply the hypothesis of the proposition and you will get the contradiction, okay.

So I leave it to you to do that, right. So, well, so now let me continue with the, the main, the main result. The main result is well, you know what we are going to which is going to be the main one should say this is really the deep theorem that is , that is the theorem of Montel, okay and it is a theorem on normality, okay. And it is a deep theorem because it involves lots of things, it involves several theorems in complex analysis. And it is the key to proving the Picard's theorem, okay. So the point is that , so this is the Montel theorem and what is the Montel theorem?

The Montel theorem, it is a, it is actually a, it is a normality, it is a theorem for normality of a family and what it says is that you take a family of meromorphic functions on a domain, okay. If you know that functions in a family always omit 3 values in the extended plane, 3 distinct values in the extended plane, then the family is normal, okay. So we say to the beautiful theorem, we check that the family is normal, all you make sure is that you find 3 values in the extended plane which means you have to find 2 complex, finite values in the complex plane, the finite complex plane and 1, probably the value infinity, okay.

So you should somehow find 3 values that all the functions in the family miss, okay. And they should omit these values and if you do that then you can, then the theorem says that the family is normal, okay. So somehow omission of values is connected to normality, okay, the normality of the family is connected to omission of certain number of values by functions in the family, okay. And all, and theorem says that you can make sure that the family omits 3 values, then you are sure it is normal, okay. And mind you normality is a condition for compactness, okay.

So you can imagine that the theorem is very powerful, you are saying that a certain family of meromorphic functions on a domain, you are saying that it is compact in the sense that you know it is normally sequentially compact, that is every sequence admits a normal convergence of sequence, okay. That is a very strong property, okay and to and to deduce that all you need to see, you have to just verify that the family omits 3 values, okay, 3 values and it is really beautiful. So in particular what this means is that you know if you take a family of analytics functions on a domain, okay, if you take a family of analytics functions on a domain, okay.

And if it, and if you know that it omits 2 finite complex values, then you can immediately say that it is normal, okay. Because when you are looking at analytics functions, you can forget the 3rd value which is infinity, okay. Infinity is already omitted, okay. So this, this version of the theorem, Montel theorem is called the fundamental normality tests, okay. So if you want to check the family of analytics functions is normal on a domain, all you have to do is, you make sure that it omits, you make sure that every member of the family omits 2 fixed complex values, then you are sure that it is normal.

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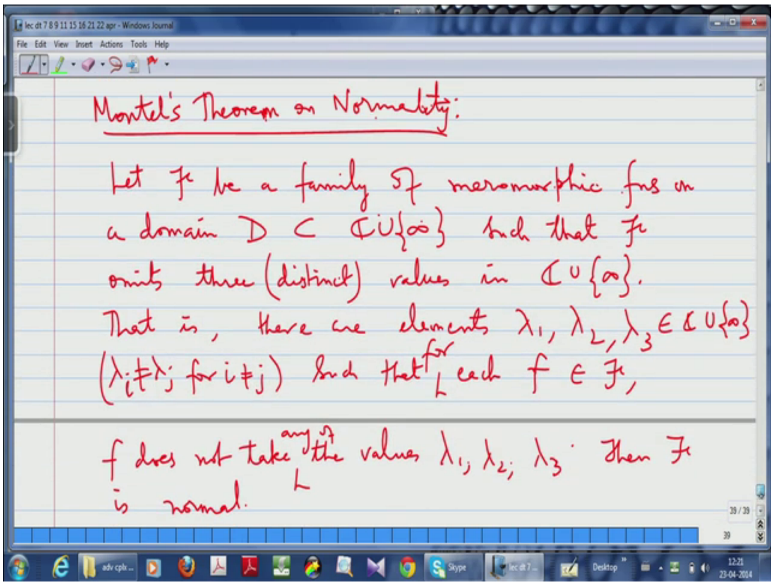

So this is a very deep theorem, so let me state it. So this is Montel's theorem on normality, so here is the theorem. And let script F be a family of meromorphic functions, functions on a domain D in the external complex plane such that script F omits 3 distinct, 3 distinct values C union infinity. So what does this mean? That is, there are elements lambda 1, lambda 2, lambda 3 in C union infinity and of course they are all distinct, lambda I not equal to lambda J for I not equal to J, okay, such that such that each such that for each f in the family, f does not take the values any of the values lambda 1, lambda 2, lambda 3, okay, these are, these are permitted values, then script F is normal.

