Advanced Complex Analysis - Part 2: Compactness of Meromorphic Functions in the Spherical Metric, Spherical Derivative, Normality, Theorems of Marty -Zalcman-Montel-Picard-Royden-Schottky

Dr. Thiruvalloor Eesamaipaadi Venkata Balaji
Department of Mathematics
Indian Institute of Technology Madras

Lecture No 4

Casorati Weierstrass Theorem; Dealing with the point at Infinity: Riemann Sphere and Stereographic Projection

Okay so let us continue with what we are trying to do, so you know we are trying to prove the final aims to prove the Picard theorems okay and the little Picard theorem will be deduced as corollary of the great Picard theorem okay and but the great Picard… The little Picard theorem is about the image of an entire function, the great Picard theorem is about the image of a deleted neighbourhood of an isolated essential singularity okay and what we are going to now 1st prove is the easiest thing to prove to begin with to get you a feel of things is the Casorati Weierstrass theorem which says that if you take any neighbourhood of an isolated essential singularity the image of that neighbourhood of course with the singularity omitted of course the image of that neighbourhood will always be a dense subset of the complex plane okay namely its closer will be the whole complex plane okay.

What this means is that it at any complex number you can always find a sequence of points in any neighbourhood of an isolated essential singularity such that the function values at those points approaches that complex number okay, so let me write that down let us prove it, the key to proving the Casorati Weierstrass theorem is the Riemann removable singularity theorem which we proved last time okay, so let me write that down.

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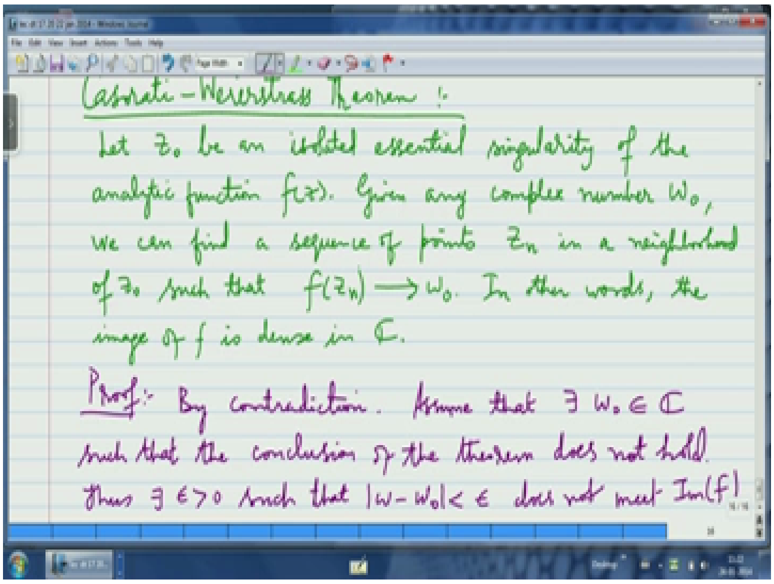

So Casorati Weierstrass theorem, so let Z naught be an isolated essential singularity of the analytic function f of Z given any complex number W naught we can find a sequence of points in a neighbourhood of Z naught such that if you take the function values at those points fZn that tends to W naught okay, so this is the this is the Casorati Weierstrass theorem okay. You take an isolated essential singularity Z naught of an analytic function f of Z and then you can always find a sequence of points in the neighbourhood of Z naught such as the function value at those points tends to the limit W naught okay so in this way, so what you are what does this mean, it means see W naught can be…f of Z n tends to W naught means that you can no matter how close you go to W naught confined function values okay.

So that means that the W naught is in the closure of the set of function values that what it means, so W naught is the limit point of the image set of f, image set of f is just the set of values of f okay and what this says is that every complex number is in the closure of the image set, the set of values that f takes okay and this is this very deep theorem because what it says is that it tells you that therefore the if you take the image of a deleted neighbourhood of the isolated essential singularity then the image is going to be huge a set which is tenses huge okay, the image is going to be dense in the complex plane.