So this is the, this is the, this is the deepest, I would say this is the deepest theorem in this course, okay, this is the most important theorem in this course, okay. To check that the family of meromorphic functions on a domain is normal, you just ensure that it omits 3 values. All the functions in the family format 3 fixed values, okay, 3 distinct fixed values. Alright, so here is, so how do we go about the proof?

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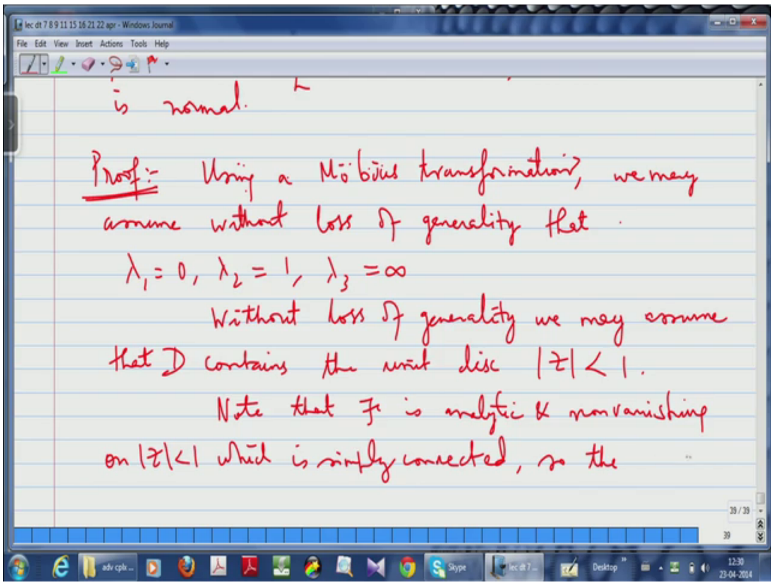

So, so the 1st thing I want to tell you is that you know, you know that fundamental property of Mobius transformations that, that you know given any 3 values, any 3 values in the extended plane, you can find the Mobius transformation that can map those 3 values to 0, 1 and infinity, okay. So you can always, I mean this is the way in which you write down the Mobius transformations in terms of cross ratios because a Mobius transformation has a fundamental property that preserves cross ratios. So it is something that you should have seen in our 1st course in complex analysis.

So you know these 3 values lambda 1 lambda 2 and lambda 3 in the extended plane, you know I can apply a Mobius transformation and make those, map those values to 0, 1 and infinity, okay and then I can compose the whole family by this, I can transform the family using this Mobius transformation. I transform the whole family by using this Mobius transformation and therefore without loss of generality, by using a Mobius transformation I can assume that the values that are omitted, 3 distinct values that are omitted are 0, 1 and infinity, okay.

So this is the 1st reduction, okay, so let me write this down. Using a Mobius a transformation, we may assume without loss of generality that lambda 1 is 0, lambda 2 is 1, lambda 3 is equal to infinity, okay, you can do this. Right, so you can assume and of course by Mobius transformation I mean bilinear transformation or linear fractional transformation, okay. So that is the 1st thing. Then the 2nd thing is that, you know the domain on which, the domain D in the extended plane where this family is defined, okay, that domain also can be, you can change that domain and scale it so that you know it is, it contains the unit disc, okay.

So the point is that, I am, I am trying, what am I trying to check, I am trying to check that this family is normal, okay, final aim is to check the family is normal. But how do I check it is normal? I check it is normal by checking it is normal at every point because normal, normality at the point means normality in open, small open discs containing that point and the property of the normally the local property, so if you check it at every point, that is if you check it in open neighbourhood of every point, that is enough to check it it is normal on the whole domain, okay.

So it is like checking analytic disc, you do not have to check, if you want to check a function analytic on whole domain, it is enough to check at every point of the domain it is analytic. So what I have to do is I will have to, I can assume that I checking normality of the family on a small disc, on a domain which is like a disc, okay. And of course, you know if I am checking at the point and infinity, all right, then I will have to take a neighbourhood of infinity which is exterior of disc in the complex plane and I will have to change the variable from z to 1 by z to make it into a disc surrounding the origin.

So in any case I can come I can always assume that I am checking normality on a disc in the complex plane and I can translate that disc to the origin and scale it so that it contains the, the disc contains the unit disc, okay. So again this is another reduction I am making, without loss of generality I will, it is enough for me to check that the family is defined on the unit disc, okay. So this is another reduction, okay. Without loss of generality we may assume that D contains the unit disc mod z less than 1, because basically because you have to check normality locally, that means you have to check normality on a disc surrounding every point.

And that this I can assume it to be unit disc, okay. Because I can always translate any small disc to the origin, okay, so that the Centre of the small disc goes to large area and then I can scale it so that it is bigger of, so that it contains the unit disc, okay. And our solutions and scaling is also Mobius transformations, so they are not going to modify the properties of the family, okay. So I, so my situation is like this, I now have a family of meromorphic functions, okay, I now have a family of meromorphic functions defined on the unit disc and what is given to me is that they omit the values 0, 1 and infinity.

I have to check that the family is normal, okay but look at the beauty of it, since the functions with the value infinity, there analytics, okay. Because you know a meromorphic function takes the value infinity only at the pole. And the moment you assume that it does not take the value infinity, all the functions have become a reality, all right. And the other thing is that all the functions are non-vanishing also because the value 0 is omitted, okay. So you are having non-vanishing analytic function on the unit disc, okay.

And they all omit the, and what is the, what is the nice thing, the nice thing now is that means if you have a non-vanishing analytic function on a simply connected domain, you can always find kth roots, okay, which is analytics. Because the reason is if you have a non-vanishing analytic function on a simply connected domain, you can find a logarithm for the function. And once you find the logarithm, multiplying that analytics logarithm by 1 by k and then taking exponential, okay, so e power 1 by k log will give you kth root of the function which is analytic.

So the advantage now is that your family has kth roots, every function in your family has kth roots for k, okay. And the trick is what, the point is that since it, since the, if a function does not take the value 1, okay, then its kth root cannot take the value which is equal to the kth of unity. If the original function does not take the value 1, then it is kth root cannot, the analytics kth root cannot take the kth root of unity as the value, okay. And what ks we will be using, we will be using 2 power ks, okay. So I am going to write that down.

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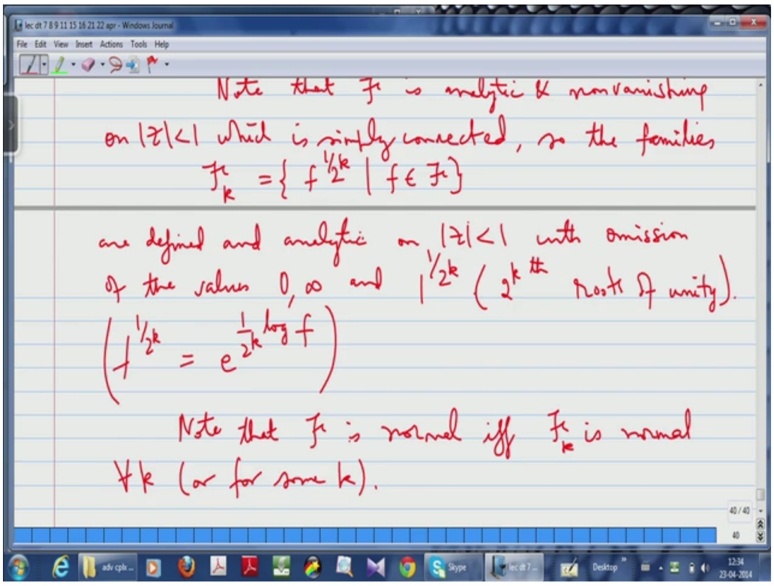

note that script F is analytic and non-vanishing on mod z less than 1 which is simply connected. So the, so the families, so I will put this F sub k, this is f to the 1 by 2 to the k where f is in script F are defined and analytic on the unit disc with omission of the values 0, infinity and 1 by 1 by 2 to the k which is 2 power kth roots of unity, okay. So, so I am, so you see this, mind you the whole point is f to the 1 by 2 power k is defined as e to the 1 by 2 to the k log f. And this log f, an analytic branch of log f exists because the domain is, the unit sphere simply connected and f never vanishes on the domain.