So that means the image closure is the whole complex plane so it is the image is huge and this is the this is kind of much weaker when compared to the (())(5:57) the great Picard theorem which says that the images actually the whole complex plane or at most a punctured plane namely it can at most omit a point okay but you must remember that that omitted point is also in the closure because it can be approached by the points okay, so this Casorati

Weierstrass theorem is weaker version of you know the great Picard theorem but it tells you it answers this question that we have been worried about namely what is the image of an analytic function okay, so let us prove this, the prove of this is going to involve it is just going to involve Riemann removable singularity theorem, the use of Riemann removable singularity theorem, so let me write this also in other words, the image of f is dense in the complex plane, this is another way of saying it okay.

So let us go on to the the proof of the theorem, so you know so you know the idea of the proof is very simple, what you do in the proof is that you prove by contradiction okay, so you assume that there is a complex value which is not in the closure of the image that means there is a complex value which is not in the limit of values in the image okay that means that the image is bounded away from a certain complex value okay you assume that and then you show that if this is the case then your function cannot have an isolated essential… Essential singularity at Z naught but the singularity has to either be removable or it has to be a pole and this is where we will be using Riemann removable singularity theorem, so this is the technique of the proof.

The taking of the proof is just by contradiction okay. You assume that a certain value is not in the limit okay and then you sure that this will imply that either Z naught is a removable singularity or it is a pole and both of these are not possible they will give contradiction because we have assumed that Z naught is an essential singularity which by definition is something that neither removable singularity nor a pole okay so that is how the proof works, so let me write down the proof, so prove so let me go to a different color uhh. Proof by contradiction, assume that there existed W naught complex value such that such that the conclusion of the theorem does not hold so you assume.

Conclusion of the theorem is that given any complex value W naught okay you can find the sequence of point Z n such that the function values at Z n approaches W naught, now you assume that that is not the case assumed that there is at least one W naught or which this does not happen okay and we will try to get a contradiction, so what does this mean? What it means is that it means that W naught cannot be approached by image points okay the another way of saying this is that you are trying to say that there is a neighbourhood of W naught okay which has nothing to do with the image okay which does not intersect the image.

So that means there is an Epsilon greater than 0 such that this open disk centred at W naught radius Epsilon is disjoint from the image of f okay that is the that is what it means okay so let me write that down so thus that exists Epsilon greater than 0 such that mod W minus W, W naught less than Epsilon does not meet the image of f, image of f Im f is just the set of values of f and set of values of f includes all values at f takes in a also in the deleted neighbourhood of the isolated singularity Z naught which we have assumed is essential, okay so what does this mean, so what do I do now?

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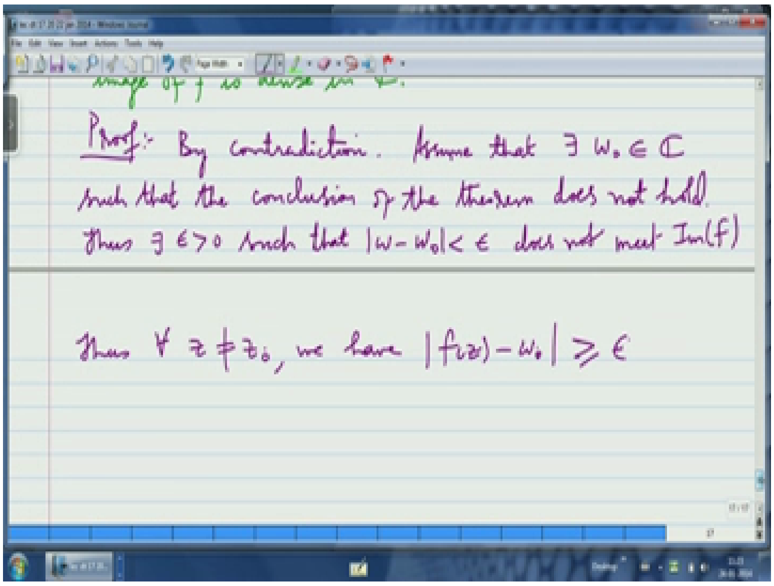