So this is something that is very very important, okay. So fine and you know, now I want you to understand what the idea is. See the idea is you know, these, so you look at these functions, okay, these functions are defined on the unit disc, okay. But then you know, what are trying, I am trying to prove they are all normal, I am trying to prove that the family script F is normal on the unit disc, okay. And mind you, that means that I can extract the normalities, just that I can extract from any sequence the normally convergent subsequence, okay.

But you see if I can extract such a, from a sequence in normally convergent subsequence, I do that also for the 2 power kth roots, okay. So it is obvious that you know the family script F is normal if and only is any of the family's script F sub k is normal for any k greater than 1, okay. So actually it amounts to show, to showing that F is normal, it is enough to show that one of the script F k is normal, okay. And therefore if you contradict the normality of script F, what happens is you are contradicting in one stroke the normalities of each of the script F sub k, okay.

And once you contradict the normality of each of the script F sub ks, zaltzmann lemma comes into the picture and gives you zoomed limit functions which is nonconstant meromorphic function on C which spherical derivative equal to1 at the origin and the spherical derivative is always bounded by 1, okay. And the beautiful thing is that, the function that you get is an entire function, okay. Because the limit of the functions from each of these families, it will not take the value infinity, so it will be analytic, and it will be defined on the whole complex plane, so it will be entire, okay.

And you will see that you can get a contradiction easily by applying Louisville's theorem, okay. So, so let me write this down. note that F is normal, normal if and only if, if and only if Fk is normal for every k or for some k, okay. So, so we are going to proceed by contradiction. What we will do is assume script F is not normal, okay because I want, we want to use the zaltzmann lemma which is the characterisation of non-normality, all right.

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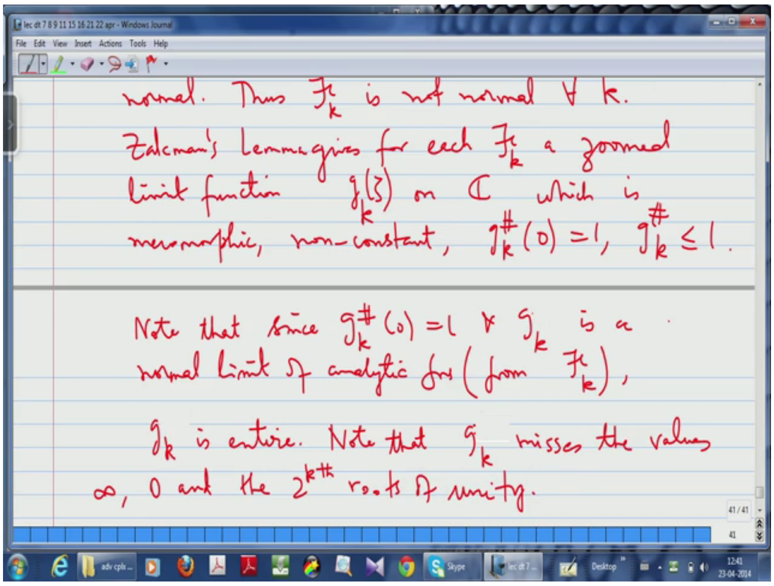

So assume F is not normal, thus F sub k is not normal for every k, all right. And zaltzmann lemma gives for each script Fk is zoomed limit function, I will call this resume the limit function as g sub k of zeta, okay, on the whole complex plane which is, which is meromorphic, nonconstant, it is spherical derivative at the origin is 1 and all it is spherical derivatives are bounded by 1, okay. This is what zaltzmann lemma tells you, zaltzmann lemma tells you that whenever a family is not normal, I get a zoomed limit function which is nonconstant meromorphic.