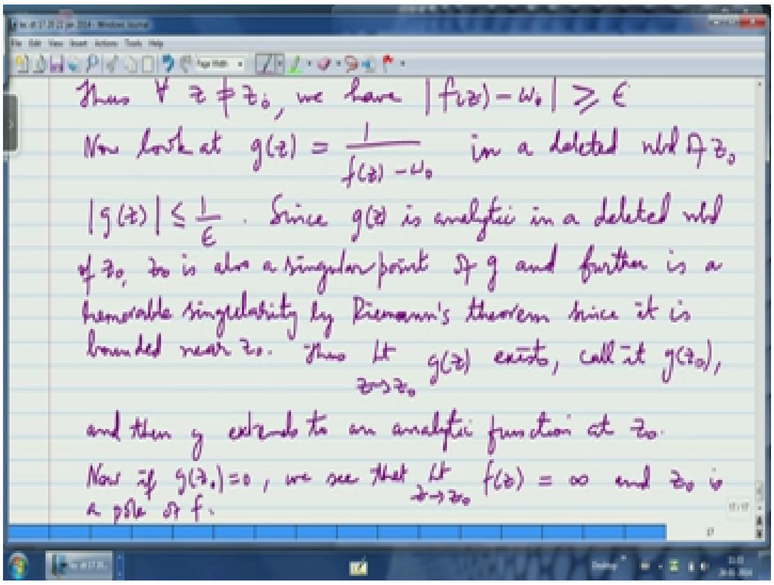

Now you see so let us rewrite that thus for all Z not equal to Z naught of course okay, we have mod f of Z minus W naught is greater than equal to Epsilon this is what it means okay. So let me again tell you what I wrote down before what I wrote down before is that the disk centred at W naught radius Epsilon does not meet the image okay, it means that there is no point in the image whose distance from W naught is less than Epsilon, so it means that if you take any point in the image the distance from that point to W naught is at least Epsilon greater than Epsilon that is what I have written down, so when you write mod f when I write mod f Z minus W naught is greater than not equal to Epsilon I am actually saying that the distance between f of Z which is the point to the image of f the value of f at Z and W naught is at least Epsilon.

Now so you know the advantage with having this kind of thing is that see whenever you have a function which is bounded away from 0 okay, what is the advantage of having a function on bounded away from 0? The advantage of having a function bounded away from 0 is that you can invert it okay it is reciprocal make sense okay, so here f Z minus W naught mind you is also function f of Z is a function, f Z minus W naught is just a function f of Z added to the constant minus W naught and adding this does not change the analyticity of the function except at the point Z naught where which of course we are not going to worry about okay.

Mind you when I write f of Z I am of course assuming Z is in the domain of analysis the of f it is implicit and of course Z is not Z naught because at Z naught is not a point where f is analytic to begin with I have assume that Z naught is an essential singularity it is a singular point okay, so the point is that since I have mod f minus W naught is greater than Epsilon 1 by f minus W naught makes sense as function okay, so and what it tells you is that that function in a deleted neighbourhood of Z naught is bounded by 1 by Epsilon okay that is what it says, so now look at g of Z defined to be 1 by f of Z minus W naught okay in in a deleted neighbourhood of Z naught. Now look at this function, now you see this function mod g Z is greater than or equal to is less than or equal to 1 by Epsilon you have that that is just inverting mod f Z minus W naught greater than equal to Epsilon.

So what you have is now you have 2 things see g of Z makes sense as an analytic function okay it is because it is a reciprocal of an analytic function and it is defined where the denominator does not vanish, so f Z minus W naught never vanishes because if it vanishes then it is modulus will be 0 and vice a versa but the modulus is always bounded away from 0 it is always greater than equal to Epsilon, so f Z minus W naught never vanishes okay and mind you wherever f is analytic, f Z minus W naught is also analytic okay because it is just f of Z added to the constant minus W naught adding a constant does not change the analyticity of a function because or constant function is also analytic and sum of analytic functions is analytic okay, so f Z minus W naught is also analytic in a deleted neighbourhood of Z naught okay and it is non-zero.