And nonconstantcy is a kind of normalised or fixed by making the spherical derivative to be 1 at the origin. And my new the limit function is defined as a function on the whole plane, okay, thus doomed element function is on the whole plane. So I have this, okay. now what I want you to notice is that the 1st thing I want to tell you is that this zoomed limit function, what is, what is each source to limit function, it is a normal limit of meromorphic functions, okay. But it is a normal limits of functions from Fk, script F sub k, but mind you script F sub k are all analytic, okay.

See because we cleverly assumed one of the omitted values is infinity and therefore we are only working with analytic functions, okay. Therefore these limit functions, gks, they also have to omit the value infinity. The only other possibility is that they can be identically infinity because you know whenever you have a normal limit of analytic functions, okay, then either the normal limit is again, the limit function is again analytic or it is identically infinity, this is the only thing that is possible. So the only thing that would have happened is that these limits functions are all identically infinity, some of the limit functions, zoomed limit functions gks, they would have been identically infinity.

But even that cannot happen because if they were identically infinity, the spherical derivative would have been 0. What I have put the condition that the zaltzmann lemma tells you that the spherical derivative at the origin is 1, they are nonconstant. So what it means is that all these gks are all entire, they are all entire functions, you cooked up a family of entire functions, you have cooked up a sequence of entire functions, okay. So let me write that down, note that since the k hash of 0 is 1 and gk is a normal limit of analytic functions from the family Fk, gk is entire, it is entire.

Because it is analytic and it is defined on the whole plane, so it is entire. And the point is that it does not take the value infinity, cannot take the value infinity. See in principle it would have been a meromorphic function, it could have taken the value infinity at a poll but it can never take the value, that the only way, the only possibility is that because it is a normal limit not of just meromorphic function but it is a normal limit of analytic functions, okay, the limit can only is there be completely analytic or it can be completely identically infinity. You cannot get from limit of normal limit of analytic functions you cannot cook up a meromorphic function, this will not happen.

That is basically because of Hurwitz theorem, because if you cook up a meromorphic function, it means that your pole is popping up in the limit but it is a poll pops up, then for the reciprocal function a zero pops up. Hurwitz theorem says that zero of the limit will come from the zero of the original functions beyond a certain stage. That means the original functions, the reciprocal of the original functions, if the, if the limit takes, if the if the limit takes value 0 or if limit takes the value infinity, okay, then the reciprocal of the value will take the value 0 and the reciprocal of the original functions beyond a certain stage should have zeros, which means that beyond a certain stage the original functions should be meromorphic but they are all analytic.

Okay, so basically did Hurwitz theorem which is working behind, we had all this. So therefore each of these functions is entire and the beautiful thing is what are the values that they miss, see these, this gk will miss the value 0, infinity of cost and all the 2kth roots of unity. Because every function in Fk script Fk is supposed by construction it misses all the 2kth roots of infinity, okay. So let me write that down, note that gk misses the value 0 and infinity, 0 and the symbol 2kth roots of unity, okay. And actually what I am using here is actually Hurwitz theorem, use Hurwitz theorem.

So again let me repeat that, what does Hurwitz theorem say? You take a sequence of analytic functions, okay, suppose it converges normal to limit function, okay, a normal limit of analytic function is always analytic, okay. Or it can be identically infinity but this being identically infinity is anyways out of the picture because all the spherical derivatives are all nonzero, okay. So the limit is always an analytic function, so if you have a sequence of analytic functions that can watch normally do an analysis function, then the limit function if it has a 0, then the 0 must come by, limit of zeros of the original functions that are converging beyond a certain stage, that is Hurwitz theorem. Okay.

In other words what are you saying, what is it saying, it is saying that if the limit function takes the value 0, then the original function should also take the value 0 beyond a certain stage in a neighbourhood of the zero of the limit function, okay. And this is not only true for the value 0, it is true for any value, because they, the F, for F to take, for F of z to take the value lambda, it is the same as looking at a zero of F of z minus lambda, which is also analytic, okay. So actually what you can say is Hurwitz theorem can also be thought of as, suppose you have a sequence of analytic functions, suppose it is converging to a nonconstant analytic function, okay, then if the limit function takes any complex value, then all the original functions also should take that complex value beyond a certain stage, that is what it says.