So it is reciprocal 1 by f Z minus W naught is as well analytic in a deleted neighbourhood of Z naught okay, so what I have now is I have this function g Z it is analytic in the deleted neighbourhood of Z naught okay Z naught is a singular point but look at the last inequality, this function is bounded in a neighbourhood of Z naught. Now Riemann removable singularity theorem will tell you that Z naught has to be a removable singularity for g okay, so since so let me write that down, since g of Z is analytic in a deleted neighbourhood of Z naught, Z naught is also a singular point of g of g and further is a removable singularity by Riemann's theorem on removable singularities since it is bounded okay, so here is where we are using the Riemann's removable singularity theorem okay.

So what it means is that so it means that g can be redefined at Z naught so that you get function which is analytic at Z naught as well okay and the fact that you can define g redefine g at Z naught okay should tell you that f has to have at Z naught either a removable singularity or a pole, it cannot have an essential singularity that is the conclusion of all this and that is the contradiction to the hypothesis that Z naught is actually an essential singularity and that is how we get the proof of the theorem of the Casorati Weierstrass theorem okay, so let me write that down. Thus limit Z tends to Z naught g of Z exist call it g of Z naught and then g extends to an analytic function at Z naught okay, so this is what removable singularity means you can take the you can take the so this is probably the right time for me to recall the Riemann removable singularity theorem, what does it say?

It gives you an equivalence of 4 statements, the first statement is that the point in concern the isolated singularity concern is removable okay which is equivalent to saying by definition that the function can be extended to an analytic function at that point. The 2^{nd} condition is slightly weaker that you can extend the function continuously to that point okay namely the condition is that the limit of the function as you approach that point exist okay as a finite complex number and then the 3^{rd} condition was the condition equivalent condition that involves the Laurent expansion and that condition was that if you write out the Laurent expansion about removable singularity you actually get Taylor expansion namely there are no negative powers there is no principle part, there is no singular apart okay.

And then the 4^{th} condition which was the most amazing was the condition that the function is bounded in a neighbourhood of the singularity okay bounded of course bounded means bounded in modulus okay and I told you that that is the weakest of all the 4 conditions and that boundedness in the neighbourhood of a singularity can only happen if the singularity is a removable singularity that is essentially Riemann's removable singularity theorem.

So you know so because of that the limit Z tends to Z naught g of Z exist let us call it as g of Z naught so g extends to analytic function at Z naught, so here I am using several equivalent version of the Riemann removable singularity theorem you must realize that. Now the question is it all depends on what is the value of g of Z naught is, the point is that if g of Z naught is 0 okay then f has a pole at Z naught okay and if g of Z naught is not 0 then f has a removable singularity at Z naught okay and then thus we have manifest a contradiction to what we have assumed okay so let me write that down.

Now if g of Z naught is 0 we see that limit Z tends to Z naught f of Z has to be infinity this has to happen because you know the limit as I said tends to Z naught g of Z is 0 so the limit is Z tends to Z naught 1 by f Z minus W naught to 0 that means the denominator has to become unbounded, so that means and if f Z minus W naught has to become unbounded in modulus then f has… because W naught is just a constant f has to become unbounded in modulus and that is a condition for a pole okay, so if g of Z naught is 0 then limit Z tends to Z naught f of Z is infinity and Z naught is a pole, is a pole of f and you see Z naught is a pole of f which is not possible so that is ruled out so you have ruled out the case that g of Z naught is 0 the only other case is when g of Z naught is not zero okay.

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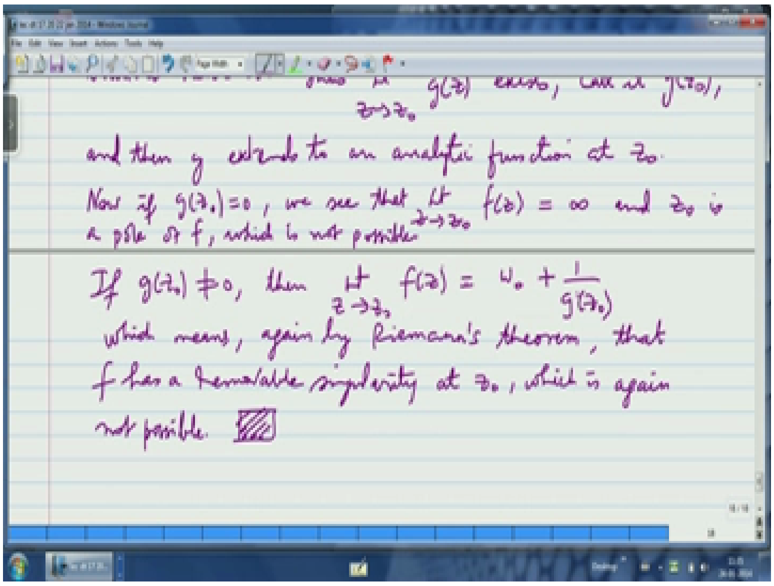

If g of Z naught is not equal to 0 then limit as Z tends to Z naught f of Z is actually W naught plus 1 by g of Z naught will get this okay which means again by Riemann's removable singularity theorem that f has a removable singularity at Z naught okay which is again not possible, so in both cases you get a contradiction and we are done okay so that that is the that brings you do the end of the proof okay so you see we are applying the you must see that we have applied the Riemann's removable singularity theorem twice we have applied it once to g and then we have applied it in one of the cases to f itself okay, fine. So this so this theorem is a very nice theorem and so what it tells you is that you take neighbourhood of an isolated essential singularity and take its image you are going to more or less fill up the whole complex plane.

You are going to get the images that is dense and we have to move towards the proof the great Picard theorem okay. Now what I am going to do next is going to deal with the point at infinity okay, so you see you see in this proof itself for example when I wrote down limit Z instead to Z naught f of Z is infinity okay I am using the point at infinity, so you would have seen the point at infinity as the extra point that is added to get 1 point compactification of the complex plane and you would also have seen it as Riemann's sphere in the 1st course and but anyway I want to recall these things because you see there is very important from now on to be able to think of infinity both in the domain of definition of the function as well as in the range of values of the function.

So you want to have a situation where you can talk of a function variable, an independent variable going to infinity and its value at for example the value of function at infinity you want to say that and you also want a function to take the value infinity okay you want to include infinity into your set of values of the independent variable and the set of values of the independent variable, so you have to deal with infinity carefully and usually in the 1st course probably this is sometimes not covered very thoroughly, so I want to just revise these things so that you are comfortable about thinking about limits at infinity and infinite limits okay, so that is what I am going to do next.

So and I need that because of the following reason, see I want to be able to think of infinity as one of the values of a function okay and I also want to be able to think of infinity as singularity okay see For example if you take an entire function okay then the point at infinity is of course approachable by infinity is of course approachable by any by any curve on the complex plane which is not bounded okay so you can always approach the point at infinity and then the question is whether the function is analytic at infinity or it is not analytic at infinity, so you want to think of infinity as a singular point okay and then the question is what kind of singularity is it because you know we have already classified singularity as either removable or pole or essential, so the question is I want be able to think of infinity the point at infinity as a singularity and question or study when they singularity is either removable singularity or a pole or essential singularity, so the point at infinity is very important.

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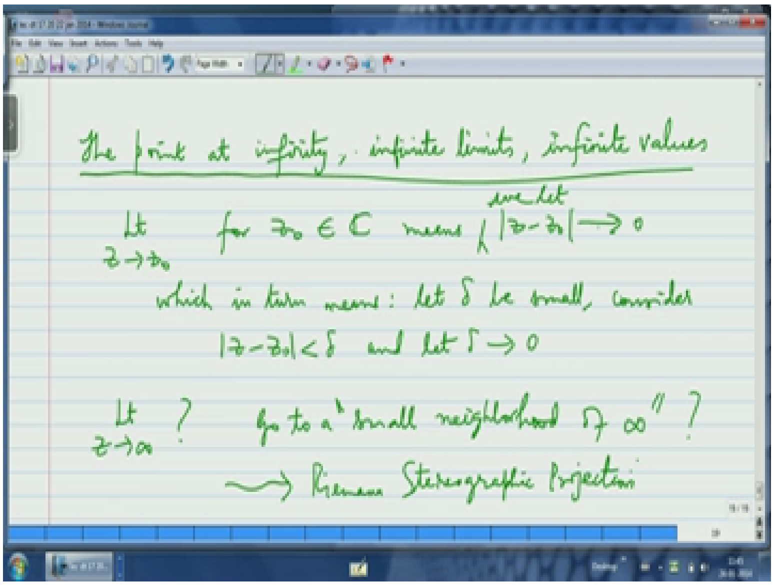

So let me go to that so I will take another color the point at infinity infinite limits, infinite values so this is what I am going to I am going to tell you about okay, so well so the idea is as follows, so what we do is let us so 1st of all let me you know the approach to everything is since we are doing calculus approach is always through limits, so let me recall what finite limit is okay as we as an independent variable approaches finite value, so you know see limit so when I write limit Z tends to Z naught okay when I write this what does it mean? For Z naught in C for let Z naught be a complex number, okay.