And in fact, Hurwitz theorem says, more in fact it says that the with the multiplicity should coincide. The multiplicity, if the limit that function takes a value with a certain multiplicity, then all the functions in the original sequence that converge to that limit function, they also should take the same value with the same multiplicity beyond a certain stage, okay, I am a neighbourhood of the point where that value is assumed to be. Okay. So this is just Hurwitz theorem, okay, mind you spherical derivative is non-negative real valued function, okay. The limit functions gk, these are all, these are the ones that are entire. And they miss the values infinity, 0 and the 2kth roots of unity, okay.

And of course I will have to make use of these 2 conditions here that the, they all have the spherical derivative, 1 and 0 and they all have the medical derivative bounded by 1. So note that all these gks having spherical derivatives bounded by 1, what does it tell you? You can apply Marty's theorem now, you look at the sequence of gks, this is a sequence of entire functions on the plane, okay. gk is the sequence of entire functions on the plane, their spherical derivatives are all bounded, therefore my Marty's theorem, that is a subsequence which will converge normally on the plane, okay. So now I am applying Marty's theorem, okay.

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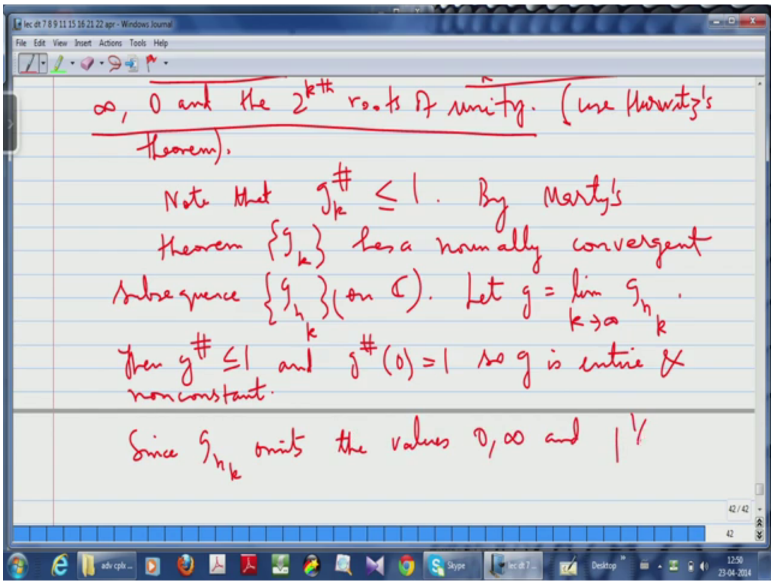

By Marty's theorem this gk has convergence of sequence, normally convergence of sequence, gnk or of course it is normally converted on the whole plane. Okay, because gk is of course it is a family of entire functions, okay. Let us take, so take such a normally convergence of sequence and take the limit, take the limit function. You will get, you will get a function g, all right. now that function g, C that function g is now a normal limit of entire functions, okay, therefore the only possibility is that it is also entire, okay. Or it is identically infinity but it cannot be identically infinity because of the spherical derivative being 1 at 0.

So the limit function is also going to be an entire functions, okay. And you will see, that is a function for which I will apply Louisville's theorem and get constancy, which is a contradiction, okay. So let, so let g be the limit as k tends to infinity of gnk, okay, then g is entire, then of course you know g Hash is also bounded by 1 and g Hash at the origin is one, so g is entire nonconstant and nonconstant, okay. So you cooked up an entire nonconstant function, okay. And see now comes the, come something very nice. See each of the gMks, the values that they omit are 0, infinity and the 2kth 2 to the nkth roots of unity, okay.