What does limit Z tends to Z naught mean, see it means that you going closer and closer to the point Z naught okay so basically what it means is that and what does it mean to say that you are going closer and closer to Z naught, if you think of Z as a moving point a variable point then you are saying that the distance Z to Z naught is becoming smaller and smaller okay so the limit Z tends to Z naught can be interpreted as limit mod Z minus Z naught tends to 0 okay that is how you can interpret it okay so this means so let me write that here means mod Z minus Z naught tends to 0 okay means we let so let me write that we are letting mod Z minus Z naught tends to 0 and mod Z minus Z naught mind you is the distance between Z and Z naught and what it means is that, if you go to definitions what is the business of letting something go to 0.

In analysis trying to let something go to 0 is the same as making it as small as possible, so that is where your Epsilon comes in, so usually we use Epsilon for the values of the function and use delta for the values of the variables, so let me use delta so the point is that you are putting you are choosing delta as small as you want and you are letting mod Z minus Z naught less than delta okay so let me write that down which in turn means let delta be small consider mod Z minus Z naught less than delta and let delta tends to 0 this is what it means you are making something small means you are actually taking values of that which are getting closer and closer to 0 okay and you know therefore you know of course all this conveys very clearly what is happening topologically mod Z minus Z naught less than delta is actually a disk, it is an open disk centred at Z naught radius delta.

It means that you are going infinitesimally small very small neighbourhood of Z naught and the smaller the delta is the smaller the neighbour is, so you are basically looking at you are just concentrating attention at a very small neighbourhood of Z naught that is what it means okay and now you know in the same way… so now this this can be used to also define what limit Z tends to infinity means okay so, so limit Z tends to infinity, what we make of this? What you make of limit as Z tends to infinity, okay so now this is the point where you will have to have a little bit of imagination okay, so the idea is that 1st of all you should be able to think of infinity as a point okay as a concrete point and the 2nd thing is that once you think of it as a concrete point in a space then you can think of limit Z going to infinity just as you thought of limit Z going to Z naught you know the limit as Z going to Z naught meant that you were going to a small neighbourhood of Z naught.

Now if you can think of infinity as a point in the same way limit Z tends to infinity means at your going to a small neighbourhood of infinity okay, so all you need is the way of thinking of infinity is a point on a space where you can think of a small neighbourhood around that point okay and the key to this is as you would have seen in the 1st course in complex analysis, the key to this is the so-called Riemann's stereographic projection okay, so let me explain that, so let me write this down as go to a small neighbourhood of infinity, now what does that mean? What does that mean? So the key is the Riemann's stereographic projection that is the key.

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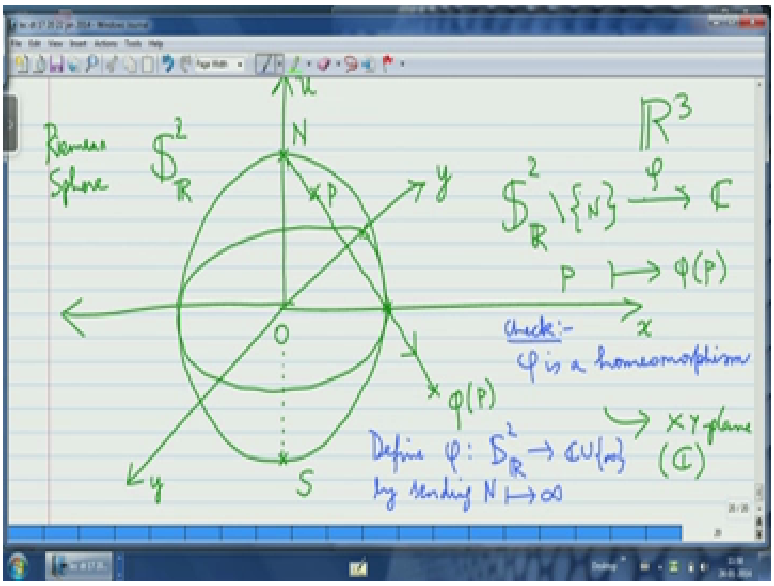