But you know as k becomes large, this 2 to the nkth roots of unity, they, so take the union of all these, that is a dense subset of the unit circle, okay. So what it means is that this g, this function g, it omits a dense subset of values of the unit circle, therefore it has to omit all values from the unit circle, that is because of the open mapping theorem. What is the open mapping theorem say? Whenever a function takes a value, it has to take all values in a small disc surrounding that, the image of every open set is an open set, for a for a nonconstant analytic function, the image of an open set is always an open set.

So g being an entire nonconstant entire function, if g omits values on a dense subset of unit circle, by the open mapping theorem g has to omit all the values of the unit circle, okay. But the unit circle disconnects the plane into 2 pieces, one is interior and their that is exterior. And therefore the image of the complex plane under g which has to be connected has to either go completely inside the unit disc or it has to go completely outside the unit disc. If it goes inside the unit disc, then you have found g is bounded entire function, so it is a constant, that is the contradiction.

If it goes completely outside the unit disc, you take 1 by g, which is also going to be entire because mind you g also omits the value 0, so 1 by g is also entire. So 1 by g will become a boundary entire function, it will become constant, therefore g will become constant. So in any case you will get g is a constant, you will get a contradiction, okay and that proves that theorem, that is all. So what you must appreciate is that if you use open mapping theorem, you use Hurwitz theorem, you have use zaltzmann lemma, you have used Marty's theorem, okay, everything is being used, okay.

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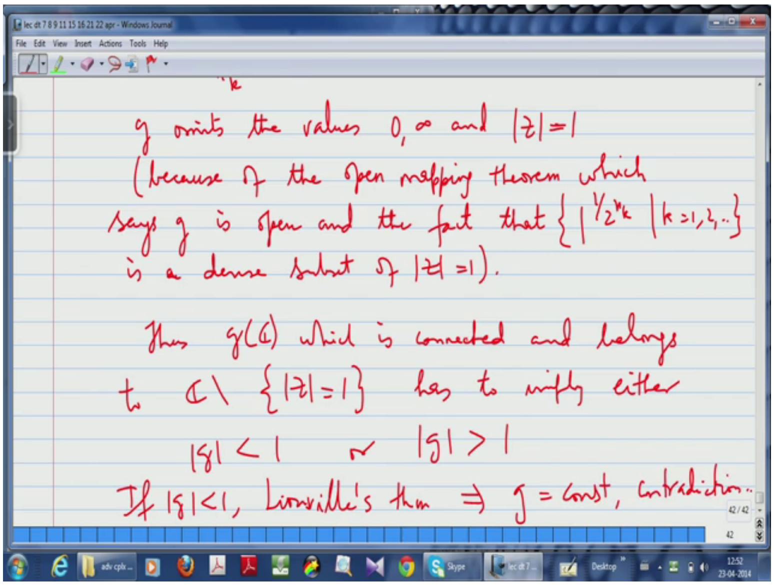

So let me write this down. Since gk, g to the nk omits the values 0, infinity and 1 by 1 by 2 to the nk, g omits 0, infinity and mod z equal to1 because of the open mapping theorem which says g is open and the fact that 1 by, the 2 power nkth roots of unity k equal to 1, 2, and so on is a dense subset of mod z equal to1, okay. Thus, g of C which is connected and belongs to complex plane minus mod z equal to1 has to imply either mod g is less than 1 or mod g is greater than 1, okay. This mod g is less than 1, Louisville's theorem implies g is equal to constant, contradiction, it is not constant because the spherical derivative at the origin is 1, okay.

And if g is, if mod g is greater than 1, then 1 by g is entire and mod 1 by g is less than 1, so again Louisville's theorem implies 1 by g is constant, which implies g is constant, again contradiction, okay. So that is it. So the family has to be normal, right. So that finishes the proof of this theorem. And as a corollary you can see that if you have a family of analytic functions on a domain, is, if you know that the family omits 2 complex values, 2 finite complex values, then it has to be normal, that is the corollary of this. This theorem that we have proved is for meromorphic functions and you are including the value infinity also, okay.

So sometimes that, that is called as, this, these conditions of omitting 3 values being omitted for family of meromorphic functions or 2 values being omitted for a family of analytic functions is called the fundamental normality criterion. So it is a condition, very simple condition to check whether a given family is normal, okay. I will stop here.