So let me recall what that this, so basically so the idea is you should have seen this, so what you do is the following. So let me draw a diagram so here is well this is the three-dimensional space, so this is my so let me draw it like this, so this is my usual x y plane, this is the origin okay and this is the x axis and this is the y axis okay or rather this is if you want to be right-handed then that is the y-axis as the negative y-axis right, so if you want the 0.1 is here 0.1 on the y-axis is here, so this point is 1, 0 this point is 0, 1. So if you think of this x y plane as usual complex plane then you have 2 coordinates x and y and of course 1, 0 is the 0.1 complex number 1, 0, 1 is the complex number I okay the point is I want to put in a 3rd axis which are normally in three-dimensional you would call the Z axis but you do not want to use Z because Z is already supposed to be x plus I y.

So you use let us see with something else for it, you use u if you want okay some books use u so let me also use it, so what we do is now you take the three-dimensional space now the snore xyz but it is xyu there is a positive u axis then what you do is that you draw this you draw this circle I mean you draw this sphere centred at the origin and radius 1 okay so let me rub these coordinates off because it will make things easier for me to draw, so you know draw this, so I have this circle I have this circle, this is the unit circle on the complex plane and then I have this 0.1 with u coordinate 1 okay and x and y coordinates 0 okay, so it will be the North pole of a sphere of this sphere centred at the origin and radius 1, so what I am going to get is am going to get something, so I am going to get something like this, so here is my sphere.

So this is this sphere is, this is the sphere that is called the Riemann's sphere, it is the sphere centred at the origin radius one unit and this point here which has coordinates 0, 0, 1 or x y and u is called the North pole so I will use the word I will put the symbol N okay and of course even if you project it all the way down you are going to get the point with ordinate 0, 0, minus 1 which is the South pole so-called South pole okay you can think of the Earth as a sphere and you have the North pole and the South pole it is just like that and so what is stereographic projection? So what it does is that so let me call this sphere let me give you a name for the sphere, so let me call this as I will put this S like a dollar symbol and put S 2 and I will put a subscript R, so this is the standard topological notation.

The 2 on top S is supposed to denote as sphere okay the 2 on top this supposed to denote the dimensions okay it is the surface of the sphere okay and mind you I am not taking the solids sphere I am only taking the surface of the sphere which is surface okay, so it is two-dimensional and by dimension I mean real dimension, so it is real two-dimensional okay and the subscript R is to remind you that this is being done in real space okay, this is being done in real space, real 3 space so the ambience space here is R 3 , so the ambience space here is R 3 in 3 space the only thing is that I am treating the x y plane as the complex plane and instead of the usual Z axis I am calling it the U axis because Z is already reserved for x plus i y of now what you do is well this is a stereographic projection there is very simple projection what it does is that it goes from the sphere I will call this sphere the Riemann's sphere okay.

So it is called the Riemann's sphere okay it goes from the Riemann's sphere minus the North pole to the complex plane to the complex plane okay and what is a map? So what you do is you take any point on the sphere okay and mind you are taking a point on the surface of the sphere okay and you are not taking the point at infinity I mean you not taking the point N you are not taking the point N, so as I just said inadvertently the point and will be the missing point at infinity okay so that will be the analogy so we will see that, so you take any point P here on this sphere other than the North pole and then what you do is you joined this thing the straight line passing through N and in P okay that straight line will go down and hit the plane at some point and that point will give you complex number because for me any point on the plane is a complex number.

I have thought of the x y plane as a complex plane and that is the complex number to which I am going to send P to okay and you can see clearly it is a project map okay, so projection map and that is why this called the stereographic projection okay. So basically what I do is that I take this line from N passing through P and then it will go and hit the let me call this as phi of P, so this is the map phi which sends P to phi of P and phi of P is a complex number, phi of P is a complex number and this is a stereographic projection, so you have if you want to think of it as a projection it is like this okay.

Now the beautiful thing is that so the beautiful thing is that this map phi is actually a Bijective map you can very will see that if P changes then phi P will change so it is an (()) (41:17) map okay and conversely give me any point on the complex plane it is so the form phi P for a unique point on this on the Riemann sphere because I can get that point by simply joining that point to the North pole that will hit the sphere at a certain point and that will be the point which will be mapped to the given point under phi okay so it is very clear that this map is Bijective okay it is very clear that this map is Bijective and in fact you can try it out as an exercise this map is actually a homeomorphism there is a topological isomorphism.

See the complex plane as a topology and this topology is also a topology that it is a same as a topology that the plane inherits as a subspace of three-dimensional space okay you take the x y plane take the complex plane x y plane and treated as plane in 3 space and you take the natural topology on R 3 you restrict that topology to the subset that is the same as the topology on the complex plane okay, so the topology on the complex plane is same as the topology that it inherits as a subspace from 3 space and this sphere is also living in 3 space, so it is also subset of the 3 space so it also has inherits a topology okay and the fact is that this map from this sphere to the plane is in fact continues Bijective map which is open and therefore you know its inverse is also continuous so it is homeomorphism it is a topological isomorphism there is a map which is continuous whose inverse is also continuous okay and of course the inverse being defined because the map is Bijective okay.

So the beautiful thing is that phi is actually homeomorphism okay so the fact that phi is a homeomorphism so let me write that down but before that let me tell you something the fact that phi is a homeomorphism tells you that therefore you can think of the whole complex plane as a punctured sphere okay see S2 minus N is a punctured sphere, it is the sphere minus the North pole and what this homeomorphism tells you, what does a homeomorphism tells you? It tells you that the 2 spaces are topologically the same, same means up to isomorphism, so when you say phi is an isomorphism, topological isomorphism namely homeomorphism you are actually saying that you get this is just another way of saying that the complex explain can be thought of topologically as a punctured sphere that is the significance of this statement okay.

So let me write that down, so here let me take another color phi is check so this check is something that is more or less obvious but you should do is phi is a homeomorphism okay I am certain many of you would have done this in the 1st course in complex analysis but it is not very difficult to do if you have not done it, So phi is a homeomorphism which tells you that the complex plane can be thought of as a punctured sphere okay, now the punctured sphere this is only 1 point namely the North pole and now you know you have you wanted a point at infinity you wanted to attach to the complex plane a point at infinity but the point is where do you attach it?

I mean you cannot see it okay but then if you look at this picture you think of the complex plane like a punctured sphere now what is the extra point that you will have to add to make it the whole sphere and mind you when you make it the whole sphere only then it becomes compact okay if you remove a point from a sphere it loses compactness okay because it will not be closed since we are in Euclidean space a subset is compact if you know only if it is closed and bounded, so this sphere this of course any subset of the sphere is of course bounded but the problem is that unless it is closed it is not compact, so the only way to make it compact is to hand that missing point that in this case it is the North pole okay.

So here comes the nice upshot of this what you do is you think of the complex plane plus the point at infinity you denote the point at infinity as with the symbol of infinity and think of it as an extra point that you add to the set of complex numbers and then what the topology you give, you give the topology which makes the natural extension of this homeomorphism phi into a homeomorphism from this sphere to the (())(46:10) complex plane which is a complex plane with that extra point at infinity added okay, so define phi from the Riemann sphere to C union infinity by sending N to the point at infinity okay.

So the North pole maps to the point at infinity and once you do this what we have done is you have given a Bijection between this sphere and C union infinity mind you C union infinity will be called the extended complex explain, it is the complex plane plus the point at infinity and therefore the complex plane plus the point at infinity is now nicely identified as a sphere okay and the advantage now is that you have therefore the point at infinity being thought of as the North pole on the sphere and it is a point on the topological space you can do topology in a neighbourhood of the North pole and think of it as doing as working in a neighbourhood of infinity okay that is how you think of infinity the point at infinity, so I will stop here.